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AUTOMORPHISMS OF COMPACT KLEIN SURFACES WITH BOUNDARY

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A Hurwitz ramification formula for morphisms of compact Klein surfaces is obtained and used to show that a compact Klein surface of genus $g \geqq 2$ with nonempty boundary cannot have more than $12(g-1)$ automorphisms.

0. **Introduction.** Let X be a compact Klein surface [1], that is, X is a compact surface with boundary together with an equivalence class of dianalytic atlases on X. A homeomorphism $f: X \rightarrow X$ of X onto itself that is dianalytic will be called an *automorphism* of X.

A natural task is to seek an upper bound for the order of the automorphism group of X when X is of (algebraic) genus $g \geqq 2$. The corresponding result for Riemann surfaces is well-known; Hurwitz [2] showed that a compact Riemann surface of genus $g \geqq 2$ cannot have more than $84(g-1)$ (orientation preserving) automorphisms. Using this result it is easy to show that the upper bound in the Klein surface case cannot be larger than $84(g-1)$. In fact, Singerman [6] has exhibited a Klein surface without boundary of genus 7 that has $504 = 84(7-1)$ automorphisms.

In this paper then we concentrate on Klein surfaces with boundary. We obtain a Hurwitz ramification formula for morphisms of Klein surfaces and show that a compact Klein surface with boundary of genus $g \geqq 2$ cannot have more than $12(g-1)$ automorphisms. We also show that the bound $12(g-1)$ is the best possible.

1. Let X be a Klein surface. The boundary of X will be denoted ∂X. Let $X^\circ = X \backslash \partial X$. X° will be called the *interior* of X.

Let $p \in X$. Then let $n_p = 1$ if $p \in \partial X$ is a boundary point of X, and let $n_p = 2$ if $p \in X^\circ$ is an interior point of X.

Now we recall the definition of a morphism of Klein surfaces [1, page 17]. Let $\mathbb{C}^+ = \{z \in \mathbb{C} \mid \operatorname{Im}(z) \geqq 0\}$, and let $\phi: \mathbb{C} \rightarrow \mathbb{C}^+$ be the folding map, so that $\phi(\alpha + \beta i) = \alpha + |\beta| i$.

DEFINITION. Let X, Y be Klein surfaces and $g: X \rightarrow Y$ a continuous map. Then g is a *morphism* if $g(\partial X) \subset \partial Y$ and if for every point $p \in X$ there exist dianalytic charts (U, z) and (V, w) at p and $g(p)$ respectively and an analytic function G on $z(U)$ such that the following diagram commutes:

$$\begin{tikzcd} U \arrow[rr, "g"] \arrow[d, "z"'] & & V \arrow[d, "w"] \\ \not{C}^{+} \arrow[r, "G"] & \not{C} \arrow[r, "\phi"] & \not{C}^{+} \end{tikzcd}$$

Let $g: X \rightarrow Y$ be a nonconstant morphism of Klein surfaces. Let $x \in X$. We can find dianalytic charts (U, z) and (V, w) at x and $g(x)$ respectively, such that $z(x) = 0 = w(g(x))$, $g(U) \subset V$, and such that $g|_U$ has the form

$$g|_U = \begin{cases} w^{-1} \circ \phi \circ (\pm z^e) & \text{if} \quad g(x) \in \partial Y \\ w^{-1} \circ (\pm z^e) & \text{if} \quad g(x) \in Y^{\circ} \end{cases}$$

where e is an integer, $e \geqq 1$ [1, pages 27–30]. The integer e is called the *ramification index* of g at x and will be denoted $e_g(x)$. We say that g is *ramified at* x if $e_g(x) > 1$; otherwise we say that g is *unramified at* x. Also, the *relative degree* of x over $g(x)$, denoted $d_g(x)$, is defined by

$$d_g(x) = \frac{n_x}{n_{g(x)}} .$$

Note that $d_g(x) = 2$ if $x \in X^{\circ}$ and $g(x) \in \partial Y$; otherwise $d_g(x) = 1$.

