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RADON PARTITIONS IN REAL LINEAR SPACES

C. M. PETTY

An intimate connection is established between primitive Radon partitions and generalized poonems of a set in a real linear space $\mathscr{L}$. It is shown that if $K \subset \mathscr{L}$ is convex then K is the convex hull of its extreme points if and only if the intersection of poonems of K is a poonem of K. Among the applications is a study of k-neighborly sets. This yields a considerable generalization of the theory of k-neighborly polytopes.

1. Basic concepts. The notion of a primitive partition introduced in R^d by Hare and Kenelly [3] may be formulated for a real linear space $\mathscr{L}$. A pair of subsets (A, B) is called a Radon partition provided $A \cap B = \varnothing$ and $\operatorname{conv} A \cap \operatorname{conv} B \neq \varnothing$. A Radon partition (A, B) is called primitive if (A', B') with $A' \subset A$, $B' \subset B$ is a Radon partition only in the case $A' = A$ and $B' = B$.

If $P \subset \mathscr{L}$ then a subset $S \subset P$ is called a poonem of P provided (a) $\operatorname{conv} S = \operatorname{conv} P \cap \operatorname{aff} S$ and (b) $\operatorname{conv} P \sim \operatorname{conv} S$ is convex. If (a) holds for some $S \subset P$, $\operatorname{conv} P \sim \operatorname{conv} S = \operatorname{conv} P \sim \operatorname{aff} S$ and if P is a closed convex set in R^d and $S \subset P$ is convex then this definition reduces to the definition of poonem given in [2, p. 20]. However, the admission of nonconvex poonems is essential for the results obtained here.

Primitive partitions and poonems are basic in that other concepts may be defined in terms of them.

DEFINITIONS 1.1. (a) A subset $P \subset \mathscr{L}$ is called a simplex provided there is no primitive partition (A, B) with $A \cup B \subset P$. (b) If F is a flat and $P \subset F$, then P is said to be in general position in F provided for any subset $S \subset P$ either S is a simplex or $\operatorname{aff} S = F$. (c) A set $P \subset \mathscr{L}$ is said to be k-neighborly, where k is a cardinal number, provided every subset $S \subset P$ with $\operatorname{card} S \leqq k$ is a poonem of P.

If K is a convex d-polytope in R^d and $1 \leqq k \leqq d$, then K is k-neighborly in the usual sense [2, Chapter 7] if and only if $P = \operatorname{vert} K$ is k-neighborly as defined here.

2. The principal theorems. We first recall or prove some basic results.

LEMMA 2.1. (a) *If (A, B) is a Radon partition, then there exists a primitive partition (A', B') with $A' \subset A$ and $B' \subset B$. If (A, B)*

is primitive, then both A and B are finite simplices, $\operatorname{conv} A \cap \operatorname{conv} B$ *is a singleton and* $\operatorname{card}(A \cup B) = \dim \operatorname{aff}(A \cup B) + 2$.

(b) *A nonempty set P is a simplex if and only if P is an affinely independent set.*

(c) *If $x \in \operatorname{aff} P$ but $x \notin P$, then there exists a simplex $T \subset P$ and a primitive partition (A, B) such that $A \cup B = \{x\} \cup T$.*

Proof. (a) Suppose (A, B) is a Radon partition and $p \in \operatorname{conv} A \cap \operatorname{conv} B$. Then p is a convex combination of a finite number of points in A and p is also a convex combination of a finite number of points in B. See [5, p. 15]. The existence of a primitive (A', B') now follows easily. For a proof of the remaining statements in (a) see [1] and [3].

(b) A finite set of (distinct) points $x_1, \cdots, x_n$ is affinely independent if the conditions $\alpha_1 x_1 + \cdots + \alpha_n x_n = 0$ and $\alpha_1 + \cdots + \alpha_n = 0$ are simultaneously satisfied only by $\alpha_1 = \cdots = \alpha_n = 0$. A nonempty set P is affinely independent if every nonempty finite subset of P is affinely independent. In the proof of (a) above, the difference of the two convex combinations which represent p yields an affine dependency. It is now easily shown that a nonempty set P is affinely dependent if and only if there exists a primitive (A, B) with $A \cup B \subset P$.

