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REDUCING SERIES OF ORDINALS

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If s is a sequence of ordinals, we denote by "$S(s)$" the set of sums (of the corresponding series) obtainable by permuting the terms of s in such a way that the length $o(s)$ is unchanged. If $o(s) = \omega$, the first tran sfinite ordinal, then a fairly well-known result of Sierpiński's states that $S(s)$ is finite, which immediately raises the question of whether there is a finite sequence r such that $S(r) = S(s)$.

It turns out in fact that such a sequence r always exists: and we are concerned in this note with proving certain generalizations of this latter result. The general problem, that of determining criteria that must be satisfied by an infinite sequence s in order that a sequence r exist with $o(r) < o(s)$ and $S(r) = S(s)$, is to the best of our knowledge still open and would appear to be no easy one.

Throughout this paper ordinals are generally denoted by lower case Greek letters, with "ω" always being reserved for the first transfinite ordinal. Each ordinal is assumed to be the set of all smaller ordinals. Cardinality (of a set) is denoted by "$|\ \ |$", and an initial ordinal (i.o.) is an ordinal $\kappa \geqq \omega$ such that $|\alpha| < |\kappa|$ whenever $\alpha < \kappa$. We usually reserve the letters "κ", "λ", "η" for i.o.'s, and for any $\alpha \geqq \omega$, we denote by "$i(\alpha)$" the i.o. κ defined by the equation $|\kappa| = |\alpha|$. If κ is an i.o., then κ^+ is the next larger i.o. Finally, for any $\alpha > 0$, we denote by "$cf(\alpha)$" the cofinality of α.

The Axiom of Choice is assumed throughout, and we assume familiarity with the elementary theory of cardinals and ordinals.

We define, for each α, an α-sequence s to be an ordinal-valued function having α as its domain: the length (or order-type) $o(s)$ of s is of course defined to be α, and we usually write an α-sequence s as "$(s_\xi)_{\xi<\alpha}$", or simply "(s_ξ)" if the value of $o(s)$ is clear. Sequences are denoted by "r", "s", $\cdots$, "z". If r is a subsequence of a sequence s, then we assume the terms of r to have been resubscripted in such a way as to make the denotation "$(r_\xi)_{\xi<o(r)}$" legitimate.

If s is a sequence and α an ordinal, then we denote by "s/α" that subsequence of s consisting precisely of those $s_\xi \geqq \alpha$. If r is a subsequence of a sequence s, then $s - r$ is the subsequence of s consisting of those s_ξ that are not terms of r. Finally, if r, s are sequences, then we denote by "$r \mathring{\cup} s$" the sequence t such that

$o(t) = o(r) + o(s)$, $t_\xi = r_\xi$ for $\xi < o(r)$, and $t_{o(r)+\xi} = s_\xi$ for $\xi < o(s)$. Clearly this last definition can be generalized to any number of sequences.

A sequence t is called an "arrangement" of a sequence s if there is a bijection $b: o(s) \to o(t)$ such that $t_{b(\xi)} = s_\xi$ for all $\xi < o(s)$. An arrangement t of a sequence s is called a "permutation" of s if $o(s) = o(t)$. We put $P^a(s) = \{t; t$ is an arrangement of $s\}$, and $P(s) = \{t; t$ is a permutation of $s\}$.

For any sequence s, we denote by "$\Sigma(s)$" the sum of the associated series:—

$$\Sigma(s) = \sum_{\xi < o(s)} s_\xi = s_0 + s_1 + \cdots + s_\xi + \cdots; \xi < o(s) .$$

We put $S^a(s) = \{\Sigma(t); t \in P^a(s)\}$, $S(s) = \{\Sigma(t); t \in P(s)\}$.

As stated in our abstract, Sierpiński showed that $|S(s)| < \aleph_0$ for every ω-sequence s: the proof is given in [5]. His result was generalized by Ginsburg in [1], and in [4] we succeeded in obtaining best upper bounds for $|S(s)|$ for every infinite value of $o(s)$.

The results obtained here follow on from those obtained in [4], and for convenience we now list the results (or parts of results) obtained there that we shall require for our present work. Firstly, however, we need to define a certain parameter.

Let s be a κ-sequence, with κ an i.o. We define $C(s)$ by

$$C(s) = \min\{\tau; |\{\xi < \kappa; s_\xi \geqq \tau\}| < |\kappa|\} .$$

R1. Let s be a sequence, r a subsequence of s, and take any α such that $\Sigma(r) + \alpha = \alpha$. Then $\Sigma(s) + \alpha = \Sigma(s - r) + \alpha$.

R2. Let κ be an i.o., and let s be a κ-sequence of positive ordinals. Then $|S^a(s)| = |\kappa^+|$.

R3. (1) For every sequence s with $o(s) \geqq \omega$, we have $|S(s)| \leqq |o(s)|$.

(2) For every $\alpha \geqq \omega$, there exists an α-sequence s with $|S(s)| = |\alpha|$ if and only if α is not a regular limit i.o.

R4. Let κ be a singular i.o., and let s be a κ-sequence. Put $\rho = C(s)$. Then $|S(s)| = |\kappa|$ if and only if $\rho = \kappa^\beta$ for some $\beta > 0$.

