PACIFIC JOURNAL OF MATHEMATICS
Vol. 59, No. 2, 1975

LUSIN AREA FUNCTIONS ON LOCAL FIELDS

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We show that over a local field, Lusin area functions and nontangential maximal functions of a regular function are equivalent in the L^p "norm" for $0 < p < \infty$. As a consequence, we have that "nice" singular integral transforms preserve H^p-spaces for $0 < p < \infty$.

1. By a local field, we mean a locally compact, nondiscrete, totally disconnected, (complete) field. Various aspects of harmonic analysis on local fields have been studied. A list of references can be found in [4]. We also refer to [4] for notation and preliminaries.

Let K be a fixed local field with the ring of integers $\mathcal{O}$. $\mathcal{O}/\mathcal{P} \cong GF(q)$ where $\mathcal{P}$ is the maximal ideal in $\mathcal{O}$ and q is a prime power. For $k \in \boldsymbol{Z}$, let $\mathcal{P}^{-k} = \{x \in K: |x| \leqq q^k\}$, $(\mathcal{O} = \mathcal{P}^0)$. $\mathcal{P}_y^{-k} = y + \mathcal{P}^{-k}$ are spheres. The Haar measure on K has been normalized so that $|\mathcal{O}| = \int_{\mathcal{O}} dx = 1$ and $|\mathcal{P}_y^{-k}| = q^k$ for all k. The theory of regular functions which are the local field analogue of harmonic functions is studied in [10] and [4]. In particular, distributions on K have been identified with regular functions on $K \times \boldsymbol{Z}$ and the regularization kernel $R_k(x) = q^{-k}\Phi_{-k}(x)$, where Φ_{-k} is the characteristic function of $\mathcal{P}^{-k}$, serves as the Poisson kernel.

Write $(\mathcal{P}_y^{-l}, k) = \{(x, k) \in K \times \boldsymbol{Z}: x \in \mathcal{P}_y^{-l}\}$. For a nonnegative integer l and $z \in K$, let $\Gamma_l(z) = \{(x, k) \in K \times \boldsymbol{Z}: |x - z| \leqq q^{k+l}\} = \bigcup_k (\mathcal{P}_z^{-(k+l)}, k)$. For a distribution f on K or a regular function $f(x, k)$ on $K \times \boldsymbol{Z}$, denote $d_k f(x) = f(x, k) - f(x, k+1)$. The *Lusin area function* of f with respect to Γ_l is given by

$$S^{(l)}f(z) = (\sum |d_k f(x)|^2)^{1/2}$$

where the sum runs over distinct $(\mathcal{P}_x^{-k}, k) \subset \Gamma_l(z)$. Write $Sf(z) = S^{(0)}f(z) = (\sum_k |d_k f(z)|^2)^{1/2}$. The *nontangential maximal function* of f with respect to Γ_l is given by

$$m^{(l)}f(z) = \sup_{(x,k) \in \Gamma_l(z)} |f(x, k)| \,.$$

Write $f^*(z) = m^{(0)}f(z) = \sup_k |(z, k)|$.

Let us suppose that $f(x, k) \to 0$ as $k \to \infty$ for each $x \in K$. Let $\|f\|_p = \sup_k \|f(\cdot, k)\|_p$ for $0 < p < \infty$. It is shown in [10] that for $1 < p < \infty$,

(1) $\quad A_p\|f\|_p \leqq \|Sf\|_p \leqq B_p\|f\|_p$ with constants $A_p, B_p > 0$.

