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RATIONAL VALUED SERIES OF EXPONENTIALS AND DIVISOR FUNCTIONS

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Recently A. Terras established some (soon to be published) relations between the values of Riemann's zeta function at consecutive positive integral argument and values of certain modified Bessel functions. By combining these relations with some previous results concerning the values of $\zeta(s)$ at odd, positive integers (Grosswald–Nachrichten Akad. Wiss. Göttingen, II Math.–Phys. Klasse 1970, pp. 9–13) it follows that certain infinite series of exponentials and divisor functions (somewhat reminiscent of Lambert series) are rational valued.

Specifically, A. Terras proved [6] that for complex ρ, for a,b natural integers and with $K_u(z)$ the modified Bessel function (notation of Watson; see [1], especially 10.2.15, page 444),

$$\zeta(2\rho)\Gamma(\rho+1)+(1-\rho)\zeta(2\rho-1)\Gamma(1/2)\Gamma(\rho-1/2) \tag{1}$$

$$=2\pi^{\rho}\sum_{a,b\geqq 1}(b/a)^{\rho-1/2}\{2\pi ab(K_{1.5-\rho}(2\pi ab)+K_{0.5+\rho}(2\pi ab))-K_{0.5-\rho}(2\pi ab)\}$$

holds, provided that $\operatorname{Re}\rho>1$. Formula (1) seems related to results of Berndt [2], especially his formula (30), but does not seem to follow trivially from it.

If in (1) we take for ρ a natural integer $m>1$, replace the Bessel functions according to classical formulae (see [1], p. 444) and perform some routine transformations, (1) is seen to imply

$$\zeta(2m-1)=\frac{(m-2)!}{(2m-2)!}\left\{\frac{(4\pi)^{2m-1}m!|B_{2m}|}{2(2m)!}\right. \tag{2}$$

$$\left.-\sum_{n=1}^{\infty}e^{-2\pi n}\sigma_{-(2m-1)}(n)\sum_{k=0}^{m}\frac{(m+k-2)!(m(m-1)+k(k-1))}{k!(m-k)!}(4\pi n)^{m-k}\right\}.$$

If we equate these representations of $\zeta(2m-1)$ to those established in [3], then we obtain some rather curious formulae, that involve the divisor functions $\sigma_k(n)=\Sigma_{d|n}d^k$ for odd, negative $k<-1$. The first few of them read

$$\sum_{n=1}^{\infty} e^{-2\pi n}\sigma_{-3}(n)\{(4\pi n)^2+2(4\pi n)\} = \pi^3/90,$$

(3)
$$\sum_{n=1}^{\infty} e^{-2\pi n}\sigma_{-5}(n)\{(4\pi n)^3+6(4\pi n)^2+12(4\pi n)\} = 2\pi^5/105,$$

$$\sum_{n=1}^{\infty} e^{-2\pi n}\sigma_{-7}(n)\{(4\pi n)^4+12(4\pi n)^3+84(4\pi n)^2+360(4\pi n)\} = 22\pi^7/525.$$

One may wish to complete these formulae with one involving $\sigma_{-1}(n)$, corresponding to $m = 1$. Direct substitution of $m = 1$ in (2) is, of course, meaningless and the correct, well known formula is indeed of a slightly different structure, namely (see **[5]** vol. 1, p. 257; see also **[4]** and **[6]**)

(3′)
$$\sum_{n=1}^{\infty} e^{-2\pi n}\sigma_{-1}(n)(4\pi n) = (\pi-3)/6.$$

A glance at (3) seems to indicate that the "natural" variable is $\nu_n = 2\pi n$. If we make the corresponding change of the summation variable and set also $\tilde{\sigma}_k(n) = \Sigma_{d|n}(2\pi d)^k$, then we obtain the somewhat simpler formulae

$$\sum_{n=1}^{\infty} e^{-\nu_n}\tilde{\sigma}_{-1}(n)\nu_n = \frac{1}{24}-\frac{1}{8\pi}$$

$$\sum_{n=1}^{\infty} e^{-\nu_n}\tilde{\sigma}_{-3}(n)(\nu_n^2+\nu_n) = 1/2^6\cdot 3^2\cdot 5$$

$$\sum_{n=1}^{\infty} e^{-\nu_n}\tilde{\sigma}_{-5}(n)(\nu_n^3+3\nu_n^2+3\nu_n) = 1/2^7\cdot 3\cdot 5\cdot 7$$

$$\sum_{n=1}^{\infty} e^{-\nu_n}\tilde{\sigma}_{-7}(n)(\nu_n^4+6\nu_n^3+21\nu_n^2+45\nu_n) = 11/2^{10}\cdot 3\cdot 5^2\cdot 7.$$

Here all second members (except, naturally, in the first identity) are rational (but, as the last one shows, not necessarily the reciprocal of an integer).

