

SUMS OF BOOLEAN SPACES REPRESENT EVERY GROUP

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For every Abelian group $(S, +)$ there exist Boolean — i.e., compact, 0-dimensional — topological spaces $X_s, s \in S$, such that $s + t = u$ if and only if X_u is homeomorphic to the disjoint union of X_s and X_t . The method of the proof of this theorem is topological, utilizing mostly properties of Čech-Stone compactifications of various spaces. A corollary, obtained from well-known dualities, is the representability of Abelian groups (in an analogous sense) by products of rings, lattices, Boolean algebras, Banach spaces or Banach algebras.

Let us recall the definition [11]:

DEFINITION. Let $(S, +)$ be a commutative semigroup, let K be a category with products \times (or sums \vee) of finite collections of objects. A mapping

$$r: S \longrightarrow \text{obj } K$$

is called a representation of S by products (or sums) in K whenever

- (i) if $s \neq s'$, then $r(s)$ is not isomorphic to $r(s')$;
- (ii) $r(s + t)$ is isomorphic to $r(s) \times r(t)$ (or $r(s) \vee r(t)$, respectively).

Many results concerning these representations can be, implicitly or explicitly, found in literature. Each finite cyclic group has a representation by products (or sums, which is the same in this case) in the category of Abelian groups [3]. Each semigroup on one generator has a representation by products in the category of modules over a suitable ring [5, 7]. Each semigroup on one generator and each Abelian group have representations by products in the category of topological spaces [10, 11] in the category of small categories [11], in a lot of subcategories of the category of graphs [1, 6], and unary universal algebras [2].

In the present note we generalize the following results of [4]: The group C_2 (i.e. the cyclic group of order 2) has a representation by products in the category of Boolean algebras or, which is the same, it has a representation by sums in the category of 0-dimensional compact Hausdorff spaces (i.e. Boolean spaces). In the present paper we prove that each Abelian group has such a representation. Thus, taking e.g. the additive group of all real numbers as the represented group, we obtain a Boolean space X which “can be multiplied by

any real number λ' such that if $\lambda \neq \lambda'$, then the spaces λX and $\lambda' X$ are nonhomeomorphic, but $(\lambda + \mu)X$ is always homeomorphic to $\lambda X \vee \mu X$ (and $(\lambda \cdot \mu)X = \lambda(\mu X)$).

1. The main theorem.

Construction 1. Given a collection $\mathfrak{X} = \{X_i; i \in I\}$ of compact Hausdorff spaces, denote by $\bigvee_{i \in I} X_i$ their disjoint union, which is a space on the set $\bigcup (X_i \times \{i\})$ with an obvious topology; denote by $s(\mathfrak{X})$ the Čech-Stone compactification $s(\mathfrak{X}) = \beta(\bigvee X_i)$ (i.e., the sum in the category of compact Hausdorff spaces). Considering I as a discrete space, we define a space $p(\mathfrak{X})$, which we obtain from βI by replacing each $i \in I$ by the space X_i : $p(\mathfrak{X})$ is a space on the set $\bigcup (X_i \times \{i\}) \cup (\beta I - I)$, whose topology on $X_i \times \{i\}$ is that of X_i and whose quotient space under the mapping $f: p(\mathfrak{X}) \rightarrow \beta I$, which carries $X_i \times \{i\}$ to $\{i\}$ and leaves $\beta I - I$ unchanged, is just the space βI .

Notice that if \mathfrak{X} is a collection of Boolean spaces then $p(\mathfrak{X})$ is also a Boolean space. Furthermore $\beta I - I$ is a subspace of $p(\mathfrak{X})$.

Let $\mathfrak{X}_1 = \{X_i; i \in I_1\}$ and $\mathfrak{X}_2 = \{X_i; i \in I_2\}$ be collections of compact spaces where I_1, I_2 are disjoint, and let $\mathfrak{X} = \{X_i; i \in I_1 \vee I_2\}$. Then $p(\mathfrak{X}_1) \vee p(\mathfrak{X}_2) = p(\mathfrak{X})$, which follows easily from the additivity of Čech-Stone compactifications.