It is easy to see that for $1 < p < \infty$

$$\|f\|_p \leqq \|f^*\|_p \leqq C_p\|f\|_p \text{ with constant } C_p > 0 . \tag{2}$$

In other words,

$$\|Sf\|_p \approx \|f\|_p \approx \|f^*\|_p \quad \text{for} \quad 1 < p < \infty . \tag{3}$$

From [4], we have that, for all nonnegative l and h,

$$\begin{aligned}\{x \in K: S^{(l)}f(x) < \infty\} &\cong \left\{x \in K: \lim_{k\to-\infty} f(x, k) \text{ exists}\right\} \\ &\cong \{x \in K: m^{(h)}f(x) < \infty\} ;\end{aligned} \tag{4}$$

i.e., the above sets are equal except possibly for a set of measure 0. Our main objective is to show that

$$\|S^{(l)}f\|_p \approx \|m^{(h)}f\|_p \quad \text{for} \quad 0 < p < \infty .$$

As a consequence, we show that "nice" singular integral transforms preserve H^p-space $(0 < p < \infty)$ which is the space of distributions whose maximal function are in L^p. The last result is the main contribution of [5].

The euclidean version of the main theorem can be found in [2] (see also [7]); its martingale version about Sf and f^* is proved in [1]. Our work has been motivated by these results. In Appendix we shall discuss briefly how our argument can be applied to certain martingales.

REMARK 1. The equivalence in L^p "norm" is interpreted in the obvious way, i.e., if one side is finite, so is the other and is bounded by a constant multiple of the former one. The restriction that $f(x, k) \to 0$ as $k \to \infty$ is needed only for the first inequality of (1) and $\|m^{(h)}f\|_p \leqq A_p\|S^{(l)}f\|_p$.

REMARK 2. A trivial modification gives us the same result for K^n, the n-dimensional vector space over K. The "Φ-inequalities" of Burkholder-Gundy [1][2] for $S^{(l)}$ and $m^{(h)}$ could also be proved.

2. We first show that $\|f^*\|_p \approx \|m^{(l)}f\|_p$ for $0 < p < \infty$.

LEMMA 1. *For* $\lambda > 0$,

$$|\{x \in K: f^*(x) > \lambda\}| \leqq |\{z \in K: m^{(l)}f(z) > \lambda\}| \leqq q^l|\{x \in K: f^*(x) > \lambda\}| .$$

Proof. $|\{f^* > \lambda\}| \leqq |\{m^{(l)}f > \lambda\}|$ is obvious since $f^* \leqq m^{(l)}f$.

Suppose $m^{(l)}f(z) > \lambda$. Then there exists $(x, k) \in \Gamma_l(z)$ such that $|f(x, k)| > \lambda$. Hence $\mathscr{P}_x^{-k} \subset \{f^* > \lambda\}$ and $z \in \mathscr{P}_x^{-(k+l)}$. Therefore

$$|\{m^{(l)}f > \lambda\}| \leqq q^l |\{f^* > \lambda\}| .$$

THEOREM 1. $\|f^*\|_p \leqq \|m^{(l)}f\|_p \leqq q^{l/p}\|f^*\|_p$ *for* $0 < p < \infty$.

Proof. This follows from Lemma 1 and the following identity:

$$\|g\|_p^p = p\int_0^\infty \lambda^{p-1}|\{g > \lambda\}|d\lambda\,, \quad 0 < p < \infty\,. \tag{5}$$

Now let us break up the proof of $\|S^{(l)}f\|_p \approx \|m^{(h)}f\|_p (0 < p < \infty)$ into several lemmas:

LEMMA 2. $\|S^{(l)}f\|_2^2 = q^l\|Sf\|_2^2 = q^l\|f\|_2^2$.

Proof. Easy and known. (See Lemma 2.8(c) of [4].)

LEMMA 3. $\|f^*\|_p \leqq A_p\|Sf\|_p$ *for* $0 < p < 2$.

Proof. By (5), it suffices to show the following estimate:

$$|\{f^* > \lambda\}| \leqq A\lambda^{-2}\int_0^\lambda t|\{Sf > t\}|dt \text{ for } \lambda > 0\,. \tag{6}$$

For a fixed $\lambda > 0$, let

$$\sigma(x) = \sup\{n: S_nf(z) > \lambda \text{ for some } z \in \mathscr{P}_x^{-(n+1)}\}$$

where $S_nf(z) = (\sum_{k\geqq n}|d_kf(z)|^2)^{1/2}$. (Convention: $\sup\varnothing = -\infty$.) For $x \in K$ with $\sigma(x) = n$, let

$$g(x, k) = \begin{cases} f(x, k) & \text{if } \quad k \geqq n+1\,, \\ f(x, n+1) & \text{if } \quad k \leqq n\,. \end{cases}$$