(c) If $S = \{x_1, \cdots, x_n\} \subset P$, then $\operatorname{aff} S = \{\alpha_1 x_1 + \cdots + \alpha_n x_n \mid \alpha_1 + \cdots + \alpha_n = 1\}$ and $\operatorname{aff} P = \bigcup \{\operatorname{aff} S \mid S \subset P \text{ is finite}\}$. It follows that $\operatorname{aff} P = \bigcup \{\operatorname{aff} T \mid T \subset P \text{ is a finite simplex}\}$. If $x \in \operatorname{aff} P$ but $x \notin P$, then there exists a minimal simplex $T \subset P$ such that $x \in \operatorname{aff} T$ but $x \notin T$. Thus, $\{x\} \cup T$ can be partitioned so that $A \cup B = \{x\} \cup T$ and (A, B) is a Radon partition. But (A, B) must be primitive since, otherwise, x would lie in the affine hull of a proper subset of T contrary to the minimal property of T.

THEOREM 2.2. *Let $S \subset P \subset \mathscr{L}$. Then the following four statements are equivalent:*

(a) *S is a poonem of P.*

(b) *If (A, B) is primitive with $A \subset \operatorname{aff} S$ and $B \subset \operatorname{conv} P$, then $B \subset \operatorname{conv} S$.*

(c) *If (A, B) is primitive with $A \subset S$ and $B \subset P$, then $B \subset \operatorname{conv} S$.*

(d) $\operatorname{conv}(P \sim \operatorname{conv} S) \cap \operatorname{aff} S = \varnothing$.

Proof. (a) $\Rightarrow$ (b). Let S be a poonem of P and suppose (A, B) is primitive with $A \subset \operatorname{aff} S$ and $B \subset \operatorname{conv} P$. Let $p \in \operatorname{conv} A \cap \operatorname{conv} B$ and let $B = B_1 \cup B_2$ where $B_1 \subset \operatorname{conv} S$ and $B_2 \subset \operatorname{conv} P \sim \operatorname{conv} S$. By definition of a poonem, $\operatorname{conv} B_2 \subset \operatorname{conv} P \sim \operatorname{conv} S = \operatorname{conv} P \sim \operatorname{aff} S$ and therefore $\operatorname{conv} B_2 \cap \operatorname{aff} S = \varnothing$. Since $p \in \operatorname{conv} A \subset \operatorname{aff} S$, it follows that $p \notin \operatorname{conv} B_2$ and therefore $B_1 \neq \emptyset$. Suppose $B_2 \neq \emptyset$. If $p \in \operatorname{conv} B_1$, then (A, B_1) would be a Radon partition contradicting (A, B) being primitive. Thus, $p \in \operatorname{conv} B = \operatorname{conv}(\operatorname{conv} B_1 \cup \operatorname{conv} B_2)$ but $p \notin \operatorname{conv} B_1 \cup \operatorname{conv} B_2$. By a basic result in real linear spaces [5, p. 16], there exist $p_i \in \operatorname{conv} B_i$ such that $p = tp_1 + (1-t)p_2$ where $0 < t < 1$. Since both p and p_1 belong to $\operatorname{aff} S$, so also does p_2. But $\operatorname{conv} B_2 \cap \operatorname{aff} S = \emptyset$. Hence $B_2 = \emptyset$ which completes the proof.

(b) $\Rightarrow$ (c) is trivial.