R5. Let κ be a successor i.o., $\kappa = \lambda^+$, let s be a κ-sequence of positive ordinals, and put $\rho = C(s)$. Then $|S(s)| = |\kappa|$ if and only if $o(s/\rho) \geqq \lambda$ and if either $cf(\rho) < \kappa$ or $\kappa^\beta < \rho < \kappa^{\beta+1}$ for some β then $o(s/\rho\kappa) \geqq \lambda$.

The problem that we wish to consider in this paper is the following:—

Let κ be an i.o., and let s be a κ-sequence of positive ordinals, with $|S(s)| < |\kappa|$. Under what conditions is there a sequence r of positive ordinals such that

(a) $S(s) = S(r)$;

(b) $o(r) < \kappa$ and either $o(r)$ is finite or $o(r)$ is an i.o.?

T1. *Let κ be an* i.o., *and let s be a κ-sequence of positive ordinals with $|S(s)| < \aleph_0$. Then there exists a finite sequence r of positive ordinals with $S(s) = S(r)$.*

Proof. Put $\rho = C(s)$, and let t be the subsequence $s - s/\rho$ of s. We wish to show that either $S(t) = \{\rho\}$ or $S(t) = \{\rho\kappa\}$.

We consider the following cases. Suppose firstly that κ is singular. Then, since s is a sequence of positive ordinals, we know that $\rho > 1$, and thus from R4 we conclude that $\kappa^\beta < \rho < \kappa^{\beta+1}$ for some β. Thus $o(t/\kappa^\beta) = \kappa$, and hence $\Sigma(u) \geqq \kappa^\beta\kappa = \kappa^{\beta+1}$ for any $u \in P(t)$. Clearly, however, we must have $\Sigma(u) \leqq \rho\kappa = \kappa^{\beta+1}$ for each such u. Therefore in this case we have $S(t) = \{\rho\kappa\}$.

Now suppose that κ is regular. If $\kappa^\beta < \rho < \kappa^{\beta+1}$ for some β, then the argument above again shows that $S(t) = \{\rho\kappa\}$.

Hence we may suppose that $\rho = \kappa^\alpha$ for some $\alpha > 0$, whence we have $\rho = \sup\{t_\xi; \xi < o(t) = \kappa\}$, and so $cf(\rho) \leqq \kappa$.

Assume that $cf(\rho) = \kappa$, take $u \in P(t)$, and let u' be any proper initial segment of u. Then $o(u') < o(u) = \kappa$, and of course $u'_\xi < \rho$ for every $\xi < o(u')$; hence $\sup\{u'_\xi; \xi < o(u')\} = \delta < \rho$. But this gives $\Sigma(u') \leqq \delta o(u') < \delta\kappa \leqq \rho$, since $\rho = \kappa^\alpha$ for some α, and thus $\Sigma(u) \leqq \rho$. However, we obviously have $\Sigma(u) \geqq \rho$. This gives $S(t) = \{\rho\}$.

Finally, assume that $cf(\rho) < \kappa$. Now for each $u \in P(t)$, we must have $\Sigma(u) = \rho\gamma$ for some γ. For if not, there must be some $u \in P(t)$ having a final segment $v \neq 0$ such that $\Sigma(v) < \rho$. Putting $\delta = \sup\{v_\xi; \xi < o(v) = \kappa\}$, we see therefore that $\delta < \rho$, and so $o(t/\delta) = \kappa$. But from the definition of δ we see that there is some $w \in P(t/\delta)$ such that w is a subsequence of $u - v$. Since $o(u - v) < \kappa$, this is a contradiction, and our claim is established. Thus take $u \in P(t)$, and let γ be such that $\Sigma(u) = \rho\gamma$.

Suppose $\gamma < \kappa$. Since $cf(\rho) < \kappa$, we thus have $cf(\rho\gamma) < \kappa$. However, u is a κ-sequence of positive ordinals, from which it is easy to show that we must have $cf(\Sigma(u)) = cf(\kappa) = \kappa$. Hence we must have $\gamma \geqq \kappa$. However, it is clear that $\Sigma(u) \leqq \rho\kappa$. Therefore $S(t) = \{\rho\kappa\}$ in this case. This establishes our claim concerning $S(t)$.

Consider the case $S(t) = \{\rho\}$; we see from the above that we must have κ regular, $cf(\rho) = \kappa$, and $\rho = \kappa^\alpha$ for some $\alpha > 0$. Put $u = s/\rho$. Now $o(u) < \kappa$, and κ is regular. Thus if we take any $v \in P(s)$ and let $w \in P^\alpha(u)$ be such that w is a subsequence of v, then w must be a subsequence of some proper initial segment v' of v. But then $v - v'$ is a nonempty final segment of some $t^\circ \in P(t)$, and so $\Sigma(v - v') = \rho, = \Sigma(t^\circ)$ and $t^\circ = v' - w \overset{\circ}{\cup} v - v'$. Now $\rho = \kappa^\alpha$, and is thus a prime component, and so $\Sigma(v' - w) + \rho = \rho$. Hence by R1 we have $\Sigma(v) = \Sigma(w) + \rho$. That is, $S(s) = \{\sigma + \rho; \sigma \in S^\alpha(u)\}$. Now for each $\xi < o(u)$ we have $u = \rho\gamma_\xi + \tau_\xi$ for some $\gamma_\xi \geqq 1$ and some $\tau_\xi < \rho$. Since ρ is a prime component, it follows from the above characterization of $S(s)$ that we have $S(s) = S(s^\#)$, where $s^\#$ is the κ-sequence defined as follows. If $s_\xi < \rho$, then $s^\#_\xi = s_\xi$; otherwise $s^\#_\xi = \rho\gamma_\xi$, where $\gamma_\xi + 1 = \min\{\psi; s_\xi < \rho\psi\}$. Thus there is no loss of generality—but considerable typographical convenience—in assuming $s = s^\#$, and so we make this assumption. But this means that for any $v, w \in P^\alpha(u)$, if $\Sigma(v) \neq \Sigma(w)$, then $\Sigma(v) + \rho \neq \Sigma(w) + \rho$. Hence from our characterization of $S(s)$ and from R2, we see that if $o(u) \geqq \omega$, then $|S(s)| \geqq \aleph_0$. Thus u must be a finite sequence.

