PACIFIC JOURNAL OF MATHEMATICS
Vol. 61, No. 1, 1975

# LINEAR OPERATORS FOR WHICH $T^*T$ AND $T + T^*$ COMMUTE

STEPHEN L. CAMPBELL

**This paper is about the bounded linear operators $T$ acting in a separable Hilbert space $\mathscr{H}$ such that $T^*T$ and $T+T^*$ commute. It will be shown that such operators are normal if they are either compact or quasinilpotent. It is conjectured that if $T^*T$ and $T+T^*$ commute, then $T=A+Q$ where $A=A^*$, $AQ=QA$, and $Q$ is quasinormal. This conjecture is shown to be equivalent to $[T^*T-TT^*]T[T^*T-TT^*]$ being hermitian.**

For bounded linear operators $X$, $Y$, let $[X, Y]=XY-YX$. Let $\theta=\{T: [T^*T, T+T^*]=0\}$. The defining condition for $\theta$ appears in the work of Embry. She has shown that if $\sigma(T^*)\cap\sigma(T)=\emptyset$ and $T$ or $T^*$ are in $\theta$, then $T$ is normal [9, p. 236]. She has also shown that if $T\in\theta$ and $[T^*T, TT^*]=0$, then $T$ is quasinormal [8, p. 459]. On the other hand if $Q$ is quasinormal, $A=A^*$, and $[A, Q]=0$, then $A+Q\in\theta$. Thus Embry's result shows that the intersection of the class (BN) $=\{T: [T^*T, TT^*]=0\}$ (see [4] and [5]) and $\theta$ is trivial, i.e., the quasinormals. In particular, there are no nonquasinormal centered [11] operators in $\theta$. These last observations are helpful when trying to construct examples of nonquasinormal operators in $\theta$ since (BN) includes all weighted shifts and most weighted translation operators. Using [13] it is also easy to see that if $T^2$ is normal and $T\in\theta$, then $T$ is normal.

It seems reasonable to make the following conjecture:

$$\text{(C)}\qquad \theta=\{A+Q: [Q, Q^*Q]=0, [Q, A]=0, A^*=A\}.$$

If (C) is true, then using the canonical form for quasinormals given in [1], it is easy to see that every operator in $\theta$ is subnormal. While we have not been able to resolve (C) we shall present several results which show that the operators in $\theta$ behave much as if they were hyponormal. In particular, we shall show that if $T\in\theta$ is compact or quasinilpotent, then it is normal. This will strengthen the result in [6] which asserts that if $T\in\theta$ and $T$ is trace class, then $T$ is normal.

Finally, let $B(\lambda)=(\lambda-T^*)(\lambda-T)=\lambda^2-\lambda(T^*+T)+T^*T$. Note that if $T\in\theta$, then the values of $B(\lambda)$ form a commutative family of normal operators.

**2. Main results.** Recall from [6] that if $T\in\theta$, then $\lambda+T\in\theta$ for real $\lambda$. Also if $T\in\theta$, then the null space of $T$, $N(T)$, is reducing.

Finally, $T \in \theta$ if and only if $T^*[T^*, T] = [T^*, T]T$.

THEOREM 1. *Suppose that $T \in \theta$ and $\lambda$ is an eigenvalue of $T$. Then the eigenspace of $T$ associated with $\lambda$ is reducing.*

*Proof.* Suppose that $T \in \theta$ and $\lambda$ is an eigenvalue. If $\lambda$ is real, we are done. Suppose that $\lambda$ is not real. Since $N(T)$ is reducing we may also assume that $T$ is one-to-one. Let $\phi$ be such that $T\phi = \lambda\phi$. Then $[T^*, T]\phi = (\lambda - T)T^*\phi$. Thus $T^*[T^*, T]\phi = [T^*, T]T\phi$ becomes $B(\lambda)T^*\phi = 0$. Since $B(\lambda)$ is normal, and $B(\lambda)^* = B(\bar{\lambda})$, we have $B(\bar{\lambda})T^*\phi = 0$. Thus

$$0 = \lambda B(\bar{\lambda})T^*\phi = B(\bar{\lambda})T^*T\phi = T^*TB(\bar{\lambda})\phi ,$$

so that $B(\bar{\lambda})\phi = 0$. But then

$$0 = B(\bar{\lambda})\phi = (\bar{\lambda} - T^*)(\bar{\lambda} - T)\phi = (\bar{\lambda} - \lambda)(\bar{\lambda} - T^*)\phi .$$

Hence $T^*\phi = \bar{\lambda}\phi$ and the eigenspace is reducing.