(c) $\Rightarrow$ (d). Let $K_1 = \operatorname{conv} S$, $D = P \sim K_1$ and $K_2 = \operatorname{conv} D$. Suppose there exists a point $p \in K_2 \cap \operatorname{aff} S$. If $p \in K_1$, then (S, D) is a Radon partition. By Lemma (2.1a), there exists a primitive (A', B') with $A' \subset S$ and $B' \subset D$. But (c) implies $B' \subset K_1$ and therefore $B' \subset D \cap K_1 = \emptyset$. Thus, $p \in \operatorname{aff} S \sim K_1$. By Lemma (2.1c), there exists a simplex $T \subset S$ and a primitive (A, B) such that $A \cup B = T \cup \{p\}$. We may assume that $p \in B$ and therefore $A \subset T \subset S$. Let $C = B \sim \{p\} \subset T$. Now $(A, C \cup D)$ is a Radon partition since $A \cap (C \cup D) = \emptyset$ and $\operatorname{conv} A \cap \operatorname{conv}(C \cup D) \supset \operatorname{conv} A \cap \operatorname{conv} B \neq \emptyset$. By Lemma (2.1a), there exists a primitive (A', B') with $A' \subset A \subset T \subset S$ and $B' \subset C \cup D \subset P$. Now (c) implies $B' \subset K_1$. Since $D \cap K_1 = \emptyset$ we have $B' \subset C \subset T$. But T is a simplex and therefore $A' \cup B' \not\subset T$. Hence, $K_2 \cap \operatorname{aff} S = \emptyset$ which completes the proof.

(d) $\Rightarrow$ (a). As above, let $K_1 = \operatorname{conv} S$, $D = P \sim K_1$ and $K_2 = \operatorname{conv} D$. If $K_1 = \emptyset$, then $S = \emptyset$ is a poonem of P and if $K_2 = \emptyset$, then $\operatorname{conv} S = \operatorname{conv} P$ which, by definition, implies S is a poonem of P. We may, therefore, assume that K_1 and K_2 are nonempty. Let $p \in \operatorname{conv} P \cap \operatorname{aff} S$. Since $p \in \operatorname{conv}(K_1 \cup K_2)$, there exist $p_i \in K_i$ such that $p = tp_1 + (1-t)p_2$ where $0 \leqq t \leqq 1$. If $t < 1$, then $p_2 \in \operatorname{aff} S$ which contradicts (d). Hence $t = 1$ and $p = p_1 \in K_1$. Thus $K_1 = \operatorname{conv} P \cap \operatorname{aff} S$ which establishes part (a) of the definition of a poonem. Now, suppose $\operatorname{conv} P \sim K_1$ is not convex. Then, there exist $r_1, r_2 \in \operatorname{conv} P \sim K_1$ such that for some α, $0 < \alpha < 1$, the point $p_0 = \alpha r_1 + (1-\alpha)r_2 \in K_1$. Since $r_i \in \operatorname{conv} P = \operatorname{conv}(K_1 \cup K_2)$, we have $r_i = (1-\beta_i)p_i + \beta_i q_i$ where $p_i \in K_1$, $q_i \in K_2$ and $0 \leqq \beta_i \leqq 1$. However, $\beta_i > 0$ since $r_i \notin K_1$. Let $\lambda = \beta_1\alpha/(\beta_1\alpha + \beta_2(1-\alpha))$, $0 < \lambda < 1$. Then, $\lambda q_1 + (1-\lambda)q_2 = \tau_0 p_0 + \tau_1 q_1 + \tau_2 p_2$ where $\tau_0 = (\beta_1\alpha + \beta_2(1-\alpha))^{-1}$, $\tau_1 = -\alpha(1-\beta_1)\tau_0$, and $\tau_2 = -(1-\alpha)(1-\beta_2)\tau_0$. Since p_0, p_1, p_2 belong to K_1 and $\tau_0 + \tau_1 + \tau_2 = 1$, the point $\lambda q_1 + (1-\lambda)q_2 \in \operatorname{aff} S$ which contradicts (d). Hence $\operatorname{conv} P \sim K_1$ is convex. This completes the proof of Theorem 2.2.

COROLLARY 2.3. *Let* $Q \subset S \subset P \subset \mathscr{L}$.

(a) *If Q is a poonem of S and S is a poonem of P, then Q is a poonem of P.*

(b) *If Q is a poonem of P, then Q is a poonem of S.*

(c) *P is a simplex if and only if every subset of P is a poonem of P.*

Proof. (a) Suppose (A, B) is primitive with $A \subset Q$ and $B \subset P$. We need only show that $B \subset \operatorname{conv} Q$. Since S is a poonem of P and $A \subset S$, we have $B \subset \operatorname{conv} S$. But Q is a poonem of S. Thus, $B \subset \operatorname{conv} Q$.