We can now define the required finite sequence r.

If $u_\xi < \rho\omega$ for every $\xi < o(u)$, we put $r = u \overset{\circ}{\cup} (\rho)$. On the other hand, if $u_\xi \geqq \rho\omega$ for some $\xi < o(u)$, then we define r as follows. $o(r) = o(u)$; if $u_\xi < \rho\omega$, then $r_\xi = u_\xi$; if $u_\xi \geqq \rho\omega$, then $r_\xi = u_\xi + \rho$.

It is not difficult to see that in each case we have

$$S(r) = \{\sigma + \rho; \sigma \in S(u)\}, = S(s) .$$

This proves our theorem for the case $S(t) = \{\rho\}$.

Consider now the case when $S(t) = \{\rho\kappa\}$ and κ is regular. Put $t^* = s - s/\rho\kappa$: we claim that $S(t^*) = \{o\kappa\}$. For if we put $u^* = t^*/\rho$, then from the regularity of κ we obtain just as before $S(t^*) = \{\sigma + \rho\kappa; \sigma \in S^\alpha(u^*)\}$, and as $\rho\kappa$ is a prime component, our claim will be established if we show that $\sigma < \rho\kappa$ for every $\sigma \in S^\alpha(u^*)$.

Thus take $v \in P^\alpha(u^*)$. Then $v_\xi < \rho\kappa$ for every $\xi < o(v) < \kappa$. Hence, as $cf(\rho\kappa) = \kappa$, we have $\sup\{v_\xi; \xi < o(v)\} = \delta < \rho\kappa$. This gives $\Sigma(v) \leqq \delta o(v) < \delta\kappa \leqq \rho\kappa$, since $\rho\kappa = \kappa^{\beta+1}$ for some β. Hence $S(t^*) = \{\rho\kappa\}$. But now we are in an analogous situation to that above, with ρ

being replaced by $\rho\kappa$. Hence the corresponding argument brings us the desired conclusion.

It remains to prove the theorem in the case in which κ is singular. We show firstly that $\sup\{s_\xi < \rho\kappa; \xi < \kappa\} < \rho\kappa$. For suppose not, and put $\lambda = cf(\rho\kappa), = cf(\kappa)$. It is easily seen that there is some permutation u of some subsequence of s such that u is an increasing λ-sequence with limit $\rho\kappa$. Thus, since $cf(\rho\kappa) = \lambda$ and $\rho\kappa = \kappa^{\beta+1}$ for some β, we have $\Sigma(u) = \rho\kappa$. By the same reasoning, if v is any cofinal subsequence of u, then $\Sigma(v) = \rho\kappa$. Thus, from the cardinal equality $|\lambda|^2 = |\lambda|$, we can deduce that $S^a(u) \supseteqq \{\rho\kappa\alpha; 1 \leqq \alpha < \lambda^+\}$. Since $\lambda < \kappa$, it now follows that $S(s) \supseteqq \{\gamma + \rho\kappa(\alpha + 1); 1 \leqq \alpha < \lambda^+\}$, for some γ. Since this contradicts $|S(s)| < \aleph_0$, we must have $\delta < \rho\kappa$, where $\delta = \sup\{s_\xi < \rho\kappa; \xi < \kappa\}$.

Putting $u = s - s/\rho\kappa$, we now claim that $S(u) = \{\rho\kappa\}$. For take $v < P(u)$, and let v' be a proper initial segment of v. Then we know that $\sup\{v'_\xi; \xi < o(v')\} \leqq \delta$, and thus we have $\Sigma(v') \leqq \delta o(v') < \delta\kappa \leqq \rho\kappa$. Hence $\Sigma(v) \leqq \rho\kappa$. But of course v has as a subsequence some permutation of t, and so $\Sigma(v) \geqq \rho\kappa$. This shows that $S(u) = \{\rho\kappa\}$.

Since $o(s/\rho\kappa) < \kappa$, it is clear that $S(s) \supseteqq \{\sigma + \rho\kappa; \sigma \in S^a(s/\rho\kappa)\}$. Define the sequence w by $o(w) = o(s/\rho\kappa)$, $w_\xi = o\kappa\alpha_\xi$, where $\alpha_\xi + 1 = \min\{\gamma; (s/\rho\kappa)_\xi < \rho\kappa\gamma\}$. Then $S(s) \supseteqq \{\sigma + \rho\kappa; \sigma \in S^a(w)\}$. As usual, we can now deduce that if $o(w) \geqq \omega$, then $|S(s)| \geqq |o(w)|^+$, and thus w, and hence $s/\rho\kappa$, is a finite sequence.