DEFINITION. A nonconstant morphism $g: X \rightarrow Y$ between two Klein surfaces will be called a *ramified r-sheeted covering* of Y if for every point $y \in Y$,

$$\sum_{x \in g^{-1}(y)} e_g(x) \cdot d_g(x) = r .$$

In fact, every nonconstant morphism between two compact Klein sufaces is a ramified r-sheeted covering for some r [1, page 102].

Now let X, Y, and T be Klein surfaces, $g: X \rightarrow Y$ and $f: Y \rightarrow T$ be nonconstant morphisms. Then $f \circ g: X \rightarrow T$ is a nonconstant morphisms [1, page 19]. Also, if g is a ramified r-sheeted covering of Y and f is a ramified m-sheeted covering of T, then it is easily seen that $f \circ g$ is a ramified mr-sheeted covering of T.

Let X be a Klein surface. We will denote the automorphism group of X by $\mathrm{Aut}(X)$. If X is orientable, we will denote the subgroup of orientation preserving automorphisms by $\mathrm{Aut}^+(X)$.

THEOREM 1. *Let X be a compact Klein surface and let $G \subset \mathrm{Aut}(X)$ be a finite group of automorphisms of X. Then the quotient space $\Phi = X/G$ has a unique dianalytic structure such that the canonical map $\pi: X \rightarrow \Phi$ is a morphism of Klein surfaces. Moreover, if $|G| = r$,* *then π is a ramified r-sheeted covering of Φ.*

Proof. Alling and Greenleaf have shown that Φ has a unique dianalytic structure such that π is a morphism [1, pages 52–56]. Actually, in the case of a finite group action (they consider the action of a discontinuous group), their proof shows that π is a ramified r-sheeted covering of Φ.

2. Let Y be a compact Klein surface, and let E be the field of all meromorphic functions on Y. E is an algebraic function field in one variable over $\boldsymbol{R}$, and as such has an *algebraic genus* g. We will refer to this nonnegative integer g as the *genus* of the compact Klein surface Y. In case Y is a Riemann surface, g is equal to the topological genus of Y. For more details, see [1].

Henceforth the term Klein surface will be reserved for those Klein surfaces X that are not Riemann surfaces, that is, for those X that are nonorientable or have nonempty boundary or both.

Let X be a compact Klein surface. Let (X_c, π, σ) be the complex double of X, that is, X_c is a compact Riemann surface, $\pi: X_c \to X$ is an unramified 2-sheeted covering of X, and σ is the unique anti-analytic involution of X_c such that $\pi = \pi \circ \sigma$. For more details, see [1, pages 37–40]. It is well-known that the genus of X is equal to the genus of its complex double X_c. The complex double also has the following important property [1, page 39]:

PROPOSITION 1. *Let M be a compact Riemann surface, X a compact Klein surface, and $f: M \to X$ a nonconstant morphism. Then there exists a unique analytic map $\rho: M \to X_c$ such that $\pi \circ \rho = f$.*

We use the complex double to obtain a Hurwitz ramification formula for morphisms of compact Klein surfaces.

THEOREM 2. *Let X and Y be compact Klein surfaces* (that are not Riemann surfaces), *and let $f: X \to Y$ be a ramified r-sheeted covering of Y. Let g be the genus of X, γ the genus of Y. Then*

$$2g - 2 = r(2\gamma - 2) + \sum_{x \in X} n_x(e_f(x) - 1)\ .$$

Proof. Let (X_c, π, σ) and (Y_c, ν, τ) denote the complex doubles of X and Y respectively. By Proposition 1, there exists a unique analytic map $\tilde{f}: X_c \to Y_c$ such that the following diagram commutes:

$$\begin{tikzcd} X_c \arrow[r, "\tilde{f}"] \arrow[d, "\pi"'] & Y_c \arrow[d, "\nu"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

$f \circ \pi = \nu \circ \tilde{f}$ is a ramified $2r$-sheeted covering of Y. But $\tilde{f}$ is a nonconstant analytic mapping between compact Riemann surfaces. Thus $\tilde{f}$ is a ramified m-sheeted covering of Y_c for some m [3, page 15]. Since ν is a 2-sheeted covering, clearly $m = r$. Then, since a Klein surface and its complex double have the same genus, the classical Hurwitz ramification formula [3, page 16] gives

$$(2g - 2) = r(2\gamma - 2) + \sum_{p \in X_c} (e_{\tilde{f}}(p) - 1) .$$

Let $p \in X_c$ and note that $e_{\tilde{f}}(p) = e_f(\pi(p))$, since $e_{\tilde{f}}(p) = e_{\nu \circ \tilde{f}}(p) = e_{f \circ \pi}(p) = e_f(\pi(p))$.

