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FREE PRODUCTS IN THE CATEGORY OF k_w-GROUPS

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This paper provides a short proof of the existence of free-products in the category k_w-groups.

1. **Introduction.** Graev defined free products in the category of Hausdorff topological groups, and gave a lengthy proof of their existence using norms [2]. Hulanicki showed that free products exist in the category of compact groups [3]. S. A. Morris pointed out that the later result extends to the category of almost periodic groups [5]. The proof in the above mentioned paper is correct, except that it should rely on Graev's proof for the existence of free products in the category of Hausdorff topological groups and should not on the erroneous proof in that paper.

The method of proof of this paper has been adopted by Ordman to prove the existence of free products in the category of k-groups [6].

2. **Definitions and statement of the main result.**

DEFINITION 1. a k_w-space is a Hausdorff topological space X with compact subsets X_n such that: (i) $X = \bigcup_{n=1}^{\infty} X_n$; (ii) $X_{n+1} \supset X_n$ for all n; (iii) a subset A of X is closed in X, if and only if $A \cap X_n$ is compact for all n.

By a k_w-decomposition $X = \bigcup X_n$, we mean that X_n have properties (i), (ii) and (iii).

A k_w-group G, is a topological group which is also a k_w-space. We denote by KG the category of k_w-groups.

DEFINITION 2. Let G_1 and G_2 be two k_w-groups. Then, $G_1 * G_2$ is their free product in the category of k_w-groups if the following axioms hold: (i) the underlying group of $G_1 * G_2$ is their free product as groups. (ii) G_1 and G_2 are topological subgroups of $G_1 * G_2$. (iii) If γ_i: $G_i \to H$, $i = 1, 2$, are continuous homomorphisms into the k_w-group H, then they extend uniquely to a continuous homomorphism Γ: $G_1 * G_2 \to H$.

We will refer to the following concept frequently.

DEFINITION 3 (Graev [1]). The Hausdorff topological group $F(X)$ is a free topological group with basis (X, e) a pointed topological space, if the following axioms hold: (i) X is a subspace of $F(X)$.

(ii) $X - e$ generates $F(X)$ freely as a group. (iii) For any continuous map $\phi: X \longrightarrow G$, where G is a Hausdorff topological group, and $\phi(e)$ equals the identity of G, then ϕ extends uniquely to a continuous homomorphism $\Phi: F(X) \longrightarrow G$.

The main result. Free products exist in the category of k_w-groups.

3. **Proof of the main result.** Let $\psi: |F(G_1 \vee G_2)| \longrightarrow |G_1| * |G_2|$ be the homomorphism induced by the inclusion map of $G_1 \vee G_2$ into $|G_1| * |G_2|$, where $G_1, G_2 \in KG$, $\vee$ denotes the disjoint union identifying distinguished points, $|\ |$ is the forgetful functor into the category of groups and $*$ denotes the free product in the category of groups, in addition to its use in Definition 2. Denote by K the kernel of ψ with the subspace topology of $F(G_1 \vee G_2)$. Our first object is to prove that K is a closed subset. For then, $|G_1| * |G_2|$ with the identification topology of $F(G_1 \vee G_2)$ via ψ, is a k_w-group, which we denote by $G_1 *' G_2$.

To carry out the proof of the just mentioned fact we need some further notation. The map $h: (G_1 \vee G_1^{-1} \vee G_2 \vee G_2^{-1})^n \longrightarrow F(G_1 \vee G_2)$ is defined to send words in $G_1 \vee G_2$ whose length is at most n, to their reduced form. This map is obviously continuous, since it is induced by the multiplication map of $F(G_1 \vee G_2)$. We also use the following notation: $Z^n = Y_1^n \times \cdots \times Y_n^n$, where $Y_l^n = (L_{q_l}^n)^{p_l}$, $p_l = \pm 1$, $q_l = 1, 2$ and $G_i = \bigcup_{j=1}^{\infty} L_i^j$ are k_w decompositions, $i = 1, 2$.

