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SOME MATRIX TRANSFORMATIONS ON ANALYTIC SEQUENCE SPACES

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Let A denote the space of all complex sequences a such that if z is a complex number and $|z| < 1$ then $\sum a_n z^n$ converges, and B the space of all complex sequences b for which there is a complex number z such that $|z| > 1$ and $\sum b_n z^n$ converges. In this paper we characterize matrix transformations from A to B and from B to A.

M. G. Haplanov [1] has described the matrix transformations from A to A, and P. C. Tonne [4] those from A to the bounded sequences, the convergent sequences and l.

A *sequence space* is a linear space each point of which is an infinite complex sequence. If λ is a sequence space, then λ^*, the *dual* of λ, is the collection of all infinite complex sequences y such that $\sum |x_n y_n|$ converges for every x in λ. For each λ a dual system with λ^* is formed using the bilinear functional

$$Q(x, y) = \sum_{n=0}^{\infty} x_n y_n ,$$

where x is in λ and y is in λ^*. Under this duality, λ is provided with the standard weak topology.

Theorem A is a classic result of Köthe and Toeplitz [2]:

THEOREM A. *Suppose λ is a sequence space such that $\lambda = \lambda^{**}$. In order that a linear transformation from λ to a sequence space be weakly continuous, it is necessary and sufficient that it be a matrix transformation.*

In [3] O. Toeplitz studied the topological properties of the spaces A and B. The following theorem is a summary of his basic results:

THEOREM B. (1) *$A^* = B$ and $B^* = A$.*

(2) *A point set M is bounded in A [B] if and only if there exists a point y of A [B] such that $|x_n| < y_n$ whenever x is a point of M and n is a nonnegative integer.*

(3) *A point sequence is convergent in A [B] if and only if it is bounded in A [B] and coordinatewise convergent.*

THEOREM 1. *If M is an infinite matrix then the following are* *equivalent*:

(1) *M throws A into B.*

(2) *Each row and each column of M is in B, and there exist numbers t and r such that $0 < r < 1$ and $|M_{jk}| \leqq tr^{j+k}$ whenever each of j and k is a nonnegative integer.*

Proof. (1) $\rightarrow$ (2). Suppose statement (1) is true and statement (2) is not. In that case there exist increasing sequences $j_0, j_1, j_2, \cdots$ and $k_0, k_1, k_2, \cdots$ of positive integers such that if n is a nonnegative integer, then

$$|M_{j_n, k_n}| > \left(\frac{n}{n+1}\right)^{j_n + k_n},$$

and either (i) $j_n \leqq k_n$ for each nonnegative integer n or (ii) $k_n \leqq j_n$ for each nonnegative integer n.

Suppose case (i) holds. For each nonnegative integer n, let c_n denote a complex number such that

$$|c_n| = \frac{1}{|M_{j_n, k_n}|} \quad \text{and} \quad \left|\sum_{i=0}^{n} M_{j_n, k_i} c_i\right| \geqq 1 .$$

Each c_n has the property that $|c_n| < (1 + 1/n)^{2k_n}$.

For each nonnegative integer n, let ξ_n denote the point of A such that for each nonnegative integer m, $\xi_{nm} = c_i$ whenever there is an integer i such that $0 \leqq i \leqq n$ and $m = k_i$, and $\xi_{nm} = 0$ otherwise.

The point sequence ξ is bounded in A, so $M(\xi)$ is bounded in B. However, for each positive integer n,

$$\begin{aligned} |(M\xi_n)_{j_n}| &= \left|\sum_{i=0}^{k_n} M_{j_n, i}\xi_{ni}\right| \\ &= \left|\sum_{i=0}^{n} M_{j_n, k_i} c_i\right| \\ &\geqq 1 . \end{aligned}$$

This is a contradiction.

In case condition (ii) holds, M' is a matrix that throws A into B and satisfies condition (i). This is also a contradiction.

(2) $\rightarrow$ (1). If x is a point of A and j is a nonnegative integer, then

$$\begin{aligned} |(Mx)_j| &= \left|\sum_{k=0}^{\infty} M_{jk} x_k\right| \\ &\leqq tr^j \sum_{k=0}^{\infty} r^k |x_k| . \end{aligned}$$

Consequently, $\limsup_j |(Mx)_j|^{1/j} \leqq r$, and Mx is a point of B.

THEOREM 2. *If M is an infinite matrix then the following are equivalent*:

(1) *M throws B into A.*

(2) *Each row and each column of M is in A, and if $\varepsilon > 0$ there is a positive integer m such that $|M_{jk}|^{1/(j+k)} < 1 + \varepsilon$ whenever each of j and k is a nonnegative integer and $j + k \geqq m$.*

Proof. (1) $\rightarrow$ (2). Suppose statement (1) is true and statement (2) is not. In that case, there exist a positive number ε and infinitely many nonnegative-integer pairs (j, k) such that $|M_{jk}|^{1/(j+k)} > 1 + \varepsilon$.