That the eigenspaces of a hyponormal operator are reducing is well known. See, for example, [12, p. 420].

THEOREM 2. *If $T \in \theta$ and $T$ is quasinilpotent, then $T = 0$.*

*Proof.* Suppose that $T \in \theta$ and $\sigma(T) = \{0\}$. We may assume that $T$ is one-to-one if $T$ is not zero. If $T^*T(T + T^*) = 0$, we are done. Suppose then that $T^*T(T + T^*) \neq 0$. Since $\sigma(T) = \{0\}$, $B(\lambda)$ is invertible for all $\lambda \neq 0$. Let $E(\cdot)$ be the spectral measure associated with the commutative Banach $*$-algebra generated by $T^*T$ and $T + T^*$. Then there exist $E$ measurable functions $g, h$ such that

$$T^*T = \int_\Delta g(s)E(ds),\ T^* + T = \int_\Delta h(s)E(ds)$$

and $\Delta$ is a compact subset of the plane. (In fact $\Delta \subseteq \sigma(T^*T) \times \sigma(T^* + T)$.) Since $(T^*T)(T + T^*) \neq 0$, there exists $s_0 \in \Delta$, $s_0$ in the support of $E$, such that $g(s_0)$, $h(s_0)$ are in the $E$-essential ranges of $g$, $h$, respectively, and both $g(s_0)$, $h(s_0)$ are nonzero. The polynomial $\lambda^2 + h(s_0)\lambda + g(s_0)$ has at least one nonzero root. Call it $\lambda_0$. Then

$$B(\lambda_0) = \int_\Delta \left(\lambda_0^2 + h(s)\lambda_0 + g(s)\right)E(ds)$$

is not invertible which is a contradiction. Hence $T = 0$.

As an immediate consequence of Theorems 1 and 2 we get:

COROLLARY 1. *If $T \in \theta$ and $T$ is compact, then $T$ is normal.*

Our next result has two interesting corollaries.

THEOREM 3. *Suppose that $N$ is normal, $B \in \theta$, and $[N, B] = 0$. Then $N + B \in \theta$ if and only if, relative to the same orthogonal decomposition of the underlying Hilbert space, $N = N_1 \oplus N_2$, $B = B_1 \oplus B_2$, $N_1 = N_1^*$ and $B_2$ is normal.*

*Proof.* The only if part is clear. Suppose then that $T = N + B \in \theta$ where $N$ is normal, $[N, B] = 0$, and $B \in \theta$. Note that $[N, B^*] = 0$ by Fuglede's theorem. Then $[T^*, T] = [B^*, B]$, so that $T^*[T^*, T] = [T^*, T]T$ becomes $(N^* - N)[B^*, B] = 0$. Let $P$ be the orthogonal projection onto the null space of $N^* - N$. Then $PN = NP$ and $PB = BP$ since $P$ is a measurable function of $N$. Thus the range of $P$ reduces both $N$ and $B$, so that $N = N_1 \oplus N_2$, $B = B_1 \oplus B_2$ relative to $R(P) \oplus R(I - P)$. But $N_1^* = N_1$ by definition of $P$ and $B_2$ is normal since $P[B^*, B] = [B^*, B]$.