2. General result and proofs. According to [3], for odd $m > 1$,

$$\zeta(2m-1) = \frac{(2\pi)^{2m-1}}{(m-1)(2m)!}\sum_{k=0}^{(m-1)/2}(-1)^k(m-2k)\binom{2m}{2k}B_{2k}B_{2m-2k} - 2\sum_{n=1}^{\infty} e^{-2\pi n}\sigma_{-(2m-1)}(n)\left(\frac{2\pi n}{m-1}+1\right).$$

If we set this equal to (2), we obtain, after routine simplifications

$$\sum_{n=1}^{\infty} e^{-2\pi n}\sigma_{-(2m-1)}(n)\left\{\sum_{k=0}^{m-1}\frac{(m-2)!}{(2m-2)!}\frac{(m+k-2)!(m(m-1)+k(k-1))}{(m-k)!k!} - \frac{4\pi n}{m-1}\right\}$$

(5′)

$$= \frac{(2\pi)^{2m-1}}{(2m)!}\left\{\frac{2^{2m-2}}{\binom{2m-2}{m}}B_{2m} - \frac{1}{m-1}\sum_{k=0}^{(m-1)/2}(-1)^k(m-2k)\binom{2m}{2k}B_{2k}B_{2m-2k}\right\}.$$

For even m, according to [**3**],

$$\zeta(2m-1) = \frac{(2\pi)^{2m-1}}{2(2m)!}\sum_{k=0}^{m}(-1)^{k-1}\binom{2m}{2k}B_{2k}B_{2m-2k} - 2\sum_{n=1}^{\infty}e^{-2\pi n}\sigma_{-(2m-1)}(n).$$

We now set this equal to (2), simplify and obtain:

(5″)

$$\sum_{n=1}^{\infty} e^{-2\pi n}\sigma_{-(2m-1)}(n)\sum_{k=0}^{m-1}\frac{(m-2)!}{(2m-2)!}\frac{(m+k-2)!(m(m-1)+k(k-1))}{(m-k)!k!}(4\pi n)^{m-k}$$

$$= \frac{(2\pi)^{2m-1}}{(2m)!}\left\{\frac{2^{2m-2}}{\binom{2m-2}{m}}|B_{2m}| + \frac{1}{2}\sum_{k=0}^{m}(-1)^k\binom{2m}{2k}B_{2k}B_{2m-2k}\right\}.$$

Formulae (3) are the particular cases $m=2,3$ and 4 of (5′) and (5″), respectively.

Finally, in terms of $\nu_n = 2\pi n$ and $\tilde{\sigma}_k(n)$, (5′), (5″) become

$$\sum_{n=1}^{\infty} e^{-\nu_n}\tilde{\sigma}_{-(2m-1)}(n)\left\{\sum_{k=0}^{m-2}\frac{(m+k-2)!(m(m-1)+k(k-1))}{(m-k)!k!2^k}\nu_n^{m-k} + \frac{(2m-3)!}{(m-3)!}2^{2-m}\nu_n\right\}$$

(6′)

$$= \frac{1}{(2m)!}\left\{2^{m-2}m!B_{2m} + 2^{-m-1}\frac{(2m-2)!}{(m-1)!}\sum_{k=0}^{m}(-1)^k(2k-m)\binom{2m}{2k}B_{2k}B_{2m-2k}\right\}$$

for m odd; and

$$\sum_{n=1}^{\infty} e^{-\nu_n}\tilde{\sigma}_{-(2m-1)}(n)\sum_{k=0}^{m-1}\frac{(m+k-2)!(m(m-1)+k(k-1))}{(m-k)!k!2^k}\nu_n^{m-k}$$

(6″)

$$= \frac{1}{(2m)!}\left\{2^{m-2}m!|B_{2m}| + 2^{-m-1}\frac{(2m-2)!}{(m-2)!}\sum_{k=0}^{m}(-1)^k\binom{2m}{2k}B_{2k}B_{2m-2k}\right\}$$

for m even.

Formulae (4) are the particular cases $m=2,3$ and 4 of (6′), (6″), respectively, to which has been added the formula obtained from (3′) that involves $\tilde{\sigma}_{-1}(n)$.

It is, of course, easy to consolidate the formulae (5′), (5″) into a single formula and similarly for (6′), (6″); however, the corresponding single formulae (each valid now both for even and for odd m), while formally simpler, are somewhat artifical and not very revealing and are, therefore, not given here.

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