Therefore

$$\begin{aligned}(2g - 2) &= r(2\gamma - 2) + \sum_{p \in X_c} (e_f(\pi(p)) - 1) \\ &= r(2\gamma - 2) + \sum_{x \in X} n_x(e_f(x) - 1) .\end{aligned}$$

Finally, we recall how the automorphism group of a compact Klein surface can be obtained from that of its complex double [1, page 79]:

PROPOSITION 2. *Let X be a compact Klein surface with complex double (X_c, π, σ). Then*

$$\mathrm{Aut}(X) \cong \{g \in \mathrm{Aut}^+(X_c) \mid \sigma \circ g \circ \sigma = g\} .$$

COROLLARY. *If X is a compact Klein surface of genus $g \geqq 2$, then*

$$|\mathrm{Aut}(X)| \leqq 84(g - 1) .$$

Thus $\mathrm{Aut}(X)$ *is finite group.*

Proof. The genus of X_c is g, so that the corollary follows immediately from the Proposition and Hurwitz's bound for $|\mathrm{Aut}^+(X_c)|$.

3. **Applications.** Let X be a compact Klein surface of genus g, and let $G \subset \mathrm{Aut}(X)$ be a finite group of automorphisms of X of order $|G| = r$. By Theorem 1, the quotient space $\Phi = X/G$ is a compact Klein surface and the canonical map $\pi: X \to \Phi$ is a ramified r-sheeted covering of Φ. Let γ denote the genus of Φ.

Let $p \in \Phi$. We will call the set $\pi^{-1}(p)$ the *fiber above* p. If $g \in \mathrm{Aut}(X)$ then $g(\partial X) = \partial X$ and $g(X^\circ) = X^\circ$. Therefore either $\pi^{-1}(p) \subset \partial X$ or $\pi^{-1}(p) \subset X^\circ$. Equivalently, if $x, y \in X$ such that $\pi(x) = \pi(y)$, then $d_\pi(x) = d_\pi(y)$.

Let $S_x = \{g \in G \mid g(x) = x\}$ be the stabilizer subgroup of G of a point $x \in X$. We can find a dianalytic chart (U, z) at x such that $g(U) = U$ for all $g \in S_x$. Let $S'_x = \{g \in S_x \mid z \circ g \circ z^{-1}$ is analytic$\}$. Clearly S'_x is independent of the choice of (U, z). Either $S_x = S'_x$ or S'_x is a subgroup of index 2. $S_x = S'_x$ in case (i) $x \in X^\circ$ and $\pi(x) \in \Phi^\circ$ or (ii) $x \in \partial X$ and $e_\pi(x) = 1$; otherwise $S_x \neq S'_x$. The ramification index $e_\pi(x)$ is the order of S'_x in case $x \in X^\circ$ and $\pi(x) \in \partial\Phi$; otherwise $e_\pi(x)$ is the order of S_x. For more details, see [1, page 52–56]. If $\pi(x) = \pi(y)$, then clearly there are isomorphisms $S_x \cong S_y$ and $S'_x \cong S'_y$, so that $e_\pi(x) = e_\pi(y)$ in any case.

If π is ramified at a point $x \in X$ and $\pi(x) = p$, then we will say that π is *ramified above* p.

Now the quotient map $\pi\colon X \to \Phi$ is ramified above a finite number of points of Φ, say $a_1, \cdots, a_\omega$. Let k_i denote the ramification index $e_\pi(x)$ of any point x such that $\pi(x) = a_i$. We will write $n_i = n_{a_i}$.