Hence $Sg(x) \leqq \lambda$ and $Sg(x) \leqq Sf(x)$ for all x. Moreover, for $x \in \{\sigma = -\infty\} \subset \{Sf \leqq \lambda\}$, we have $g^*(x) = f^*(x)$ and $Sg(x) = Sf(x)$. On the other hand, suppose $\sigma(x) = n > -\infty$. Then there exists $z \in \mathscr{P}_x^{-(n+1)}$ such that $S_nf(z) > \lambda$. Thus $\mathscr{P}_z^{-n} \subset \{z: Sf(x) > \lambda\}$ with $x \in \mathscr{P}_z^{-(n+1)}$. Therefore we have

$$|\{x: \sigma(x) > -\infty\}| \leqq q|\{z: Sf(x) > \lambda\}|\,.$$

Now

$$\begin{aligned} |\{f^* > \lambda, \sigma > -\infty\}| &\leqq q|\{Sf > \lambda\}| \\ &\leqq 2q\lambda^{-2}\int_0^\lambda t|\{Sf > t\}|dt \end{aligned}$$

and, by Lemma 2 and (5),

$$
\begin{aligned}
|\{f^* > \lambda,\ \sigma = -\infty\}| \leqq |\{g^* > \lambda\}| &\leqq 2\lambda^{-2}\|g\|_2^2 \\
&= 2\lambda^{-2}\|Sg\|_2^2 = 4\lambda^{-2}\int_0^\infty t|\{Sg > t\}|\,dt \\
&= 4\lambda^{-2}\int_0^\lambda t|\{Sg > t\}|\,dt \\
&\leqq 4\lambda^{-2}\int_0^\lambda t|\{Sf > t\}|\,dt\,.
\end{aligned}
$$

Thus

$$
\begin{aligned}
|\{f^* > \lambda\}| &\leqq |\{f^* > \lambda,\ \sigma > -\infty\}| + |\{f^* > \lambda,\ \sigma = -\infty\}| \\
&\leqq (2q + 4)\lambda^{-2}\int_0^\lambda t|\{Sf > t\}\,dt\,.
\end{aligned}
$$

This establishes (6) and Lemma 3.

LEMMA 4. *For $l > 0$ and $0 < p < 2$,*

$$\|S^{(l)}f\|_p \leqq B_p\|m^{(l)}f\|_p\,.$$

Proof. Again, it suffices to show that for $l > 0$ and $\lambda > 0$,

$$|\{S^{(l)}f > \lambda\}| \leqq B\lambda^{-2}\int_0^\lambda t|\{m^{(l)}f > t\}|\,dt\,.$$

Let $\mu(z) = \sup\{n: |f(x, n)| > \lambda$ for some $x \in \mathscr{P}_z^{-(n+l)}\}$. For $z \in K$ with $\mu(z) = n$, we have $\mu(x) = n$ for all $x \in \mathscr{P}_z^{-(n+l)}$; and let

$$
g(z, k) = \begin{cases} f(x, k) & \text{if} \quad k \geqq n + 1\,, \\ f(x, n + 1) & \text{if} \quad k \leqq n\,. \end{cases}
$$

Hence $\{\mu = -\infty\} = \{m^{(l)}f \leqq \lambda\}$ and for $\mu(z) = -\infty$, we have $g(x, k) = f(x, k)$ if $x \in \mathscr{P}_z^{-(k+l)}$ or $(x, k) \in \Gamma_l(z)$. Thus on $\{z: \mu(z) = -\infty\}$, $S^{(l)}g(z) = S^{(l)}f(z)$ and $m^{(l)}g(z) = m^{(l)}f(z) \leqq \lambda$. Now