But then $S(s) = \{\sigma + \rho\kappa; \sigma \in S(s/\rho\kappa)\}$, and we are back in our familiar situation, and can proceed as before.

This proves our theorem.

LEMMA. *Let $\alpha \geqq \omega$ be a limit ordinal, and let s be an α-sequence of positive ordinals. Then $cf(\Sigma(s)) = cf(\alpha)$.*

Proof. Almost immediate; in fact we used a particular case of this in the proof of T1. Put $\gamma = \Sigma(s)$, and define the α-sequence (γ_ξ) by $\gamma_\xi = \sum_{\zeta<\xi} s_\zeta$. Since s is a sequence of positive ordinals, (γ_ξ) is an increasing sequence, and as α is limit, we have $\gamma = \lim_{\xi<\alpha}\gamma_\xi$. Thus $cf(\gamma) \leqq cf(\alpha)$. If on the other hand, for each $\sigma < \gamma$ we put $\alpha_\sigma = \min\{\xi < \alpha; \gamma_\xi \geqq \sigma\}$, then we have $\alpha = \sup\{\alpha_\sigma; \sigma < \gamma\}$, and so $cf(\alpha) \leqq cf(\gamma)$. Thus $cf(\gamma) = cf(\alpha)$.

T2. *Let κ be a regular* i.o., *and let s be a κ-sequence of positive ordinals. Then for no* i.o. *$\lambda < \kappa$ is there a λ-sequence r of positive* *ordinals such that* $S(s) = S(r)$.

Proof. This follows at once from the preceding lemma. For suppose that for some i.o. $\lambda < \kappa$ there is a λ-sequence r of positive ordinals with $S(s) = S(r)$. Now from the lemma we have $cf(\sigma) = cf(\kappa)$ for every $\sigma \in S(s)$, and $cf(\tau) = cf(\lambda)$ for every $\tau \in S(r)$. But then we would have $cf(\kappa) = cf(\lambda) \leqq \lambda < \kappa$, contradicting the fact that κ is regular. This proves our theorem.

We have now exhausted the cases in which s is a κ-sequence of positive ordinals and κ is a regular i.o., and so we turn to the cases in which κ is a singular i.o. These provide just slightly more variety.

T3. *Let* κ *be a singular* i.o., *and let* s *be a* κ-*sequence of positive ordinals. Then for no singular* i.o. $\eta < \kappa$ *is there an* η-*sequence* r *of positive ordinals such that* $S(s) = S(r)$.

Proof. Suppose that for some singular i.o. $\eta < \kappa$ there is an η-sequence r of positive ordinals such that $S(s) = S(r)$. Then from our lemma we know that $cf(\eta) = cf(\kappa)$; call this λ. Put $\rho = C(s)$.

From R3 we know that $|S(s)| \leqq |\eta| < |\kappa|$, and so from R4 we know that $\kappa^\beta < \rho < \kappa^{\beta+1}$ for some β, and thus, as in the proof of T1, we obtain $S(s - s/\rho) = \{\rho\kappa\}$. We claim that $C(r) = \rho\kappa$.

Put $u = s/\rho$. Then $o(u) < \kappa$, and as $S(s - u) = \{\rho\kappa\}$, it follows that for some σ we have $\sigma + \rho\kappa \in S(s) = S(r)$.

Now if $\sup\{r_\xi; \xi < \eta\} = \delta < \rho\kappa$, then for each $\tau \in S(r)$ we would have $\tau \leqq \delta\eta < \delta\kappa \leqq \rho\kappa$, a contradiction. Thus $\sup\{r_\xi; \xi < \eta\} \geqq \rho\kappa$. Suppose that $o(r/\rho\kappa) = \eta$. We wish to show that this implies that for each $v \in P(r)$, we have $\Sigma(v) = \rho\kappa\alpha$ for some limit ordinal α. Now firstly, since $o(r/\rho\kappa) = \eta = o(v)$ and η is initial, v must have a cofinal subsequence w with $w \in P(r/\rho\kappa)$, which shows that we cannot have $\Sigma(v) = \rho\kappa\alpha + \gamma$ for some α and some $\gamma < \rho\kappa$.

Suppose that $\Sigma(v) = \rho\kappa(\alpha + 1)$ for some α; thus $\Sigma(v') = o\kappa$ for some final segment v' of v. But as $o(r/\rho\kappa) = \eta$, it follows that some final segment w' of some $w \in P(r/\rho\kappa)$ must be a subsequence of v', and thus $\Sigma(v') \geqq \Sigma(w') \geqq \rho\kappa\eta$, a contradiction. Hence we must have $\Sigma(v) = \rho\kappa\alpha$ for some limit ordinal α.

However, we have seen that $\gamma + \rho\kappa \in S(s) = S(r)$ for some γ, and so we must have $o(r/\rho\kappa) < \eta$. Thus $C(r) \leqq \rho\kappa$. Assume $C(r) =$

$\delta < \rho\kappa$. Then $o(r/\delta) < \eta$, and so $r/\delta \overset{\circ}{\cup} r - (r/\delta) \in P(r)$, which shows that some $\tau \in S(r)$ has positive remainder $\Sigma(r - (r/\delta)) \leqq \delta\eta < \rho\kappa$. Using the fact that $o(s - u) = \kappa$ and $S(s - u) = \{\rho\kappa\}$, however, we see easily that no $\sigma \in S(s)$ has a remainder ψ with $0 < \psi < \rho\kappa$. This shows that we cannot have $C(r) < \rho\kappa$, and hence proves our claim that $C(r) = \rho\kappa$.