Fix a_i. First suppose that if $\pi(x) = a_i$, then the relative degree $d_\pi(x) = 1$, *i.e.*, $n_x = n_{a_i} = n_i$. Then there are r/k_i points in the fiber $\pi^{-1}(a_i)$, and

$$\begin{aligned}\sum_{x \in \pi^{-1}(a_i)} n_x(e_\pi(x) - 1) &= \frac{r}{k_i}\cdot n_i \cdot (k_i - 1)\\ &= r n_i\left(1 - \frac{1}{k_i}\right).\end{aligned}$$

Now suppose that if $\pi(x) = a_i$, then $d_\pi(x) = 2$, so that $n_x = 2$, $n_i = 1$. In this case there are $r/2k_i$ points in the fiber $\pi^{-1}(a_i)$, and

$$\begin{aligned}\sum_{x \in \pi^{-1}(a_i)} n_x(e_\pi(x) - 1) &= \frac{r}{2k_i}\cdot 2 \cdot (k_i - 1)\\ &= r n_i\left(1 - \frac{1}{k_i}\right).\end{aligned}$$

Therefore the Hurwitz ramification formula (Theorem 2) can be rewritten in the following form:

$$(*) \qquad \frac{2g - 2}{r} = 2\gamma - 2 + \sum_{i=1}^{\omega} n_i\left(1 - \frac{1}{k_i}\right).$$

Henceforth we assume that X is of genus $g \geqq 2$. Then, by the corollary to Proposition 2, $\mathrm{Aut}(X)$ is a finite group, so that in our calculations here we *can* let $G = \mathrm{Aut}(X)$. The calculations will be divided into several cases.

A. $\gamma \geqq 1$.

First suppose that $\gamma \geqq 2$. Then, immediately from (*), we have $(2g-2)/r \geqq 2$. Thus $r \leqq g-1$.

Now suppose $\gamma = 1$. Then $\omega \neq 0$, and

$$\frac{2g-2}{r} \geqq n_1\left(1-\frac{1}{k_1}\right) \geqq 1-\frac{1}{k_1} \geqq \frac{1}{2}\,.$$

Hence $r \leqq 4(g-1)$.

B. $\gamma = 0$, three lemmas.

Recall that there are two compact Klein surfaces of genus zero, the disc D and the real projective plane B. Each has a unique dianalytic structure [1, pages 59–60].

Note that with $\gamma = 0$, (*) implies that $\omega \geqq 2$.

In the following lemmas we will assume that the Klein surface X has nonempty boundary. Then the quotient space Φ has nonempty boundary, and since $\gamma = 0$, Φ is the disc D (with its unique dianalytic structure).

LEMMA 1. *Suppose $\partial X \neq \varnothing$. If π is ramified at a boundary point $x \in \partial X$, then the ramification index $e_\pi(x) = 2$.*

Proof. Let $e = e_\pi(x)$. $\pi(x) \in \partial D$, of course.

We can find dianalytic charts (U, z) and (V, w) at x and $\pi(x)$ respectively, such that $z(x) = 0 = w(\pi(x))$, $\pi(U) \subset V$, and such that

$$\pi|_U = w^{-1} \circ \phi \circ (\pm z^e)$$

$e \geqq 2$, since π is ramified at x. Suppose $e > 2$. $z(U)$ is an open subset of $\mathbb{C}^+$ about the origin. Thus for a small enough real number $t > 0$, both the points $\xi_1 = t$, $\xi_2 = t\exp(2\pi i/e)$ belong to $z(U)$. Then $z^{-1}(\xi_1) \in \partial X$ and $z^{-1}(\xi_2) \in X^\circ$, and clearly $\pi(z^{-1}(\xi_1)) = \pi(z^{-1}(\xi_2))$. But for each point $p \in D$, either $\pi^{-1}(p) \subset \partial X$ or $\pi^{-1}(p) \subset X^\circ$. Thus we have a contradiction. Therefore $e = 2$.

LEMMA 2. *Suppose $\partial X \neq \varnothing$. If π is ramified above a boundary point of D, that is, $a_k \in \partial D$ for some k, then at least two of the fibers $\pi^{-1}(a_i) \subset \partial X$. Further the number of ramified fibers contained in ∂X is even.*

Proof. Suppose $a_k \in \partial D$ for some k.