Case (i). Suppose there exist infinitely many such integer pairs such that $j \leqq k$. Let r denote a number such that $(1 + \varepsilon)r > 1 + \varepsilon/2$.

Let (j_0, k_0) denote a nonnegative integer pair such that

$$j_0 \leqq k_0 \quad \text{and} \quad |M_{j_0,k_0}|^{1/(j_0+k_0)} > 1 + \varepsilon\ .$$

Let $c_0 = r$. Then

$$c_0 |M_{j_0,k_0}|^{1/k_0} \geqq c_0 |M_{j_0,k_0}|^{1/(j_0+k_0)} > (1 + \varepsilon)r > 1 + \frac{\varepsilon}{2}\ .$$

Let (j_1, k_1) denote a nonnegative integer pair such that $j_1 \leqq k_1$, $j_0 < j_1$, $k_0 < k_1$, and

$$\sum_{i=k_1}^{\infty} |M_{j_0,i}| r^i < [(1 + \varepsilon)r]^{k_0} - \left[1 + \frac{2}{\varepsilon}\right]^{j_0}\ .$$

Let c_1 denote a complex number such that $|c_1| = r$ and

$$|M_{j_1,k_1} c_1^{k_1}| \leqq |M_{j_1,k_0} c_0^{k_0} + M_{j_1,k_1} c_1^{k_1}|\ .$$

Continue this process in the following way: For each positive integer n, after choosing j_n, k_n, and c_n, let (j_{n+1}, k_{n+1}) denote a nonnegative integer pair such that $j_{n+1} \leqq k_{n+1}$, $j_n < j_{n+1}$, $k_n < k_{n+1}$, and

$$\sum_{i=k_n+1}^{\infty} |M_{j_n,i}| r^i < [(1 + \varepsilon) r]^{k_n} - \left[1 + \frac{\varepsilon}{2}\right]^{j_n},$$

and then let c_{n+1} denote a complex number such that $|c_{n+1}| = r$ and

$$|(c_{n+1})^{k_{n+1}}(M_{j_{n+1},k_{n+1}})| \leqq \left|\sum_{i=1}^{n+1} c_i^{k_i}(M_{j_{n+1},k_i})\right|\ .$$

Now, for each nonnegative integer n, let ξ_n denote the point of B such that for each nonnegative integer m,

$$\xi_{nm} = c_i^{k_i}$$

whenever there is an integer i such that $0 \leqq i \leqq n$ and $m = k_i$, and

$$\xi_{nm} = 0$$

otherwise.

The point sequence ξ is bounded in B, so $M(\xi)$ is bounded in A. However, for each positive integer n,

$$\begin{aligned} |(M\xi_n)_{j_n}| &= \left| \sum_{i=0}^{n} c_i^{k_i}(M_{j_n,k_i}) + \sum_{i=n+1}^{\infty} c_i^{k_i}(M_{j_n,k_i}) \right| \\ &\geqq \left| \sum_{i=0}^{n} c_i^{k_i}(M_{j_n,k_i}) \right| - \sum_{i=n+1}^{\infty} |c_i^{k_i}(M_{j_n,k_i})| \\ &\geqq \left[1 + \frac{\varepsilon}{2} \right]^{j_n}. \end{aligned}$$

Consequently, $|(M\xi_n)_{j_n}|^{1/j_n} \geqq 1 + \varepsilon/2$. This contradicts the fact that $M(\xi)$ is a bounded subset of A.

Case (ii). Suppose there exist infinitely many such integer pairs (j, k) such that $j \geqq k$. In that case, M' throws B into A and satisfies the assumption of case (i). This is also a contradiction.

(2)$\rightarrow$(1). Suppose x is a point of B. Let t and r denote numbers such that $0 < r < 1$ and $|x_n| \leqq tr^n$ for each nonnegative integer n. If ε is a positive number so small that $(1 + \varepsilon)r < 1$, and m is a positive integer such that $|M_{jk}|^{1/(j+k)} < 1 + \varepsilon$ whenever each of j and k is a nonnegative integer and $j + k \geqq m$, then for each nonnegative integer p,

$$\begin{aligned} \left| \sum_{k=0}^{\infty} M_{m+p,k} x_k \right| &\leqq \sum_{k=0}^{\infty} \left| M_{m+p,k} \right| tr^k \\ &\leqq (1 + \varepsilon)^{m+p} \frac{t}{1 - (1 + \varepsilon)r}. \end{aligned}$$

Therefore,

$$\limsup_p \left| \sum_{k=0}^{\infty} M_{m+p,k} x_k \right|^{1/(m+p)} \leqq 1 + \varepsilon.$$

It follows that

$$\limsup_j \left| \sum_{k=0}^{\infty} M_{jk} x_k \right|^{1/j} \leqq 1,$$

that is, Mx is a point of A.

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