LEMMA 2. *Let $\mathfrak{X} = \{X_\lambda; \lambda \in L\}$ and $\mathfrak{Y} = \{Y_\mu; \mu \in M\}$ be collections of compact Hausdorff spaces. Let h be a homeomorphism of $p(\mathfrak{X})$ onto $p(\mathfrak{Y})$ which maps $\beta L - L$ onto $\beta M - M$. Then there exist finite $L' \subset L, M' \subset M$ and a bijection φ of $L - L'$ onto $M - M'$ such that, for each $\lambda \in L - L', h$ maps X_λ onto $Y_{\varphi(\lambda)}$.*

Proof. (a) For each $\lambda \in L$ (or $\mu \in M$) denote $k(\lambda) = \{\mu \in M; h(X_\lambda) \cap Y_\mu \neq \emptyset\}$ (or $m(\mu) = \{\lambda \in L; h^{-1}(Y_\mu) \cap X_\lambda \neq \emptyset\}$). Since h maps $\beta L - L$ onto $\beta M - M$, each $k(\lambda)$ as well as each $m(\mu)$ is finite. Denote by L_1 (or M_1) the set of all $\lambda \in L$ (or $\mu \in M$) such that $\text{card } k(\lambda) \geq 2$ (or $\text{card } m(\mu) \geq 2$, respectively).

(b) We show that L_1 and M_1 are finite. Let us suppose that L_1 is infinite. Since, for each $\lambda \in L, \bigcup_{\mu \in k(\lambda)} m(\mu)$ is finite, we can find an infinite $L_2 \subset L_1$ such that $k(\lambda) \cap k(\lambda') = \emptyset$ whenever $\lambda, \lambda' \in L_2, \lambda \neq \lambda'$. For each $\lambda \in L_2$ choose different $\gamma_\lambda, \delta_\lambda$ in $k(\lambda)$. Denote by C (or D) the closure of the set $\{\gamma_\lambda; \lambda \in L_2\}$ (or $\{\delta_\lambda; \lambda \in L_2\}$) in βM . Choose $x \in \bar{L}_2 - L_2$, where \bar{L}_2 is the closure of L_2 in βL . Then necessarily $h(x) \in C$ as well as $h(x) \in D$, but $C \cap D = \emptyset$.

(c) Put $L' = L_1 \cup \bigcup_{\mu \in M_1} m(\mu)$, $M' = M_1 \cup \bigcup_{\lambda \in L_1} k(\lambda)$. Then L', M' are finite and if $\lambda \in L - L', k(\lambda)$ has precisely one element, say $\varphi(\lambda)$, and $m(\varphi(\lambda)) = \{\lambda\}$. Since h is a homeomorphism onto $p(\mathfrak{Y})$, it maps

X_λ onto $Y_{\varphi(\lambda)}$.

THEOREM 3. *Each Abelian group has a representation by sums in the category of Boolean spaces and continuous mappings.*

Proof. (a) Let n be a natural number, C_n be a cyclic group of order n , its elements are considered as $\bar{0}, \bar{1}, \dots, \overline{n-1}$, where \bar{k} is the set of all integers congruent to k modulo n . Let m be an infinite cardinal number. We construct a special representation r_m of C_n . Denote by P_m the one-point compactification of a discrete space of the cardinality m . If $k \in \{0, \dots, n-1\}$, denote by \mathfrak{X}_k the collection $\{X_\lambda; \lambda \in N\}$, where $N = \{1, 2, \dots\}$,

$$X_1 = P_m \vee P_m \vee \dots \vee P_m \text{ } k\text{-times, } (X_1 = \emptyset \text{ if } k = 0),$$

$$\lambda \neq 1 \text{ } X_\lambda = P_m \vee P_m \vee \dots \vee P_m \text{ } n\text{-times.}$$