If $\pi^{-1}(a_k) \subset \partial X$, then let $x \in \partial X$ such that $\pi(x) = a_k$. $e_\pi(x) = 2$ by Lemma 1, and it is easy to see that there is an interior point $q \in X^\circ$ such that $\pi(q) \in \partial D$ (find charts as in the proof of Lemma 1 and look at $\xi = t\exp(\pi i/2)$ for small enought t). Thus regardless of whether $\pi^{-1}(a_k) \subset \partial X$ or $\pi^{-1}(a_k) \subset X^\circ$, there is an interior point $q \in X^\circ$ such that $\pi(q) \in \partial D$.

Now $\pi(\partial X)$ is a compact and hence closed subset of ∂D. Also, $\partial D \backslash \pi(\partial X) \neq \emptyset$. Topologically ∂D is just a circle, of course. Therefore $\pi(\partial X)$ is a finite union of closed intervals.

It is easy to see that if p is an end-point of one of these closed intervals, then π is ramified above p and $\pi^{-1}(p) \subset \partial X$. The number of such end-points is clearly even and not less than two.

LEMMA 3. *Suppose X is orientable and $\partial X \neq \emptyset$. If $G \subset \mathrm{Aut}^+(X)$, then π is ramified only above interior points of D.*

Proof. Let $x \in X$, and consider the stablizer subgroup S_x and its subgroup S'_x. Since $G \subset \mathrm{Aut}^+(X)$, $S_x = S'_x$, directly from the definition of S'_x. Consequently, if $x \in X^\circ$ then $\pi(x) \in D^\circ$ (π may or may not be ramified at x), and if $x \in \partial X$ then $e_\pi(x) = 1$. Hence π is ramified only above interior points of D.

C. $\gamma = 0$, ramification above Φ° only

Suppose $a_1, \cdots, a_w \in \Phi^\circ$ are interior points of Φ. Then $n_i = 2$ for each i, and by (*)

$$\frac{2g-2}{r} = -2 + 2\sum_{i=1}^{\omega}\left(1 - \frac{1}{k_i}\right)$$

or

$$\frac{g-1}{r} = \omega - 1 - \frac{1}{k_1} - \cdots - \frac{1}{k_\omega}\,. \tag{1}$$

Again we see that $\omega \geqq 2$.

Suppose $\omega \geqq 3$. Since $k_i \geqq 2$ for each i, by (1)

$$\frac{g-1}{r} \geqq \omega - 1 - \frac{\omega}{2} \geqq \frac{1}{2}\,.$$

Hence $r \leqq 2(g-1)$.

Suppose $\omega = 2$. $k_1 = k_2 = 2$ is not a possibility, since that would imply $g = 1$. Clearly then

$$\frac{g-1}{r} \geqq 2 - 1 - \frac{1}{2} - \frac{1}{3} = \frac{1}{6}\,.$$

Hence $r \leqq 6(g-1)$.

These calculations have already yielded two interesting results:

THEOREM 3. *Let X be a compact Klein surface without boundary* *of genus $g \geqq 2$. If G is a group of automorphisms of X such that X/G is the real projective plane B, then*

$$|G| \leqq 6(g-1) .$$

Proof. $\partial B = \varnothing$, so the the theorem follows from calculations of §C.

THEOREM 4. *Let X be a compact orientable Klein surface with boundary of genus $g \geqq 2$. Then*

$$|\mathrm{Aut}^+(X)| \leqq 6(g-1)$$

and

$$|\mathrm{Aut}(X)| \leqq 12(g-1) .$$

Proof. The first fact follows from the calculations of sections A and C and Lemma 3.

Either $\mathrm{Aut}(X) = \mathrm{Aut}^+(X)$ or $\mathrm{Aut}^+(X)$ is a subgroup of $\mathrm{Aut}(X)$ of index two. Thus the first fact implies the second.

D. $\gamma = 0$, ramification above $\partial\Phi$, $\partial X \neq \varnothing$.

Now we assume that X is a Klein surface *with boundary*. Then the quotient space Φ is the disc D (with its unique dianalytic structure).