Proofs of (b) and (c) are easily obtained.

COROLLARY 2.4. *Let $K \subset \mathscr{L}$ be convex and let $P_c(K)$ be the set of all convex poonems of K. If $P_c(K)$ is partially ordered by inclusion, then $P_c(K)$ is a complete lattice where the greatest lower bound of a family of convex poonems is their intersection.*

Proof. The empty set $\varnothing$ and K itself are convex poonems of K. We need only show that the intersection $H = \bigcap K_\alpha$ of a family $\{K_\alpha\}$ of convex poonems of K is a poonem of K. Suppose (A, B) is primitive with $A \subset H$ and $B \subset K$. Since K_α is a poonem of K and $A \subset K_\alpha$, we have $B \subset K_\alpha$. Thus $B \subset H$ and therefore H is a poonem of K which completes the proof.

Even for convex K, the intersection of poonems need not be a poonem. In this direction, the following theorem with $P = K$ yields a new characterization of those convex sets K for which $K = \operatorname{conv}(\operatorname{ext} K)$.

THEOREM 2.5. *Let $P \subset \mathscr{L}$. Then* $\operatorname{conv} P = \operatorname{conv}(\operatorname{ext}(\operatorname{conv} P))$ *if and only if the intersection of an arbitrary family of poonems of P is a poonem of P.*

Proof. Suppose $\operatorname{conv} P = \operatorname{conv}(\operatorname{ext}(\operatorname{conv} P))$. Let $\{S_\alpha\}$ be a family of poonems of P and let $S = \bigcap S_\alpha$. Suppose (A, B) is primitive with $A \subset S$ and $B \subset P$. By Theorem 2.2, $B \subset \bigcap \operatorname{conv} S_\alpha$. Let $b \in B$. If $b \in \operatorname{ext}(\operatorname{conv} P)$, then $b \in \operatorname{ext}(\operatorname{conv} S_\alpha)$ for each α and it follows that $b \in S$. Now suppose $b \notin \operatorname{ext}(\operatorname{conv} P)$. Since $b \in P \subset \operatorname{conv} P = \operatorname{conv}(\operatorname{ext}(\operatorname{conv} P))$, there exists by Lemma 2.1a, a primitive $(\{b\}, T)$ with $T \subset \operatorname{ext}(\operatorname{conv} P)$. For each α, $\operatorname{conv}(B \cup S_\alpha) = \operatorname{conv} S_\alpha$ and therefore, by definition of a poonem, $B \cup S_\alpha$ is also a poonem of P. Since $\{b\} \subset B \cup S_\alpha$, Theorem 2.2 implies $T \subset \operatorname{conv}(B \cup S_\alpha) = \operatorname{conv} S_\alpha$. Hence, for each α, $T \subset S_\alpha$ and therefore $T \subset S$. The latter implies $b \in \operatorname{conv} S$. Thus, $B \subset \operatorname{conv} S$ and S is a poonem of P by Theorem 2.2.

Now suppose that the intersection of an arbitrary family of poonems of P is a poonem of P. Let $p \in P$. We will show that $p \in \operatorname{conv}(\operatorname{ext}(\operatorname{conv} P))$. Let $\{S_\alpha\}$ be the family of all poonems of P which contain the point p and let $S = \bigcap S_\alpha$. By hypothesis, S is the smallest poonem of P which contains the point p. Now $S \sim \{p\} \subset \text{ext}(\text{conv } S)$. For suppose $q \in S$, $q \neq p$. If $\text{conv}(S \sim \{q\}) = \text{conv } S$, then $S \sim \{q\}$ is a poonem of P which contains the point p contrary to the definition of S. Hence $q \in \text{ext}(\text{conv } S)$. By Corollary 2.3a, we have $\text{ext}(\text{conv } S) \subset \text{ext}(\text{conv } P)$. Thus, if $p \notin \text{ext}(\text{conv } S)$ then $p \in \text{conv}(S \sim \{p\})$ and, in either case, $p \in \text{conv}(\text{ext}(\text{conv } P))$. Hence, it follows that $\text{conv } P = \text{conv}(\text{ext}(\text{conv } P))$.