COROLLARY 2. *If $T \in \theta$, $\lambda + T \in \theta$, and $\lambda$ is not real, then $T$ is normal.*

COROLLARY 3. *If $T \in \theta$ and $T$ is completely nonnormal, then there does not exist any nonhermitian normal operator $N$ such that $[T, N] = 0$ and $T + N \in \theta$.*

3. **Block matrix representation.** If Conjecture (C) is true, then if $T \in \theta$ and $T$ is completely nonnormal, $T$ must have a lower triangular block matrix representation with all zero entries except on the diagonal and first subdiagonal. All diagonal entries are the same self-adjoint operator $A$, and all subdiagonal entries are the same positive operator $P$. This decomposition follows easily from the work of Brown on quasinormal operators [1].

It is easy to compute what subspace the first block corresponds to. It is the closure of the range of $T^*T - TT^*$. Morrel has developed a decomposition for operators $T$ which have a subspace of $N[T^*T - TT^*]$ invariant [10]. Applying this to $T \in \theta$ yields a lower triangular block representation for $T$ provided that $T^*T - TT^*$ is not one-to-one. If this approach is to verify Conjecture (C) then it will be necessary and sufficient to show that $[T^*T - TT^*]T[T^*T - TT^*]$ is hermitian.

THEOREM 4. *Suppose that $T \in \theta$ is completely nonnormal. If $[T^*, T]T[T^*, T]$ is hermitian, then $T = A + Q$ where $[A, Q] = 0$, $A =$*

$A^*$, $[Q, Q^*Q] = 0$.

*Proof*. Suppose that $T \in \theta$ is completely nonnormal and $[T^*, T]T[T^*, T]$ is hermitian. If $[T^*, T]$ is one-to-one we have $T = T^*$ and are done. Assume then that $[T^*, T]$ is not one-to-one. Since $T$ is nonnormal we have $[T^*, T] \neq 0$. Thus from [10] we get that

$$(1) \qquad \begin{bmatrix} A_1 & 0 & 0 & \cdot \\ B_1 & A_2 & 0 & \cdot \\ 0 & B_2 & A_3 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

on $\mathscr{H} = \sum_{i=0}^{l} \oplus H_i$, $H_0 = \overline{R([T^*, T])}$, $l \leqq \infty$, $\dim H_i \geqq \dim H_{i+1}$. By assumption $A_1 = A_1^*$. But then $[T^*, T] = B_1^* B_1$ so that $B_1$ is one-to-one. Using the fact that $H_0 = \overline{R([T^*, T])}$ one gets by direct computation from (1) that

$$(2) \qquad B_i^* A_{i+1} = A_i B_i^* , \quad A_{i+1}^* A_{i+1} + B_{i+1}^* B_{i+1} = B_i B_i^* + A_{i+1} A_{i+1}^*$$

for $i = 1, 2, \cdots$ where $A_{l+1} = B_{l+1} = 0$ if $l < \infty$. Furthermore, by definition of the $H_i$ we have $B_i$ has dense range so that $B_i^*$ is one-to-one. Now since $T^*[T^*, T] = [T^*, T]T$ we have that $A_1 B_1^* B_1 = B_1^* B_1 A_1$, or $B_1^* A_2 B_1 = B_1^* A_2^* B_1$ Since $B_1$ is one-to-one with dense range we get that $A_2 = A_2^*$. But then from (2), we see that $B_2^* B_2 = B_1 B_1^*$ and $B_2$ is one-to-one. Thus from $B_2^* A_3 = A_2 B_2^*$ we get that $B_2^* A_3 B_2 = A_2 B_2^* B_2 = A_2 B_1 B_1^* = B_1 A_1 B_1^* = B_1 B_1^* A_2$. Hence $A_3 = A_3^*$ and $[A_2, B_2^* B_2] = 0$. Suppose now that $A_i = A_i^*$, $[A_i, B_i^* B_i] = 0$, $B_{i+1}^* B_{i+1} = B_i B_i^*$, and $B_i$ is one-to-one with dense range for $i \leqq k$. Then $B_{k+1}$ is one-to-one with dense range. Also $B_k^* A_{k+1} B_k = A_k B_k^* B_k$ and hence $A_{k+1}^* = A_{k+1}$. Thus $B_{k+2}^* B_{k+2} = B_{k+1} B_{k+1}^*$ so that $B_{k+2}$ is one-to-one with dense range. But then $A_{k+1} B_{k+1}^* B_{k+1} = A_{k+1} B_k B_k^* = B_k A_k B_k^* = B_k B_k^* A_{k+1}$. Hence $[A_{k+1}, B_{k+1}^* B_{k+1}] = 0$.