$$
\begin{aligned}
|\{S^{(l)}f > \lambda,\ \mu > -\infty\}| &\leqq |\{m^{(l)}f > \lambda\}| \\
&\leqq 2\lambda^{-2}\int_0^\lambda t|\{m^{(l)}f > t\}|\,dt\,,
\end{aligned}
$$

and by Lemma 2 and (5),

$$
\begin{aligned}
|\{S^{(l)}f > \lambda,\ \mu = -\infty\}| \leqq |\{S^{(l)}g > \lambda\}| &\leqq \lambda^{-2}\|S^{(l)}g\|_2^2 \\
&= q^l\lambda^{-2}\|g\|_2^2 \leqq q^l\lambda^{-2}\|m^{(l)}g\|_2^2 \\
&\leqq q^l\lambda^{-2}\cdot 2\int_0^\infty t|\{m^{(l)}g > t\}|\,dt \\
&\leqq 2q^l\lambda^{-2}\int_0^\lambda t|\{m^{(l)}f > t\}|\,dt
\end{aligned}
$$

Hence

$$|\{S^{(l)}f > \lambda\}| \leqq 2(q^l + 1)\lambda^{-2}\int_0^\lambda t|\{m^{(l)}f > t\}|dt\,.$$

Therefore Lemma 4 is proved.

LEMMA 5. *For $l \geqq 0$ and $2 < p < \infty$,*

$$\|S^{(l)}f\|_p \leqq C_p\|f\|_p\,.$$

Proof. Suppose $p > 4$ and let r be the conjugate index of $p/2$. Thus $1 < r < 2$. Consider a fixed $k \in \boldsymbol{Z}$. For $x \in K$, let $\{x_i\}_{i=1}^{q^l}$ be the distinct coset representatives such that $\mathscr{P}_{x_i}^{-(k-l+1)} \subset \mathscr{P}_x^{-(k+1)}$. For $g \in L^r$ with $\|g\|_r = 1$, we have

$$\begin{aligned}\int_K \sum_{i=1}^{q^l}|d_k f(x_i)|^2|g(x)|dx &= \sum_i \int_K |d_k f(x_i)|^2|g(x, k+1)|dx\\ &= \sum_i \int_K |d_k f(x_i)|^2|g(x_i, k+1)|dx\\ &= q^l\int_K |d_k f(x)|^2|g(x, k+1)|dx\,.\end{aligned}$$

Hence it follows from this, Hölder's inequality, (1) and (2) that

$$\begin{aligned}\int_K [S_n^{(l)}f(x)]^2|g(x)|dx &= \sum_{k\geqq n}\int_K \sum_{i=1}^{q^l}|d_k f(x_i)|^2|g(x)|dx\\ &= \sum_{k\geqq n} q^l\int_K |d_K f(x)|^2|g(x, k+1)|dx\\ &\leqq q^l\int_K [S_n f(x)]^2 g^*(x)dx\\ &\leqq q^l\|S_n f\|_p^2\|g^*\|_r\\ &\leqq B_p\|f\|_p^2\end{aligned}$$

where B_p depends only on p and q. Thus

$$\begin{aligned}\|S_n^{(l)}f\|_p^2 = \|[S_n^{(l)}f]^2\|_{p/2} &= \sup_{g\in L^r, \|g\|_r=1}\left|\int_K [S_n^{(l)}f(x)]^2 g(x)dx\right|\\ &\leqq B_p\|f\|_p^2\,.\end{aligned}$$

Therefore $\|S^{(l)}f\|_p \leqq C_p\|f\|_p$ for $4 < p < \infty$.

Apply the Marcinkiewicz interpolation theorem to this and Lemma 2, we have

$$\|S^{(l)}f\|_p \leqq C_p\|f\|_p \quad \text{for} \quad 2 < p < \infty\,.$$

THEOREM 2. *For $l, h \geqq 0$ and $0 < p < \infty$,*

$$\|S^{(l)}f\|_p \approx \|m^{(h)}f\|_p\,.$$

Proof. The case of $p = 2$ is obvious.

