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ON THE EXISTENCE OF GLOBAL CLASSICAL SOLUTION OF INITIAL-BOUNDARY VALUE PROBLEM FOR $\square u - u^3 = f$

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In this paper we shall give a sufficient condition under which an initial-boundary value problem for $\square u - u^3 = f$ has a global classical solution.

1. Introduction. Let $\Omega \subset R^3$ be an open bounded domain with sufficiently smooth boundary $\partial\Omega$. In this note we are concerned with the existence of a global solution of the initial-boundary value problem:

$$\frac{\partial^2}{\partial t^2} u - \Delta u + \gamma u^3 = f(x, t) \quad \text{for} \quad x \in \Omega,\ t > 0,$$

$$(*) \quad u(x, 0) = u_0(x),\ \frac{\partial}{\partial t} u(x, 0) = u_1(x),\ x \in \Omega,$$

and

$$u(x, t)|_{\partial\Omega} = 0 \quad \text{for} \quad t \geqq 0,$$

where Δ is the Laplacian in R^3 and $\gamma = -1$.

For the equation $(*)$ with $\gamma = +1$ instead of -1, as is well known, the existence of a global classical solution was proved by J. Sather **[4]**. His method, however, depends largely on the monotonically increasingness of the term u^3, and is not applicable to our problem in its original form.

On the other hand D. H. Sattinger **[5]** introduced the concept of potential well (stable set) to show the existence of global but generalized solutions of the initial-boundary problems of hyperbolic equations with non-monotonic nonlinear terms, though in the case $f(x, t) \equiv 0$. The method of potential well is useful also for nonlinear partial differential equations of other types (Lions **[2]**, Tsutsumi **[6]**).

Now, a local existence of a classical solution for $(*)$ is known (Ebihara **[1]**), but that of a global one seems to be unknown and the aim of this note is to give it by combining the method of Sather's with the one of Sattinger's.

Roughly speaking our result is: Let $\{u_0, u_1\}$ belong to the stable set and be sufficiently smooth, and moreover let f have small norm and be smooth. Then $(*)$ admits a global classical solution.

Though we treat only a typical equation with also typical nonlinear term, our method should be applicable to more general ones.

2. Preliminaries. Throughout this paper the functions considered are all real valued and the notations are as usual (e.g. Lions **[2]**, Mizohata **[3]**). In this section we offer some lemmas which will be used later.

LEMMA 1 (Sobolev). (i) *If* $1 \leqq q \leqq 6$, *we have*

$$\|u\|_{L^q(\Omega)} \leqq C_0(q, \Omega)\|u\|_{H^1(\Omega)} \quad \text{for} \quad u \in \mathring{H}^1(\Omega).$$

(ii) *If k is a nonnegative integer, we have*

$$|u|_{C^k(\bar{\Omega})} \leqq C_1(k, \Omega)\|u\|_{H^{2+k}(\Omega)} \quad \text{for} \quad u \in H^{2+k}(\Omega).$$

For brevity we use the notations $|\cdot|$, $\|\cdot\|$, $|\quad|_q$ for $\|\cdot\|_{L^2(\Omega)}$, $\|\nabla\cdot\|_{L^2(\Omega)}$, $|\cdot|_{L^q(\Omega)}$, respectively.

We define 'kinetic' and 'potential' energies associated with our equation by the functionals

$$K(u) = \int_\Omega \tfrac{1}{2}|u_t(x, t)|^2\,dx = \tfrac{1}{2}|u_t(t)|^2$$

and

$$J(u) = \int_\Omega (\tfrac{1}{2}|\nabla u(x, t)|^2 - \tfrac{1}{4}u^4)dx = \tfrac{1}{2}\|u(t)\|^2 - \tfrac{1}{4}|u(t)|_4^4,$$

and according to **[5]** we put

$$d = \inf_{\substack{u \in \mathring{H}^1(\Omega) \\ u \neq 0}} J(\lambda_1 u),$$

where $\lambda_1 = \lambda_1(u)$ ($\geqq 0$) is the first value of $\lambda \geqq 0$ at which $J(\lambda u)$ begins to decrease. Then with the aid of Lemma 1, we have (see also Tsutsumi **[6]**):

LEMMA 2. *The number d satisfies*

$$0 < \frac{1}{4C_0^4(4, \Omega)} \leqq d < \infty.$$

Now the potential well W is defined as

$$W = \{u \in \mathring{H}^1(\Omega) \mid 0 \leqq J(\lambda u) < d \quad \text{for} \quad 0 \leqq \lambda \leqq 1\}.$$