We also assume that there is ramification above ∂D. By Lemma 2, at least two of the fibers $\pi^{-1}(a_i) \subset \partial X$. We may suppose that this is the case for a_1 and a_2. Then $k_1 = k_2 = 2$ by Lemma 1. $n_1 = n_2 = 1$, of course, so by (*)

$$\frac{2g-2}{r} = -1 + \sum_{i=3}^{\omega} n_i\left(1 - \frac{1}{k_i}\right) . \tag{2}$$

Therefore $\omega \geqq 3$ in this case.

First suppose $\omega \geqq 5$. Then by (2)

$$\frac{2g-2}{r} \geqq -1 + (\omega - 2)\cdot\frac{1}{2} \geqq \frac{1}{2} .$$

Thus $r \leqq 4(g-1)$.

Next suppose $\omega = 4$. There are three cases to consider, depending on whether there are 0, 1, or 2 of the points a_3 and a_4 on the boundary of D.

If $a_3, a_4 \in D^\circ$, then

$$\frac{2g-2}{r} \geqq -1 + 2\cdot\frac{1}{2} + 2\cdot\frac{1}{2} = 1 ,$$

and $r \leqq 2(g-1)$.

If one of the two points, say a_3, is a boundary point and $a_4 \in D^\circ$, then

$$\frac{2g-2}{r} \geqq -1 + \frac{1}{2} + 2 \cdot \frac{1}{2} = \frac{1}{2},$$

and $r \leqq 4(g-1)$.

If $a_3, a_4 \in \partial D$, then note that $k_3 = k_4 = 2$ is not a possibility. Clearly then

$$\frac{2g-2}{r} \geqq -1 + \frac{1}{2} + \frac{2}{3} = \frac{1}{6},$$

and $r \leqq 12(g-1)$.

Finally, suppose $\omega = 3$. Then from (2) we see that $n_3 = 2$, i.e., $a_3 \in D^\circ$. Then

$$\frac{2g-2}{r} = 1 - \frac{2}{k_3}.$$

Hence $k_3 \geqq 3$ and $r \leqq 6(g-1)$ in this case.

A review of the calculations of §§ A, C, and D gives our main result:

THEOREM 5. *Suppose X is a compact Klein surface with boundary of genus $g \geqq 2$. Then*

$$|\mathrm{Aut}(X)| \leqq 12(g-1).$$

4. **Sharpness of the bounds.** Here we consider three compact Klein surfaces of low genus and determine their automorphism groups directly.

EXAMPLE 1. Let Y be a sphere with 3 holes, with the holes placed around the equator, centered around the vertices of an inscribed equilateral triangle. Y is an orientable Klein surface of genus 2. Y has a group (isomorphic to the dihedral group D_3) of orientation-preserving automorphisms of order 6. Reflection in the plane of the equator is an orientation-reversing automorphism. Y therefore has $12 = 12(2-1)$ automorphisms. The automorphism group is just $C_2 \times D_3$, where C_2 denotes the cyclic group of order 2.

EXAMPLE 2. Let X be a sphere with 6 holes, with the holes centered around the vertices of an inscribed regular octahedron. X is an orientable Klein surface of genus 5. X has a group of automorphisms isomorphic to the complete symmetry group (including reflections) of the regular octahedron, which is $C_2 \times S_4$. Thus X has $48 = 12(5 - 1)$ automorphisms.

EXAMPLE 3. Let X be the Klein surface of Example 2, and let $\tau: X \to X$ denote the antipodal map. The quotient space $W = X/\tau$ is a real projective plane with 3 holes, a nonorientable Klein surface of genus 3. By considering the action of $C_2 \times S_4$ on X, it is easy to see that there is a group of automorphisms of W isomorphic to S_4.

Thus the bounds obtained in Theorems 4 and 5 are best possible. The bound $12(g - 1)$ is attained for both orientable and nonorientable surfaces. Theorem 3 was obtained incidentally in our proof of Theorem 4. We do not know if the bound of Theorem 3 is the best possible.

In a forthcoming article [5] we study those finite groups that act as a group of $12(g - 1)$ automorphisms of a compact Klein surface of genus $g \geqq 2$ with nonempty boundary. There we exhibit several infinite families of values of g for which there is a compact Klein surface with boundary of genus g that has $12(g - 1)$ automorphisms.

5. Nevertheless it is possible to improve the bound $12(g - 1)$ for a large number of topological types of Klein surfaces. Our main tool is a theorem of Maskit.