If $0 < p < 2$, then, from Lemma 3, Lemma 4 and Theorem 1, we have for $l > 0$,

$$\begin{aligned} \|f^*\|_p &\leqq A_p\|Sf\|_p \leqq A_p\|S^{(l)}f\|_p \\ &\leqq A_pB_p\|m^{(l)}f\|_p \approx \|f^*\|_p . \end{aligned}$$

If $2 < p < \infty$, then, by Theorem 1, (3) and Lemma 5,

$$\begin{aligned} \|m^{(h)}f\|_p &\approx \|f^*\|_p \approx \|f\|_p \approx \|Sf\|_p \\ &\leqq \|S^{(l)}f\|_p \leqq C_p\|f\|_p . \end{aligned}$$

Therefore $\|S^{(l)}f\|_p \approx \|m^{(h)}f\|_p$ for $0 < p < \infty$ and the proof of the theorem is completed.

REMARK 3. The above argument simplifies the extension argument as used in §2 of [4] and is essentially similar to the decomposition argument of [5]. It is also a sort of stopping time argument for martingales relative to a regular stochastic basis. (See Appendix.) The main result (with respect to "truncated cones") could be used to show (4)—the Fatou-Calderón-Stein theorem, in a similar manner as in [2].

3. Let π be a (multiplicative) unitary character on K^* such that it is homogeneous of degree 0 and is ramified of degree $h \geqq 1$. Denote $Q(x) = c\pi(x)|x|^{-1}$ where $c = 1/\Gamma(\pi)$. (See [9] for details about Γ-function.) Let $Q_n = R_n * Q$ and $Q_n^N = Q_n\Phi_{-N}$ for $N \geqq n + h$. For a distribution f on K or a regular function $f(x, k)$ on $K \times \boldsymbol{Z}$, we note that $Q_n^N * f(x, k) = Q_k^N * f(x, k) = Q^N * f(x, k)$ for $n \leqq k \leqq N - h$. Define

$$(T_\pi f)(x, k) = \lim_{N\to\infty} Q^N * f(x, k) \quad \text{for} \quad (x, k) \in K \times \boldsymbol{Z} .$$

If $f \in L^p(K)$, $1 \leqq p < \infty$, then this is just a sort of singular integral transform as been studied in [8], [11] and [4].

For $0 < p < \infty$, let $H^p(K)$ be the space of all distributions f on K whose maximal function $f^* \in L^p(K)$ with the H^p "norm" $\|f^*\|_p$. From [5], we know that for $f \in H^p$, $(T_\pi f)(x, k)$ is a well-defined regular function. The regularization of the corresponding distribution is just $(T_\pi f)(x, k)$. Moreover, the following is also shown:

THEOREM 3. *T_π preserves H^p-spaces for $0 < p < \infty$. That is, $\|(T_\pi f)^*\|_p \approx \|f^*\|_p$ for $0 < p < \infty$.*

We show here how this result can be obtained as a consequence of Theorem 2.

LEMMA 6. *$S^{(h)}f(z) = S^{(h)}T_\pi f(z)$ for all $z \in K$.*

Proof. For a fixed $k \in \boldsymbol{Z}$ and $x \in K$,

$$d_k T_\pi f(x) = T_\pi f(x, k) - T_\pi f(x, k+1) = T_\pi d_k f(x) .$$

For each $m \in \boldsymbol{Z}$, let ε_m^i, $i = 1, 2, \cdots, (q-1)q^{h-1}$, be coset representatives of $\mathscr{P}^{-(m-h+1)}$ in $\{t: |t| = q^{m+1}\}$. Then

$$\begin{aligned} T_\pi f(x, k) &= c \int_{|t| > q^k} f(x-t) \frac{\pi(t)}{|t|} dt \\ &= c \sum_{m=k}^{\infty} q^{-(m+1)} \int_{|t| = q^{m+1}} f(x-t)\pi(t) dt \\ &= c q^{-h} \sum_{m=k}^{\infty} \sum_{i=1}^{(q-1)q^{h-1}} \pi(\varepsilon_m^i) f(x - \varepsilon_m^i, m - h + 1) . \end{aligned}$$