Then we have:

LEMMA 3 (Sattinger). *The set W is bounded in $\mathring{H}^1(\Omega)$.*

For convenience we say the initial data $\{u_0, u_1\}$ belongs to the stable set if

$$u_0 \in W \quad \text{and} \quad K(u_1) + J(u_0) < d.$$

Here we state our hypotheses on the initial values u_0, u_1, and inhomogeneous term f. For this, let us consider the eigenfunctions $\{\psi_k\}$ for the Laplacian Δ with zero boundary condition:

$$\psi_k \in \mathring{H}^1(\Omega) \quad \text{and} \quad \Delta\psi_k = \mu_k \psi_k \quad \text{in} \quad \Omega \quad (k = 1, 2, \cdots),$$

where μ_k is the eigenvalue for ψ_K.

With respect to the regularity of ψ_K, it is well knwon that $\{\psi_K\}$ is involved in $H^6(\Omega)$ (recall $\partial\Omega$ is sufficiently smooth).

We introduce the spaces of the admissible initial data as follows:

$$V_j \equiv \{\text{closed linear extension of the eigenfunctions } \{\psi_K\} \text{ in } H^{6-2j}\}, \quad j = 0, 1,$$

and assume

A$_1$. $$u_0 \in V_0 \cap W \quad \text{and} \quad u_1 \in V_1.$$

Regarding the energy source function it is required that

A$_2$. $$f \in C^4([0, \infty); L^2) \bigcap_{k=1}^{4} C^{4-k}([0, \infty); H^k \cap \mathring{H}^{k-1}).$$

Finally we assume $f \in L^1([0, \infty); L^2)$ and

A$_3$. $$E_0 + 2\sqrt{E_0 + \left(\int_0^\infty |f(t)|\, dt\right)^2} \int_0^\infty |f(t)|\, dt < d,$$

where

$$E_0 = K(u_1) + J(u_0) \quad \text{(total energy of the initial data)}.$$

Note that A_1 and A_3 imply $\{u_0, u_1\}$ belongs to the stable set.

3. Theorem. In this section we prove:

THEOREM. *Under the assumption* A_1, A_2, *and* A_3, *the problem* (*) *has a classical solution* $u(x, t) \in C^2(\bar{\Omega} \times [0, \infty))$.

Proof. The Galerkin's method is employed. Let $\{u_{0m}\}$ and $\{u_{1m}\}$ be sequences such that

$$(1) \qquad u_{0m} = \sum_{i=1}^{m} \alpha_{im} \psi_i \to u_0 \quad \text{in} \quad H^6 \cap H^1, \quad \text{and} \quad u_{1m} = \sum_{i=1}^{m} \beta_{im} \psi_i \to u_1 \quad \text{in} \quad H^4.$$

This is possible by the assumption A_1. By A_1, A_3, and the continuity of $K(u)$ and $J(u)$ with respect to H^1-topology, we may assume

$$(2) \qquad u_{0m} \in W$$

and

$$(3) \qquad K(u_{1m}) + J(u_{0m}) + 2\sqrt{K(u_{1m}) + J(u_{0m}) + \left(\int_0^\infty |f|\, dt\right)^2} \int_0^\infty |f|\, dt < d.$$

Let us consider the approximate solutions:

$$(4) \qquad u_m(t) = u_m(\cdot, t) = \sum_{k=1}^{m} \lambda_k^m(t) \psi_k \qquad (k = 1, 2, \cdots, m)$$

which are determined by the following system of ordinary differential equations:

$$(5) \qquad (D_t^2 u_m(t), \psi_K) + ((u_m(t), \psi_K)) - (u_m^3(t), \psi_K) = (f(t), \psi_K)$$

with initial values

$$(6) \qquad \begin{cases} u_m(0) = u_{0m} \\ D_t u_m(0) = u_{1m}, \end{cases}$$

where (,) denotes L^2-innerproduct and $((\cdot, \cdot))$ denotes $(\nabla \cdot, \nabla \cdot)$.

Clearly $u_m(t)$ exists in some interval, say, in $[0, t_m]$, $t_m > 0$. Multiplying (5) by $\dot{\lambda}_k^m = D_t \lambda_k^m$ and summing over k from 1 to m, we obtain

$$(7) \qquad K(u_m'(t)) + J(u_m(t)) = K(u_{1m}) + J(u_{0m}) + \int_0^t (f(\tau), u_m'(\tau))d\tau$$
$$\text{for} \qquad t \in [0, t_m],$$

where $'$ denotes D_t.