THEOREM 2.6. *Let* $S \subset P \subset \mathscr{L}$. *If* $P = \text{ext}(\text{conv } P)$, *then the following three statements are equivalent*:

(a) *S is a poonem of P.*

(b) *If (A, B) is primitive with $A \subset S$ and $B \subset P$, then $B \subset S$.*

(c) $\text{conv}(P \sim S) \cap \text{aff } S = \varnothing$.

Proof. This follows directly from Theorem 2.2 with two observations. First, if $B \subset P \cap \text{conv } S$, then the hypothesis implies $B \subset S$. Second, the hypothesis implies $P \sim S = P \sim \text{conv } S$.

The above theorem is a generalization of a result of M. Breen [1, Theorem 4], who established the equivalence of (a) and (b) when P in the vertex set of a convex polytope in R^d.

COROLLARY 2.7. *Let* $P \subset \mathscr{L}$ *satisfy* $P = \text{ext}(\text{conv } P)$.

(a) *If S_1 and S_2 are poonems of P with* $\text{aff } S_1 = \text{aff } S_2$, *then* $S_1 = S_2$.

(b) *If P is in general position in* $\text{aff } P$, *then every poonem of P other than P itself is a simplex.*

Proof. (a) This follows from the equivalence of (a) and (c) in Theorem 2.6.

(b) Suppose $S \subset P$ is not a simplex. By Definition 1.1b we have $\text{aff } S = \text{aff } P$. If S is a poonem of P, then $S = P$ by part (a).

3. **Applications.** Given $S \subset P \subset \mathscr{L}$, we first consider the problem of the existence of a Radon partition (A, B) with $A \cup B = P$ and $S \subset A$. This problem was solved by Hare and Kenelly [3] for finite sets P in general position in R^d. The results here form a substantial generalization.

THEOREM 3.1. *Let* $S \subset P \subset \mathscr{L}$. *Then there exists a Radon partition (A, B) with $A \cup B = P$ and $S \subset A$ if and only if either $P \sim S$ is not a simplex or* $\text{conv } S \cap \text{aff}(P \sim S) \neq \varnothing$.

Proof. Suppose there exists a Radon partition (A, B) with $A \cup$

$B = P$ and $S \subset A$. Then there exists a primitive (A', B') with $A' \subset A$ and $B' \subset B \subset P \sim S$. If $A' \subset P \sim S$, then $P \sim S$ is not a simplex by Definition 1.1a. Therefore, we may assume that $S \cap A' \neq \varnothing$. Suppose $\operatorname{conv} S \cap \operatorname{aff}(P \sim S) = \varnothing$. Then since $\operatorname{conv}(P \sim \operatorname{conv}(P \sim S)) \cap \operatorname{aff}(P \sim S) \subset \operatorname{conv} S \cap \operatorname{aff}(P \sim S)$, by Theorem 2.2, $P \sim S$ is a poonem of P and since $B' \subset P \sim S$ we have $A' \subset \operatorname{conv}(P \sim S) \subset \operatorname{aff}(P \sim S)$. Hence, $A' \cap \operatorname{conv} S = \varnothing$ which contradicts $S \cap A' \neq \varnothing$. Thus, $\operatorname{conv} S \cap \operatorname{aff}(P \sim S) \neq \varnothing$.

Now suppose $P \sim S$ is not a simplex. Then there exists a primitive (A', B') with $A' \cup B' \subset P \sim S$. Let $A = A' \cup S$ and $B = P \sim A$. Then (A, B) is a Radon partition with $A \cup B = P$ and $S \subset A$. Finally suppose $\operatorname{conv} S \cap \operatorname{aff}(P \sim S) \neq \varnothing$. If $P \sim S$ is a poonem of P, then $\operatorname{conv}(P \sim S) = \operatorname{conv} P \cap \operatorname{aff}(P \sim S) \supset \operatorname{conv} S \cap \operatorname{aff}(P \sim S) \neq \varnothing$, and consequently $(S, P \sim S)$ is a Radon partition. On the other hand, if $P \sim S$ is not a poonem of P, then there exists a primitive (A', B') with $A' \subset P \sim S$ and $B' \subset P$. Let $A = P \sim A'$, $B = A'$. Then (A, B) is a Radon partition with $A \cup B = P$ and $S \subset A$.