Thus

$$T_\pi d_k f(x) = c q^{-h} \sum_{i=1}^{(q-1)q^{h-1}} \pi(\varepsilon_k^i) f(x - \varepsilon_k^i, k - h + 1) . \tag{7}$$

Now let $g(x)$ be the restriction of $d_k f(x)$ on $z + \mathscr{P}^{-(k+1)}$ for any fixed z. Hence from (7) we see that $T_\pi g(x)$ is also supported on $z + \mathscr{P}^{-(k+1)}$. By Plancherel's theorem, since $|\pi| = 1$, we have

$$\| T_\pi g \|_2 = \| (T_\pi g)^\wedge \|_2 = \| \pi^{-1} \hat{g} \|_2 = \| \hat{g} \|_2 = \| g \|_2 .$$

That is,

$$\sum_{i=1}^{q^h} | d_k f(x_i) |^2 = \sum_{i=1}^{q^h} | d_k T_\pi f_\pi(x_i) |^2$$

where x_i, $i = 1, 2, \cdots, q^h$, are coset representatives of $\mathscr{P}^{-(k-h+1)}$ in $\mathscr{P}_z^{-(k+1)}$. Thus summing this up with respect to k, we have

$$S^{(h)} f(z) = S^{(h)} T_\pi f(z) .$$

Proof of Theorem 3. It follows immediately from Theorem 2 and Lemma 6 that for $0 < p < \infty$,

$$\| f^* \|_p \approx \| S^{(h)} f \|_p = \| S^{(h)} T_\pi f \|_p \approx \| (T_\pi f)^* \|_p .$$

Appendix. Let $(\Omega, \mathscr{A}, P)$ be a probability space and $\{\mathscr{A}_n\}_{n=1}$ a nondecreasing sequence of sub-σ-fields of $\mathscr{A}$. Let $f = \{f_n\}_{n \geq 1}$ be a real-valued) martingale relative to $\{\mathscr{A}_n\}_{n \geq 1}$ and $\{d_k\}_{k \geq 1}$ be the difference sequence of f. For a nonnegative integer l, write

$$m^{(l)} f = \sup_n E(|f_{n+l}| \mid \mathscr{A}_n)$$

and $S^{(l)} f = [\sum_{k>l} E(d_k^2 \mid \mathscr{A}_{k-l})]^{1/2}$. $f^* = m^{(0)} f = \sup_n |f_n|$ is the maximal function of f and $Sf = S^{(0)} f = [\sum_{k>0} d_k^2]^{1/2}$ is the square function of f. Burkholder and Gundy [1] proved that for a large class of martingales,

$$\|Sf\|_p \approx \|f^*\|_p \quad \text{for} \quad 0 < p < \infty \,. \tag{8}$$

However examples (in [1]) show that

$$\|S^{(l)}f\|_p \approx \|m^{(h)}f\|_p \quad \text{for} \quad 0 < p < \infty \tag{9}$$

fails to hold. Nevertheless by a slight modification of the previous argument, we can show that this is true for martingales relative to a *regular* stochastic basis (after Chow [6]).

Indeed, the crucial part of the proof is to consider the following stopping time:

$$\mu(x) = \inf\{n: E(|f_{n+l}| \mid \mathscr{A}_n) < \lambda\} \quad (\lambda > 0)\,.$$

Together with the regularity of the stochastic basis and (8), we get (9) by a similar argument as before.

We remark that our argument gives a simplified proof of (8) for martingales relative to a regular stochastic basis. Also the argument used in Lemma 5 similar to the one in [3] provides a new proof of that

$$\|sf\|_p \leqq C_p\|f\|_p \quad \text{for} \quad p > 2$$

where $sf = S^{(1)}f = [\sum_{k>1} E(d_k^2 \mid \mathscr{A}_{k-1})]^{1/2}$ is the *conditioned* square function of the martingale f (relative to any stochastic basis).

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Received February 20, 1975.

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