But $\rho\kappa = \kappa^{\beta+1} = \eta^{\alpha}$ for some α. Since η is singular, we can apply R4 and deduce that $|S(s)| = |S(r)| = |\eta|$.

Suppose that we have $o(u) < \eta$. Since η is a singular i.o., we have $o(u) < \mu$ for some i.o. $\mu < \eta$. We claim that $|S(s)| \leqq |\mu|$. Take $s^{\circ} \in P(s)$, and let $u^{\circ} \in P^{a}(u)$ be such that u° is a subsequence of s°. Now if u° is not cofinal with s°, we can show, by using R1 in an argument exactly similar to that used in the proof of T1, that $\Sigma(s^{\circ}) = \Sigma(u^{\circ}) + \rho\kappa$. Now assume that u° is cofinal with s°. If $\Sigma(u^{\circ}) = \rho\kappa\alpha + r$ for some α and some γ with $0 < \gamma < \rho\kappa$, then if we let u' be the shortest initial segment of u° with $\Sigma(u') \geqq \rho\kappa\alpha$, we must have u' contained as a subsequence in some proper initial segment s' of s°. However, it is easily seen that $\Sigma(s^{\circ} - u') = \rho\kappa$, whence we can use R1 again to obtain $\Sigma(s^{\circ}) = \Sigma(u') + \rho\kappa = \rho\kappa(\alpha + 1)$. But clearly $u^{\circ} \overset{\circ}{\cup} s^{\circ} - u^{\circ} \in P(s)$ and $\Sigma(u^{\circ} \overset{\circ}{\cup} s^{\circ} - u^{\circ}) = \rho\kappa(\alpha + 1)$.

Finally, suppose that $\Sigma(u^{\circ}) = \rho\kappa\alpha$ for some α.

We claim that in this case $\Sigma(s^{0}) = \Sigma(u^{\circ}) = \rho\kappa\alpha$. For let v be a proper initial segment of s°, and let w be the longest initial segment of u° such that w is a subsequence of v. Then $v - w$ is a proper initial segment of $s^{\circ} - u^{\circ}$, and so $\Sigma(v - w) < \rho\kappa$. But as u° is cofinal with s° and $\Sigma(u^{\circ}) = \rho\kappa\alpha$, we certainly have $\Sigma(u^{\circ} - w) \geqq \rho\kappa$. Therefore $\Sigma(v - w) + \Sigma(u^{\circ} - w) = \Sigma(u^{\circ} - w)$, and so by R1, $\Sigma(v) + \Sigma(u^{\circ} - w) = \Sigma(w) + \Sigma(u^{\circ} - w) = \Sigma(u^{\circ})$, whence $\Sigma(v) < \Sigma(u^{\circ})$. This shows that $\Sigma(s^{\circ}) \leqq \Sigma(u^{\circ})$, and of course we must have $\Sigma(s^{\circ}) \geqq \Sigma(u^{\circ})$. Putting these three pieces together, we obtain

$$S(s) = \{\rho + \rho\kappa; \sigma \in S^{a}(u)\} \cup \{\rho\kappa\alpha \in S^{a}(u); cf(\rho\kappa\alpha) = \lambda\}\,.$$

But then $|S(s)| \leqq |S^{a}(u)|2 \leqq |\mu| < |\eta|$, as claimed.

Since we have already seen that $|S(s)| = |\eta|$, this shows that we must have $o(u) \geqq \eta$.

Put $u^{*} = u - u/\rho\kappa$; we wish to show that $\sup\{u^{*}_{\xi}; \xi < o(u^{*})\} = \rho\kappa$. For suppose that $\sup\{u^{*}_{\xi}; \xi < o(u^{*})\} = \delta < \rho\kappa$; then in the usual way we can show that $\sigma < \rho\kappa$ for every $\sigma \in S^a(u^*)$, whence it follows without much trouble that $S(s - s/\rho\kappa) = \{\rho\kappa\}$. If now we have $o(s/\rho\kappa) < \eta$, we can repeat the above argument to conclude that $|S(s)| < |\eta|$, and thus we must have $o(s/\rho\kappa) \geqq \eta$. But as $S(s) \supseteqq \{\sigma + \rho\kappa; \sigma \in S^a(s/\rho\kappa)\}$, it is not too difficult to see that this gives $|S(s)| \geqq |\eta^+|$, again a contradiction. Therefore we must have $\sup\{u^*_\xi\ \xi < o(u^*)\} = \rho\kappa$, and we already know that $o(u) \geqq \eta$. We now show that we must have $o(u^*) \geqq \eta$.

Suppose that we have $o(s/\rho\kappa) \geqq \eta$. Taking any $\sigma \in S^a(u^*)$, we have $S(s) \supseteqq \{\sigma + \gamma + \rho\kappa; \gamma \in S^a(s/\rho\kappa)\}$, and it is not difficult to see that this gives $|S(s)| \geqq |S^a(s/\rho\kappa)| \geqq |\eta^+|$. As we have thus contradicted $|S(s)| = |\eta|$, it must be the case that $o(s/\rho\kappa) < \eta$. We know, however, that $o(u)(=o(s/\rho)) \geqq \eta$. Hence, since η is an i.o., we must have $o(u^*)(=o(s - s/\rho\kappa)) \geqq \eta$.