COROLLARY 3.2. *Let P be in general position in* aff P *and let S be a nonempty subset of P. Then there exists a Radon partition (A, B) with $A \cup B = P$ and $S \subset A$ if and only if* $\operatorname{conv} S \cap \operatorname{aff}(P \sim S) \neq \varnothing$.

Proof. This follows directly from Theorem 3.1 with the observation that if $P \sim S$ is not a simplex then $\operatorname{aff}(P \sim S) = \operatorname{aff} P$ and therefore $\operatorname{conv} S \cap \operatorname{aff}(P \sim S) \neq \varnothing$.

We now turn to the study of k-neighborly sets. For $P \subset \mathscr{L}$, we define the cardinal number $h(P)$ by

3.3. $h(P) = \sup\{\operatorname{card}(A \cup B) \mid (A, B) \text{ is primitive and } A \cup B \subset P\}$.
It is understood that $h(P) = 0$ if P is a simplex. From Lemma 2.1a we have

3.4. $h(P) \leqq \dim(\operatorname{aff} P) + 2$.

THEOREM 3.5. *Let $P \subset \mathscr{L}$ and let k be a cardinal number. Then the following three statement are equivalent:*

(a) *P is k-neighborly*

(b) *Each subset $S \subset P$ with* card $S = h(P)$ *is k-neighborly*

(c) *There exists no primitive (A, B) with $A \cup B \subset P$ such that* min (card A, card B) $\leqq k$.

Proof. (a) $\Rightarrow$ (b). By Corollary 2.3b, if P is k-neighborly then every subset of P is k-neighborly.

(b) $\Rightarrow$ (c). Suppose there exists a primitive (A, B) with $A \cup B \subset P$

and $\text{card}\, A \leqq k$ (and therefore $k \geqq 1$). Since $\text{card}\,(A \cup B) \leqq h(P) \leqq \text{card}\, P$, there exists $Q \subset P$ with $A \cup B \subset Q$ and $\text{card}\, Q = h(P)$. By hypothesis, Q is k-neighborly and hence A is a poonem of Q. However, Theorem 2.6 applied to $A \subset Q$ gives $B \subset A$ which is a contradiction.

(c) $\Rightarrow$ (a). Let $S \subset P$ with $\text{card}\, S \leqq k$. By hypothesis, there is no primitive (A, B) with $A \subset S$ and $B \subset P$. Hence, S is a poonem of P by Theorem 2.2. This completes the proof.

THEOREM 3.6. *Let $P \subset \mathscr{L}$ and let k be a cardinal number.*

(a) *If k is finite, $k < \text{card}\, P$, and every subset $S \subset P$ with $\text{card}\, S = k$ is a poonem of P, then P is k-neighborly.*

(b) *If k is transfinite, $k \leqq \text{card}\, P$, and every subset $S \subset P$ with $\text{card}\, S = k$ is a poonem of P, then P is a simplex.*

Proof. (a) We may assume $k \geqq 2$ for otherwise the proof is trivial. We first show that $\text{conv}\, P = \text{conv}\,(\text{ext}\,(\text{conv}\, P))$. Let $p \in P$ and let $S \subset P$ with $\text{card}\, S = k$ contain the point p. Since S is finite, $p \in \text{conv}\,(\text{ext}\,(\text{conv}\, S))$ and since S is a poonem of P, $\text{ext}\,(\text{conv}\, S) \subset \text{ext}\,(\text{conv}\, P)$ by Corollary 2.3a. Thus, $\text{conv}\, P = \text{conv}\,(\text{ext}\,(\text{conv}\, P))$. In the above argument, the intersection of all such $S \subset P$ is $\{p\}$ since $k < \text{card}\, P$. By Theorem 2.5, $\{p\}$ is a poonem of P and hence $P = \text{ext}\,(\text{conv}\, P)$.