By use of this equation we verify:

$$(8) \qquad u_m(t) \in W \quad \text{for} \quad \forall\, t \in [0, t_m].$$

Indeed, suppose that (8) is false and let t^* be the smallest time for that $u_m(t^*) \notin W$. Then in virtue of the continuity of $u_m(t)$ we see $u_m(t^*) \in \partial W$ and hence we have (**[2]**, **[5]**, **[6]**)

$$(9) \qquad J(u_m(t^*)) = d.$$

On the other hand, setting $M = \sup_{t\in[0,t^*]} |u_m'(t)|$, (7) implies

$$\tfrac{1}{2} M^2 \leqq K(u_m) + J(u_{0m}) + M \int_0^\infty |f(t)|\, dt.$$

Here we have used the fact that $J(u) \geqq 0$ if $u \in W$. From this we have

$$M \leqq 2 \sqrt{K(u_{1m}) + J(u_{0m}) + \left(\int_0^\infty |f(t)|\, dt\right)^2}.$$

Hence,

$$\begin{aligned} J(u_m(t^*)) &\leqq K(u_m'(t^*)) + J(u_m(t^*)) \\ &= K(u_{1m}) + J(u_{0m}) + \int_0^{t^*} (f(\tau), u_m'(\tau))d\tau \\ &\leqq K(u_m) + J(u_{0m}) + M \int_0^\infty |f(t)|\, dt \\ &\leqq K(u_{1m}) + J(u_{0m}) \\ &\qquad + 2\sqrt{K(u_{1m}) + J(u_{0m}) + \left(\int_0^\infty |f|\, dt\right)^2} \int_0^\infty |f|\, dt \\ &< d \quad (\text{by } A_3), \end{aligned}$$

which is a contradiction to (9). Thus (8) is valid.

By (8) and Lemma 3, $|u'_m|$ and $\|u_m\|_{\dot{H}^1}$ are in fact majorized by a constant independent of m and we conclude that $u_m(t)$ exists in $[0, \infty)$ and the inequality

$$|D_t u_m(t)|^2 + \|u_m(t)\|^2 \leqq C_0 \quad \text{for} \quad t \in [0, \infty) \tag{10}$$

holds.

This is the key estimate for our arguments and the estimations of higher derivatives of u_m are carried out on the basis of (10). For the problem (*) with $\gamma = 1$, we note, this is easily derived from the monotone increasingness of u^3.

Now we proceed to consideration of higher derivatives of u_m, which is the same as Sather's **[4]** and sketched briefly.

For arbitrarily fixed $T > 0$, the estimations

$$|D_t^{k+1} u_m(t)|^2 + \|D_t^k u_m(t)\| \leqq C_K(T) \tag{11}$$
$$\text{for} \quad k = 1, 2, 3, 4 \quad t \in I = [0, T],$$

hold, where $C_K(T)$ are constants depending on T but independent of m. Indeed by the linearity of (5) with respect to ψ_k we obtain

$$\begin{aligned} &(D_t^{2+j} u_m(t), D_t^{j+1} u_m(t)) + ((D_t^j u_m(t), D_t^{j+1} u_m(t)) \\ &- (D_t^j u_m^3, D_t^{j+1} u_m) = (D_t^j f, D_t^{j+1} u_m). \end{aligned} \tag{12}$$

(12) with $j = 1$ implies

$$\begin{aligned} \frac{d}{dt}\{|D_t^2 u_m(t)|^2 + \|D_t u_m\|^2\} &= 2(3u_m^2 D_t u_m, D_t^2 u_m) + 2(D_t f, D_t^2 u_m) \\ &\leqq 3(|u_m^2 D_t u_m|^2 + |D_t^2 u_m|^2 \\ &\qquad + (|D_t f|^2 + |D_t^2 u_m|^2) \\ &\leqq \text{const.}\,(|u_m|_6^4 |D_t u_m|_6^2 + |D_t^2 u_m|^2) + |D_t f|^2 \\ &\leqq \text{const.}\,(\|u_m\|^4 \|D_t u_m\|^2 + |D_t^2 u_m|^2) + |D_t f|^2 \\ &\leqq \text{const.}\,(|D_t^2 u_m|^2 + \|D_t u_m\|^2) + |D_t f|^2, \end{aligned}$$

here we used Hölder's inequality, Lemma 1 and (10). Applying the Gronwall's lemma we get

$$\begin{aligned} &|D_t^2 u_m(t)|^2 + \|D_t u_m(t)\|^2 \\ &\leqq \left\{|(D_t^2 u_m)_0|^2 + \|(D_t u_m)_0\|^2 + \int_0^T |D_t f|^2 \, dt\right\} \times e^{\text{const } T} \quad \text{for} \quad \forall\, t \in I, \end{aligned} \tag{13}$$

where $(D_t^K u_m)_0$ denotes the value of $D_t^K u_m(x, t)$ at $t = 0$.