Suppose now that for some $\delta < \rho\kappa$, we have $o(u^*/\delta) < \eta$. Then for any $\sigma \in S^a(u^* - u^*/\delta)$ we have $\sigma \leqq \delta\mu^+ < \delta\kappa \leqq \rho\kappa$, where $\mu = i(o(u^*))$, from which it follows that $S(s - s/\delta) = \{\rho\kappa\}$. But the assumption $o(u^*/\delta) < \eta$ gives $o(s/\delta) < \eta$, since we have seen that we must have $o(s/\rho\kappa) < \eta$. We can now deduce in the normal way that $|S(s)| \leqq |S^a(s/\delta)|2 \leqq |i(o(s/\delta))^+| < |\eta|$, once more contradicting the proven equality $|S(s)| = |\eta|$.

We have thus demonstrated that the equality $|S(s)| = |\eta|$ implies that we must have $o(u^*/\delta) \geqq \eta$ for every $\delta < \rho\kappa$.

Now $cf(\rho\kappa) = \lambda$, and as $\lambda < \eta$, we have the cardinal equation $|\lambda||\eta| = |\eta|$. It follows from this and the fact that $o(u^*/\delta) \geqq \eta$ for every $\delta < \rho\kappa$ (the formal proof is perfectly straightforward but rather tedious), that for each $\alpha < \eta^+$ there exists an increasing subsequence v^α of some $v \in P^a(u^*)$ such that

(i) $o(v^\alpha) = \lambda$,

(ii) $\lim_{\xi<\lambda} v^\alpha_\xi = \rho\kappa$, and

(iii) v^α and v^γ have no common term for $\alpha < \gamma < \eta^+$.

But it now follows from this that for each α with $\eta \leqq \alpha < \eta^+$, there is $\sigma \in S(s)$ such that σ has $\rho\kappa(\alpha + 1)$ as a remainder. Since this implies that $|S(s)| \geqq |\eta^+|$, we have obtained a final contradiction, which thus proves our theorem.

T4. *Let κ be a singular* i.o., *and let s be a κ-sequence of positive ordinals with $|S(s)| < |\kappa|$. Put $\lambda = cf(\kappa)$, $\rho = C(s)$.*

(1) *If $\eta < \kappa$ is an* i.o. *with $\eta \neq \lambda$, then there is no η-sequence r of positive ordinals such that $S(s) = S(r)$.*

(2) *There is a* λ*-sequence* r *of positive ordinals such that* $S(s) = S(r)$ *if and only if* $\sup\{s_\xi < \rho\kappa; \xi < \kappa\} < \rho\kappa$ *and* $o(s/\rho\kappa) < \lambda$.

Proof. (1) Let η be an i.o. with $\eta < \kappa$ and $\eta \neq \lambda$. If η is singular, then the result follows from T3. Thus assume η regular, and suppose that r is an η-sequence of positive ordinals such that $S(s) = S(r)$. It now follows from our lemma that we must have $cf(\eta) = cf(\kappa)$. Since η is regular, this gives the contradiction $\eta = \lambda$.

(2) Suppose that the conditions hold, and put $u = s/\rho$, $t = s/\rho\kappa$. Since $|S(s)| < |\kappa|$, R4 tells us that $\kappa^\beta < \rho < \kappa^{\beta+1}$ for some β, and thus we have $S(s - u) = \{\rho\kappa\}$. Now from the condition $\sup\{(s - t)_\xi; \xi < o(s - t)\} = \delta < \rho\kappa$, we obtain $\sigma \leqq \delta\mu < \delta\kappa \leqq \rho\kappa$, where $\mu = i(o(u - t))^+$, for every $\sigma \in S^a(u - t)$, whence we deduce that $S(s - t) = \{\rho\kappa\}$. However, since $o(t) < \lambda = cf(\kappa)$, no $t^\circ \in P^a(t)$ can be a cofinal subsequence of any $s^\circ \in P(s)$. Thus from R1 we can conclude in the usual manner that $S(s) = \{\sigma + \rho\kappa; \sigma \in S^a(t)\}$.

Now since $\lambda = cf(\rho\kappa)$, there is an increasing λ-sequence v with $\lim_{\xi<\lambda} v_\xi = \rho\kappa$. As $\rho\kappa = \kappa^{\beta+1}$, it follows that $\Sigma(v) = \rho\kappa$. But λ is regular, and so we may apply Ginsburg's result from [1] to obtain $|S(v)| = 1$, and hence conclude that $S(v) = \{\rho\kappa\}$. Consider the λ-sequence $r = t \overset{\circ}{\cup} v$. As $o(t) < \lambda$ and λ is regular, we have from R1 that $S(r) = \{\sigma + \rho\kappa; \sigma \in S^a(t)\} = S(s)$.

Now assume that there is a λ-sequence r of positive ordinals such that $S(s) = S(r)$.