Now suppose there exists a primitive (A, B) with $A \cup B \subset P$ and $\text{card}\, A \leqq k$. Let $S \subset P$ with $A \subset S$, $\text{card}\, S = k$ and such that $P \sim S$ contains a point of B. By hypothesis and Theorem 2.6 we have $B \subset S$ which is a contradiction. Hence, by Theorem 3.5, P is k-neighborly.

(b) We first show that $P = \text{ext}\,(\text{conv}\, P)$. Suppose there exists a primitive $(\{a\}, B)$ where $a \in P$ and $B \subset P$. Let $B = \{b_1, \cdots, b_n\}$ where $n \geqq 2$. Then $a = \sum_{i=1}^{n} \lambda_i b_i$, $0 < \lambda_i < 1$, and $\sum_{i=1}^{n} \lambda_i = 1$. Let $B_1 = B \sim \{b_1\}$. Since $\{a\}$ and $\text{conv}\, B_1$ are disjoint convex sets, by a separation theorem for real linear spaces [5, p. 20], there exist complementary convex sets C and D such that $a \in C$, $B_1 \subset D$ and $C \cup D = \mathscr{L}$, $C \cap D = \varnothing$. Either $\text{card}\,(C \cap P) \geqq k$ or $\text{card}\,(D \cap P) \geqq k$. Suppose $\text{card}\,(C \cap P) \geqq k$. By hypothesis, there exist a poonem S of P with $a \in S$, $S \subset C$ and $\text{card}\, S = k$. By Theorem 2.2, we have $B \subset \text{conv}\, S \subset C$ which is a contradiction. Now suppose $\text{card}\,(D \cap P) \geqq k$. By hypothesis, there exists a poonem S of P with $a \in S$, $S \sim \{a\} \subset D$ and $\text{card}\, S = k$. By Theorem 2.2, we have $B \subset \text{conv}\, S$. Hence, $B \subset \text{conv}\,(\{a\} \cup D)$. Therefore, for each $b_i \in B$, there exists $d_i \in D$ such that for some t_i, $0 < t_i \leqq 1$, $b_i = t_i d_i + (1 - t_i)a$. It follows that $a = (\sum_{i=1}^{n} \lambda_i t_i)^{-1} \sum_{i=1}^{n} (\lambda_i t_i) d_i \in D$ which is a contradiction. Hence $P = \text{ext}\,(\text{conv}\, P)$.

Now suppose there exists a primitive (A, B) with $A \cup B \subset P$. By hypothesis, there exists a poonem S of P with $A \subset S$, $B \cap S = \emptyset$ and card $S = k$. By Theorem 2.6, we have $B \subset S$ which is a contradiction. Hence P is a simplex.

THEOREM 3.7. *Let* $P \subset \mathscr{L}$ *be k-neighborly and let* $S \subset P$.
(a) *If* $2k + 1 \geqq h(S)$, *then* S *is a simplex.*
(b) *If* $2k - 1 \geqq \dim(\operatorname{aff} S)$, *then* S *is a simplex.*
(c) *If* $2k = \dim(\operatorname{aff} S)$, *then* S *is in general position in* aff S.

Proof. (a) If S is not a simplex, then there exists a primitive (A, B) with $A \cup B \subset S$. Since $2k + 1 \geqq h(S) \geqq \operatorname{card} A + \operatorname{card} B$, either card $A \leqq k$ or card $B \leqq k$. But this contradicts Theorem 3.5.

(b) This follows from 3.4 and part (a).

(c) Suppose there exists $Q \subset S$ which is not a simplex. Then k must be finite and by part (a) we have $h(Q) \geqq 2k + 2$. Using 3.4 we have $\dim(\operatorname{aff} S) \geqq \dim(\operatorname{aff} Q) \geqq 2k = \dim(\operatorname{aff} S)$. Thus $\dim(\operatorname{aff} S) = \dim(\operatorname{aff} Q) = 2k$. Since k is finite and $\operatorname{aff} S \supset \operatorname{aff} Q$ we have $\operatorname{aff} S = \operatorname{aff} Q$ and, by Definition 1.1b, S is in general position in aff S.