$\|(D_t u_m)_0\|$ is obviously uniformly bounded in m. For the bound of $\|(D_t^2 u_m)_0\|$, set $t=0$ in (5) to get

$$((D_t^2 u_m)_0-(\Delta u_m)_0-(u_m^3)_0-(f)_0,\ \psi_K)=0, \qquad 1\leqq k\leqq m,$$

and hence

$$|(D_t^2 u_m)_0|=|\Delta u_{m0}+P_m f_0+P_m u_{m0}^3|,$$

where P_m is the orthogonal projection onto the m-dimensional subspace of L^2 with basis $\{\psi_1, \psi_2, \cdots, \psi_m\}$. This implies, with the aid of Lemma 1, the uniformly boundedness of $|(D_t^2 u_m)_0|$.

Combining these uniform estimates of initial values with (13), we obtain (11) for $k=1$.

The succession of similar procedure gives (11) for $k=2,3,4$.

Now by the standard arguments of the approximate solutions we conclude, after the extraction of suitable subsequence if necessary, the following:

$$D_t^k u_m \to D_t^k u \quad \text{in} \quad L^2(\Omega\times I) \quad \text{for} \quad 0\leqq k\leqq 4,$$

$$D_t^k u_m(t)\to D_t^k u(t) \quad \text{in} \quad L^2(\Omega) \quad \text{uniformly for} \quad t\in I,\ 0\leqq k\leqq 3,$$

$$D_j D_t^k u_m(t)\to D_j D_t^k \quad \text{weakly in} \quad L^2,\ 0\leqq j,\ k\leqq 3,\ t\in I,$$

$$\text{where } D_j \text{ denotes } \frac{\partial}{\partial x_j},$$

$$D_t^k u(t)\in \mathring{H}_1, \qquad 0\leqq k\leqq 3,$$

$$D_t^k u_m(t)\to D_t^4 u(t) \quad \text{weakly in} \quad L^2,\ t\in I,$$

$$|D_t^4 u(t)-D_t^4 u(\tau)|\leqq \text{const.}\,|t-\tau|,$$

$$|D_j D_t^k u(t)-D_j D_t^k(\tau)|\leqq \text{const.}\,|t-\tau|, \quad 0\leqq j, \quad k\leqq 3,$$

$$|D_t^k u_m(x,t)|\leqq \text{const.} \quad 0\leqq k\leqq 3, \quad (x,t)\in\bar{\Omega}\times I,$$

and

$$D_t^k u_m^3(t)\to D_t^k u^3(t) \quad \text{in} \quad L^2 \quad \text{uniformly in} \quad t\in I, \quad 0\leqq k\leqq 2.$$

The limit-function u satisfies of course:

$$(D_t^{2+j} u(t), v)+((D_t^j u(t), v))-(D_t^j u^3(t), v)=(D_t^j f, v)$$
$$\text{for} \quad \forall\, v\in H^1, \quad \forall\, t\in I, \quad 0\leqq\forall\, j\leqq 2.$$

Moreover applying the well-known regularity results concerning weak solution of elliptic equation, we obtain finally

$$u \in C^4(I, H_0) \bigcap_{k=1}^{4} C^{4-k}(I, H^k \cap \mathring{H}^1).$$

Since $\bigcap_{k=0}^{2} C^k(I, H^{4-k}(\Omega)) \subset C^2(\bar{\Omega} \times I)$ holds (c.f. Lemma 1, (ii)), we conclude that u belongs to the class $C^2(\bar{\Omega} \times I)$ and is the classical solution on $\bar{\Omega} \times I$ of the problem (*).

From the arbitrariness of T and the uniqueness of the classical solution on $[0, T]$ (it is obvious) we can construct, as is usual, the classical solution u on $\bar{\Omega} \times [0, \infty)$. The proof of theorem is now completed.

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