Let X be a compact orientable Klein surface with boundary. By the *analytic genus* p of X we mean the topological genus of the compact surface X^* obtained by attaching a disc to each boundary component of X. The relationship between p and the (algebraic) genus g of X is given by

$$g = 2p + k - 1 ,$$

where k is the number of boundary components of X.

THEOREM 6. *Let X be a compact orientable Klein surface of genus g with k boundary components. If*

$$\frac{6(g-1)}{7} < k \leqq g - 3 ,$$

then

$$|\mathrm{Aut}(X)| \leqq 84(g - k - 1) < 12(g - 1) .$$

Proof. Let p be the analytic genus of X. Maskit has shown that there exists a compact Riemann surface X^* of genus p and an analytic embedding of X into X^* such that, under this embedding, every orientation-preserving automorphism of X is the restriction of an orientation-preserving automorphism of X^* [4, page 718]. Thus $|\mathrm{Aut}^+(X)| \leqq |\mathrm{Aut}^+(X^*)|$.

Now $2p = g - k + 1 \geqq 4$, so that $p \geqq 2$ and we may apply Hurwitz's bound for $|\mathrm{Aut}^+(X^*)|$. Hence $|\mathrm{Aut}(X)| \leqq 2\cdot 84(p-1) = 84(g-k-1)$.

Note that $84(g-k-1) < 12(g-1)$ if and only if $6(g-1) < 7k$.

If $g < 16$, there are no integer values of k such that $6(g-1)/7 < k \leqq g-3$. The improved bound of Theorem 6 does apply to orientable Klein surfaces of genus 16 with 13 boundary components.

For large values of g and suitable values of k, Theorem 6 gives a much better bound than Theorem 5. In fact, if $(g-k)$ is held fixed (that is, the analytic genus remains constant), Theorem 6 gives a uniform bound for the size of the automorphism group. On the other hand, there are orientable Klein surfaces with boundary of each genus $g \geqq 2$ to which Theorem 6 does not apply.

Finally, we obtain a similar result for nonorientable Klein surfaces with boundary.

THEOREM 7. *Let X be a compact nonorientable Klein surface of genus g with k boundary components. If*

$$\frac{6(g-1)}{7} < k \leqq g-2\,,$$

then

$$|\mathrm{Aut}(X)| \leqq 84(g-k-1) < 12(g-1)\,.$$

Proof. Let (X_0, ν, τ) denote the orienting double of X, that is, X_0 is a compact orientable Klein surface with $2k$ boundary components, $\nu: X_0 \rightarrow X$ is an unramified 2-sheeted covering of X, and τ is the unique antianalytic involution of X_0 such that $\nu \circ \tau = \nu$. Further the genus g' of X_0 is $g' = 2g - 1$. For more details, see [1, pages 42-43].

Suppose $f: X \rightarrow X$ is an automorphism of X. Then there exists a unique orientation-preserving automorphism $\tilde{f}$ of X_0 such that

$$\begin{tikzcd} X_0 \arrow[r, "\tilde{f}"] \arrow[d, "\nu"'] & X_0 \arrow[d, "\nu"] \\ X \arrow[r, "f"] & X \end{tikzcd}$$

commutes [1, page 42]. Hence $|\mathrm{Aut}(X)| \leqq |\mathrm{Aut}^+(X_0)|$.

Let p' be the analytic genus of X_0.

$$p' = \frac{(2g-1) - 2k + 1}{2} = g - k \geqq 2\,.$$

Then, using Maskit's theorem as in the proof of Theorem 6, we have that

$$|\mathrm{Aut}\,(X)| \leqq |\mathrm{Aut}^+\,(X_0)| \leqq 84(p' - 1) = 84(g - k - 1)\,.$$

As before, $84(g - k - 1) < 12(g - 1)$ if and only if $6(g - 1) < 7k$.

Note that the improved upper bound in Theorem 7 is the same as in Theorem 6. The bound is applicable to a larger range of values of g and k in the nonorientable case, however.

The lowest genus to which Theorem 7 applies is the case of nonorientable Klein surfaces of genus 9 with 7 boundary components.

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