The sets, $h[(L_1^n \vee (L_1^n)^{-1} \vee L_2^n \vee (L_2^n)^{-1})^n]$ which consist of all reduced words in $L_1^n \vee L_2^n$, whose length is at most n, with the supspace topology of $F(G_1 \vee G_2)$, form a k_w-decomposition of $F(G_1 \vee G_2)$ [4]. Thus in order to show that K is closed, it suffices to show that $K \cap h(Z^n)$ is compact for any n.

Suppose Z^n is of the form $(L_{i_0}^n)^{l_1} \times \cdots \times (L_{i_0}^n)^{l_n}$, i_0 fixed, 1 or 2. Denote by $\mu: \times G_{i_0} \longrightarrow G_{i_0}$ the map induced by the multiplication in G_{i_0}, and denote by $\phi: Z^n \longrightarrow \times G_{i_0}$ the map induced by the inclusion of $L_{i_0}^n$ into G_{i_0}. By the continuity of the maps h, ϕ, μ, the set $K \cap h(Z^n) = h(\ker \mu\phi)$ is compact. If Z^n is not of the above form, $K \cap h(Z^n)$ will be, by an induction argument, a product of compact sets.

Next we prove the $G_1 *' G_2$ is a free product. Let $t': F(G_1 \vee G_2) \longrightarrow H$ be the extension of $t_i: G_i \longrightarrow H$, $i = 1, 2$, two continuous homomorphisms into the k_w-group H. The homomorphism ψ factors through t', i.e., the following diagram commutes:

$$\begin{tikzcd} F(G_1 \vee G_2) \arrow[r, "\psi"] \arrow[d, "t'"'] & G_1 *' G_2 \arrow[dl, "t"] \\ H & \end{tikzcd}$$

Since ψ is an identification map, t is continuous. The uniqueness of t follows from the algebraic structure.

We still have to show that $G_1 *' G_2$ induces the original topology on $|G_i|$. Let $g: G_1 *' G_2$ be induced by l_{G_1} and the constant map on G_2. Since $g^{-1}(U) \cap G_1 = U$, if U is open in G_1, U is also open in G_1', where G_1' is $|G_1|$ with the subspace topology of $G_1 *' G_2$. Suppose that V' is open in $G_1 *' G_2$ and let $V = V' \cap G_1'$. Denote by $\chi: F(G_1) \to F(G_1 \vee G_2)$ the map induced by the inclusion, and $p: F(G_1) \to G_1$ the map which sends a word to its reduced form. Graev proved in [1] that p is an open map. Since $V = p(\psi\chi)^{-1}(V')$, V is open in G_1. This proves that $G_1 *' G_2$ induces the original topology on $|G_1|$. We conclude that $G_1 *' G_2$ is the free product of G_1 and G_2 in KG.

4. **Remarks.** In our proof that $G_1 *' G_2$ satisfies property (iii) of Definition 2 we used only the fact that H is a Hausdorff topological group. Thus $G_1 *' G_2$ is also the free product of G_1 and G_2 in the category of Hausdorff topological groups.

We point out that the free product in KG is actually the co-product in KG. This follows from the uniqueness property of co-products.

References

1. M. I. Graev, *Free topological groups*, Izv. Akad. Nauk SSSR. Ser. Mat., **12** (1948), 279-324, (Russian). English Transl., Amer. Math. Soc. Transl. no. 35, (1951).

2. ———, *On free products of topological groups*, (Russian), Izv. Akad. Nauk. SSSR. Ser. Mat., **14** (1950), 343-354.

3. Hulanicki, *Isomorphic embeddings of free products of compact groups*, Colloq. Math., **16** (1967), 235-241.

4. J. Mack, S. A. Morris, E. T. Ordman, *Free topological groups and the projective dimension of a locally compact abelian group*, Proc. Amer. Math. Soc., (to appear).

5. S. A. Morris, *Free products of topological groups*, Bull. Austral. Math. Soc., **4** (1971), 17-29.

6. E. T. Ordman, *Free Product in the Category of k-Groups*, (to appear).

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