If $l < \infty$, then the $l$th equation is $A_{l+1}^* A_{l+1} = B_l B_l^* + A_{l+1} A_{l+1}^*$. As before we get $A_{l+1}^* = A_{l+1}$ and hence $B_l = 0$. But then $B_i = 0$ for all $i$ which is a contradiction of the nonnormality of $T$. Thus $l = \infty$. Now let

$$A = \begin{bmatrix} A_1 & 0 & & \\ 0 & A_2 & & \\ & & \cdot & \\ & & & \cdot \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 0 & \cdot \\ B_1 & 0 & 0 & \cdot \\ 0 & B_2 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

Then $B^*A = AB^*$ from (2). But $A = A^*$ so that $[B, A] = 0$. Hence $B = T - A \in \theta$. However $B^*[B^*, B] = 0$ so that $B^*(B^*B) = (B^*B)B^*$ and $B$ is quasinormal as desired.

3. **Comments.** The conclusion of Theorem 1, that eigenspaces are reducing, appears in the work of Berberian. Using Theorem 1, it follows immediately from [3, p. 276] that if $T \in \theta$, $\sigma(T)$ is countable, and $T$ is reduction-isoloid [3, p. 277], then $T$ is normal.

In studying nonnormal operators one usually picks off a normal summand and studies the completely nonnormal operator that is left. Theorem 1 tells us that any condition which provides for eigenvalues is incompatible with the complete nonnormality of a $T \in \theta$. Thus one can prove results such as [2, p. 190], [3, p. 277].

THEOREM 5. *If $T \in \theta$ is completely nonnormal and $T$ is also $(G_1)$ or restriction convexoid, then $\sigma(T)$ has no isolated points.*

Finally, we note that the restriction of a $T \in \theta$ to an invariant subspace is not necessarily in $\theta$. The quasinormal operator in [7] whose restriction to an invariant subspace is not quasinormal is an example.

## REFERENCES


1. Arlen Brown, *On a class of operators*, **4** (1953), 723-728.
2. S. K. Berberian, *Some conditions on an operator implying normality*, Math. Ann., **184** (1970), 188-192.
3. S. K. Berberian, *Some conditions on an operator implying normality, II*, Proc. Amer. Math. Soc., **26** (1970), 277-281.
4. Stephen L. Campbell, *Linear operators for which $T^*T$ and $TT^*$ commute*, Proc. Amer. Math. Soc., **34** (1972), 177-180.
5. ———, *Linear operators for which $T^*T$ and $TT^*$ commute*, II, Pacific J. Math., **53** (1974), 355-361.
6. ———, *Operator-valued inner functions analytic on the closed disc*, II, Pacific J. Math., (to appear).
7. ———, *Subnormal operators with non-trivial quasinormal extensions*, Acta Sci. Math. Szeged. (to appear).
8. Mary R. Embry, *Conditions implying normality in Hilbert space*, Pacific J. Math., **18** (1966), 457-460.
9. ———, *A connection between commutativity and separation of spectra of operators*, Acta Sci. Math. Szeged., **32** (1971), 235-237.
10. Bernard B. Morrel, *A decomposition for some operators*, Indiana Univ. Math. J., **23** (1973), 497-511.
11. Bernard B. Morrel and P. S. Muhly, *Centered operators*, Studia Math., **LI** (1974), 251-263.
12. C. R. Putnam, *Spectra of polar factors of hyponormal operators*, Trans. Amer. Math. Soc., **188** (1974), 419-428.
13. H. Radjavi and P. Rosenthal, *On roots of normal operators*, J. Math. Anal. and Appl., **34** (1971), 653-664.


Received May 28, 1975 and in revised form July 28, 1975.

NORTH CAROLINA STATE UNIVERSITY—RALEIGH