Suppose firstly that $\sup\{s_\xi < \rho\kappa; \xi < \kappa\} = \rho\kappa$. Define u, t as above, and put $\mu = i(o(u - t))$. Then we must have $\lambda \leqq \mu < \kappa$, and there exists an increasing λ-subsequence v, $\Sigma(v) = \rho\kappa$, of some $w \in P^a(u - t)$ with $o(w) = \mu$. But then we have $t \overset{\circ}{\cup} w - v \overset{\circ}{\cup} v^\circ \overset{\circ}{\cup} s - u \in P(s)$ for every $v^\circ \in P^a(v)$, from which it follows that $S(s) \supseteqq \{\gamma + \rho\kappa(\alpha + 1); 1 \leqq \alpha < \lambda^+\}$, where $\gamma = \Sigma(t \overset{\circ}{\cup} w - v)$. But this gives $|S(s)| \geqq |\lambda^+|$, whereas by R3 we have $|S(s)| = |S(r)| \leqq |\lambda|$.

Hence we must have $\sup\{s_\xi < \rho\kappa; \xi < \kappa\} < \rho\kappa$, whence we can show in the usual way that $S(s - t) = \{\rho\kappa\}$.

But now we must have $S(s) \supseteqq \{\sigma + \rho\kappa; \sigma \in S^a(t)\}$, and from the definition of t we obtain from this $|S(s)| \geqq |S^a(t)| = |i(o(t))^+|$. Therefore, since $|S(s)| = |S(r)| \leqq |\lambda|$, this gives $o(t) < \lambda$, as required.

This proves our theorem.

Thus far we have looked at the problem of "reducing" a given series of positive ordinals to a shorter series of positive ordinals, the reduction leaving the set of permutation-sums invariant, and we have obtained a complete solution to this problem whenever the length of the original series is an i.o. and the length of the new series is either finite or an i.o.

We now wish to consider the analogous problem obtained by removing the restriction that the terms of the second series be positive. This situation is, naturally, a little more complicated than the previous one, and in one case we have as yet been unable to determine satisfactory criteria.

T5. *Let* κ *be a regular* i.o., *and let* s *be a* k*-sequence of positive ordinals such that* $|S(s)| \geqq \aleph_0$. *Then there is no* i.o. λ *such that for some* λ*-sequence* r *with* $o(r/1) < \lambda$ *we have* $S(s) = S(r)$.

Proof. Let κ, s be as described, and suppose that for some i.o. λ and some λ-sequence r, we have $o(r/1) < \lambda$ and $S(s) = S(r)$. Put $\eta = i(o(r/1))$; then $\eta^+ \leqq \lambda$, and it is obvious that $S(r) = S^a(t)$, where for convenience we are taking $t \in P^a(r/1)$ with $o(t) = \eta$. Put $\rho = C(s)$, $u = s/\rho$. Then from the proof of T1 we know that either $S(s - u) = \{\rho\}$ or $S(s - u) = \{\rho\kappa\}$, depending upon the exact value of ρ. In the first case we have $S(s) = \{\sigma + \rho; \sigma \in S^a(u)\}$, and in the second case we have $S(s) = \{\sigma + \rho\kappa; \sigma \in S^a(u)\}$. We assume the former; the argument used in the latter case is exactly similar.

Thus each $\sigma \in S(s)$ has ρ as a remainder; in fact, by examining the proof of T1, we can see that ρ is the smallest positive remainder of each $\sigma \in S(s)$. Take any $\xi < \eta$, and let $t^* \in P^a(t)$ be such that $o(t^*) = \eta + 1$, and $t^*_\eta = t_\xi$. Then $\Sigma(t^*) = \gamma + t_\xi$ for some γ, whence it follows from $S^a(t) = S(s)$ that t_ξ has smallest positive remainder ρ.

Now consider $\Sigma(t) \in S^a(t) = S(s)$: from our characterization of $S(s)$, we see that $\Sigma(t)$ must have smallest positive remainder ρ, whence it follows from the fact η is a limit ordinal that t has some final segment t' with $\Sigma(t') = \rho$. However, $o(t') = \eta$, and so $t'_\xi < \rho$ for every $\xi < \eta$, contradicting the proven fact that t_ξ has remainder ρ for every $\xi < \eta$. This proves our theorem.

T6. *Let* κ *be a singular* i.o. *and let* s *be a* κ*-sequence of positive ordinals, with* $\aleph_0 \leqq |S(s)| < |\kappa|$.

(1) *We can never have* $|S(s)| = \aleph_0$.

(2) *If either* $|S(s)| \geqq \aleph_2$ *or* $cf(\kappa) \geqq \omega_1$, *then for no* i.o. $\eta < \kappa$

is there an η*-sequence* r *such that* $o(r/1) < \eta$ *and* $S(s) = S(r)$.

Proof. (1) Put $\rho = C(s)$, $u = s/\rho$. Then from $|S(s)| < |\kappa|$ we have $\kappa^\beta < \rho < \kappa^{\beta+1}$ for some β, and $S(s - u) = \{\rho\kappa\}$. Furthermore, we know from the proof of T3 that

$$S(s) = \{\sigma + \rho\kappa; \sigma \in S^a(u)\} \cup \{\rho\kappa\alpha \in S^a(u); cf(\rho\kappa\alpha) = cf(\kappa)\} .$$

From $|S(s)| \geqq \aleph_0$ we therefore obtain $o(u) \geqq \omega$, whence we have $|S(s)| \geqq |i(o(u))^+| > \aleph_0$

(2) Suppose that for some i.o. $\eta < \kappa$ there is an η-sequence t of positive ordinals such that $S(s) = S^a(t)$.