THEOREM 3.8. *Let* $P \subset \mathscr{L}$ *be an infinite set which is in general position in* aff P *and let* k *be a cardinal number such that* $2k \leqq \dim \operatorname{aff} P$. *Then there exists an infinite subset of* P *which is k-neighborly.*

Proof. If $\dim \operatorname{aff} P = 1$, then any subset of the line aff P is in general position in the line but any such subset is also O-neighborly. If dim aff P is infinite, then P must be a simplex. For if (A, B) is primitive with $A \cup B \subset P$, then $A \cup B$ is not a simplex nor is $\operatorname{aff}(A \cup B) = \operatorname{aff} P$. We may therefore assume that $d = \dim \operatorname{aff} P$ is a positive integer greater than 1 and we will show that there exists an infinite subset of P which is $[d/2]$-neighborly. Since P is in general position in aff P, each subset $S \subset P$ consisting of $d + 2$ points determines a unique primitive (A, B) with $A \cup B = S$. Thus the set all such subsets $S \subset P$ may be partitioned into $[d/2] + 1$ mutually exclusive classes C_i according to the value $i = \min(\operatorname{card} A, \operatorname{card} B)$. By the infinite version of Ramsey's theorem [4, p. 82], P contains an infinite subset Q such that every subset of Q consisting of $d + 2$ points belongs to the same C_i. By use of Gale diagrams, M. A. Perles has shown [2, p. 120] that if $d + 3$ points of R^d are in general position then some $d + 2$ of them are the vertices of a $[d/2]$-neighborly polytope. Hence, for the particular C_i referred to above, we have $i = [d/2] + 1$. Consequently, by Theorem 3.5, Q is $[d/2]$-neighborly. This completes the proof.

We conclude with some example which are pertinent to the general situation considered here.

EXAMPLES 3.9. (a) There exist nondenumerable sets P in R^d, $d \geqq 1$, which are $[d/2]$-neighborly. Let $x(t) = (t, t^2, \cdots, t^d) \in R^d$ and let $P = \{x(t) \mid t \text{ real}\}$. Each subset of P with $d + 2$ points is $[d/2]$-neighborly [2, p. 61]. Since $h(P) = d + 2$, P itself is $[d/2]$-neighborly by Theorem 3.5.

(b) Let $\mathscr{L}$ be the real linear space of all real-valued functions defined on the set of positive integers. Let $x(t) = (t, t^2, \cdots) \in \mathscr{L}$ and let $P = \{x(t) \mid t \text{ real}\}$. Then P is a simplex. This follows from Lemma 2.1b by showing that every nonempty finite subset of P is an affinely independent set. The latter may be established by a proof similar to that given in [2, p. 62] where the corresponding problem for the moment curve is considered.

(c) Let $\mathscr{L}$ be an infinite dimensional real linear space. For each finite cardinal number k there exists $P_k \subset \mathscr{L}$ with $\operatorname{aff} P_k = \mathscr{L}$ such that P_k is k-neighborly but not $(k + 1)$-neighborly. For $k = 0$, we may take $P_0 = \mathscr{L}$. Now let $k \geqq 1$ and let $H \subset \mathscr{L}$ be a Hamel basis for $\mathscr{L}$. Let $\{p_1, \cdots, p_k\}$ and $\{q_1, \cdots, q_k\}$ be disjoint subsets of H and let $p_0 = -\sum_{i=1}^k p_i$, $q_0 = -\sum_{i=1}^k q_i$. Define $P_k = \{p_0, q_0\} \cup H$. If $A = \{p_0, p_1, \cdots, p_k\}$ and $B = \{q_0, q_1, \cdots, q_k\}$, then (A, B) is primitive, the origin being the unique point in $\operatorname{conv} A \cap \operatorname{conv} B$. Since $\operatorname{aff} P_k$ contains the origin, we have $\operatorname{aff} P_k = \mathscr{L}$. Moreover, (A, B) is the only primitive, with $A \cup B \subset P_k$ which may be verified by studying the possible affine dependencies in P_k. Thus, by Theorem 3.5, P_k is k-neighborly but not $(k + 1)$-neighborly.

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UNIVERSITY OF MISSOURI—COLUMBIA