We assume firstly that $|S(s)| \geqq \aleph_2$. Then from R2 it follows that $\eta \geqq \omega_1$. But then of course there exists a limit ordinal β with $\eta \leqq \beta < \eta^+$ and $cf(\beta) \neq cf(\kappa)$.

Take $t^\circ \in P^a(t)$ with $o(t^\circ) = \beta$; then from our lemma we have $cf(\Sigma(t^\circ)) = cf(\beta)$. But by hypothesis, $\Sigma(t^\circ) \in S(s)$, whence by our lemma again, $cf(\Sigma(t^\circ)) = cf(\kappa)$: contradiction.

Now suppose that $cf(\kappa) \geqq \omega_1$; thus by the lemma, $cf(\sigma) \geqq \omega_1$ for every $\sigma \in S(s)$. But if $\eta = \omega$, then the lemma would tell us that for some $\tau \in S^a(t)$ we have $cf(\tau) = \omega$.

Thus we must have $\eta \geqq \omega_1$ in this case also, and we can thus repeat the above argument to obtain a contradiction.

This proves our theorem.

That the conditions imposed upon κ and s cannot be eliminated completely is demonstrated by the following example.

Let t be the $\omega_\omega 2$-sequence defined by $t_n = \omega_\omega^2$ for $n < \omega$, and $t_\xi = \xi$ for ξ with $\omega \leqq \xi < \omega_\omega 2$. Now take $s \in P^a(t)$ with $o(s) = \omega_\omega$.

We have of course $C(s) = \omega_\omega 2$, and from our general characterization of $S(s)$ when $o(s)$ is a singular i.o., we see that $S(s) = \{(\omega_\omega^2)\alpha; \omega \leqq \alpha < \omega_1\}$.

But of course if we let t° be the initial segment of t with $o(t^\circ) = \omega$, then we obviously have $S^a(t^\circ) = S(s)$.

On the other hand, if κ is a singular i.o. with $cf(\kappa) = \omega$, and s is a κ-sequence of positive ordinals with $|S(s)| = \aleph_1$, then it is not necessarily true that there is an ω-sequence r of positive ordinals with $S(s) = S^a(r)$.

To see this, let us define the $\omega_\omega 2$-sequence t by $t_n = \omega_\omega \omega_n$ for $n < \omega$, $t_\xi = \xi$ for ξ with $\omega \leqq \xi < \omega_\omega 2$, and take $s \in P^a(t)$ with $o(s) = \omega_\omega$. Once again we have $C(s) = \omega_\omega 2$, and it is not difficult to see that $S(s) = \{(\omega_\omega^2)\alpha; 1 \leqq \alpha < \omega_1\}$.

Suppose that there is an ω-sequence r of positive ordinals with $S(s) = S^a(r)$. By taking $u \in P^a(r)$ with $o(u) = \omega + 1$ and $u_\omega = r_n$, we see that we must have $r_n \geqq \omega_\omega^2$ for each $n < \omega$. Since this implies that $\sigma \geqq (\omega_\omega^2)\omega$ for every $\sigma \in S^a(r)$, however, we have a contradiction.

T7. *Let κ be a singular* i.o. *with $cf(\kappa) = \omega$, and let s be a κ-sequence of positive ordinals with $|S(s)| = \aleph_1$. Put $\rho = C(s)$, $u = s - s/\rho$. Then there is a sequence t of positive ordinals with $o(t) < \omega_1$, such that $S(t \overset{\circ}{\cup} u) = S(s)$.*

Proof. Since for some γ we have $S(s) \supseteqq \{\gamma + \sigma + \rho\kappa; \sigma \in S^a(s/\rho\kappa)\}$, it is clear that the condition $|S(s)| = \aleph_1$ forces $o(s/\rho\kappa) < \omega_1$. Put $v = s/\rho - s/\rho\kappa$, and suppose that $o(v/\delta) \geqq \omega_1$ for every $\delta < \rho\kappa$. Then by familiar arguments, we can show that for each α with $1 \leqq \alpha < \omega_2$, there exist $\sigma \in S(s)$ having remainder $\rho\kappa(\alpha + 1)$. Since this of course implies that $|S(s)| \geqq \aleph_2$, we must have $o(v/\delta) < \omega_1$ for some $\delta < \rho\kappa$, and hence $o(s/\delta) < \omega_1$.

However, by the usual process we can show that $S(s - s/\delta) = \{\rho\kappa\}$, whence it follows easily that $S(s/\delta \overset{\circ}{\cup} u) = S(s)$.

We conclude this paper by remarking that if we allow s to have zero terms, then nothing else of interest emerges.

For suppose that s is a κ-sequence for some i.o. κ, and that $s/1 \neq s$. Obviously, if $o(s/1) = \kappa$, then $S(s/1) = S(s)$: hence we may assume $o(s/1) < \kappa$, when we have $S(s) = S^a(s/1)$. If $i(o(s/1))^+ = \kappa$, then $|S(s)| = |\kappa|$, and any "reduction" is either trivial or impossible. Thus assume $i(o(s/1))^+ = \lambda < \kappa$. Obviously there is a λ-sequence r with $S(r) = S^a(s/1)$, and the question of whether for some i.o. $\eta \geqq \lambda$ there is an η-sequence r of positive ordinals with $S(r) = S^a(s/1)$ reduces to the questions already investigated. Clearly (R3) for no i.o. $\eta < \lambda$ is there an η-sequence r of positive ordinals with $S(r) = R^a(s/1)$.

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