Put $r_m(\bar{k}) = p(\mathfrak{X}_k)$. We show that r_m is really a representation. Clearly, $r_m(\bar{k}_1) \vee r_m(\bar{k}_2)$ is homeomorphic to $r_m(\overline{k_1 + k_2})$. Let us suppose that for some $0 \leq k_1 < k_2 \leq n-1$ there exists a homeomorphism h of $r_m(\bar{k}_1)$ onto $r_m(\bar{k}_2)$. We show that $h(\beta N - N) = (\beta N - N)$. In fact, the other points of $r_m(\bar{k}_1)$ and of $r_m(\bar{k}_2)$ are either isolated, or they have a neighbourhood \mathcal{Z} such that $\text{card } \mathcal{Z} = m$ and for any other neighbourhood \mathcal{V} , $\mathcal{Z} - \mathcal{V}$ is finite. None of the points of $\beta N - N$ has this property. Now, use Lemma 2.

(b) Let I be a subset of the set of all natural numbers. Let $\{C_n; n \in I\}$ be a collection of finite cyclic groups (C_n has the order n). Let C be their product. Let $M = \{m_i; i \in I\}$ be a collection of infinite cardinal numbers such that $m_i \neq m_{i'}$, whenever $i \neq i'$. We construct a special representation r_M of C . If $x = \{x_i; i \in I\}$ is a point of C denote $\mathfrak{X}_x = \{r_{m_i}(x_i); i \in I\}$. Put $r_M(x) = s(\mathfrak{X}_x)$. We show that r_M is really a representation. $r_M(x) \vee r_M(x')$ is clearly homeomorphic to $r_M(x + x')$. If $x, x' \in C$ and $x \neq x'$, there exists $i \in I$ such that $r_{m_i}(x_i)$ is not homeomorphic to $r_{m_i}(x'_i)$. Since $m_i \neq m_{i'}$, whenever $i \neq i'$, $r_M(x)$ cannot be homeomorphic to $r_M(x')$.

(c) Let p be a positive prime. We denote by R_p the additive group of all p -adic rational numbers, i.e. numbers of the form k/p^n , k an integer, n a natural number. Denote by L the set of all primes bigger than p . R_p is isomorphic to a subgroup of $\prod_{k \in L} C_k$. There exists a one-to-one homomorphism $\varphi: R_p \rightarrow \prod_{k \in L} C_k$ such that $\varphi(1) = \{1, 1, \dots\}$.

(d) Let p be a positive prime. As usual, we denote by C_{p^∞} the group R_p modulo integers, elements of C_{p^∞} are considered as \bar{x} , where x is a p -adic rational number, $0 \leq x < 1$, and \bar{x} is the set of all p -adic rational numbers congruent to x modulo integers. We construct

a special representation of C_{p^∞} . Let L, φ, M, r_M have the same meaning as in (b), (c). If $\bar{x} \in C_{p^\infty}$, $0 \leq x < 1$, put $\bar{x}_x = \{X_n; n \in N\}$, $X_1 = r_M(\varphi(x))$, $X_n = r_M(\varphi(1))$ for all $n > 1$. Put $q_M(\bar{x}) = p(\bar{x}_x)$. We show that q_M is a representation of C_{p^∞} . $q_M(\bar{x}) \vee q_M(\bar{x}')$ is homeomorphic to $q_M(\overline{x+x'})$, or to $q_M(\overline{x+x'-1})$ whenever $x+x' \geq 1$ because $r_M(\varphi(x)) \vee r_M(\varphi(x'))$ is homeomorphic to $r_M(\varphi(x+x'))$. Let \bar{x}, \bar{x}' be p -adic rational numbers, $0 \leq x < x' < 1$. Let us suppose that there exists a homeomorphism h of $q_M(\bar{x})$ onto $q_M(\bar{x}')$. Let us prove that $h(\beta N - N) = \beta N - N$. As follows from the construction of q_M , $\beta N - N$ can be characterized by neighbourhoods of points: let z be a point of $q_M(\bar{x})$ (proceed analogously for $q_M(\bar{x}')$). Then $z \in \beta N - N$ if and only if every neighbourhood \mathcal{U} of z contains a subspace of the form $\bigvee_{k \in L} P(m_k)$, where each $P(m_k)$

(1) is homeomorphic to the one-point compactification of a discrete space of power m_k ,

(2) is not contained in any subspace of $q_M(\bar{x})$, homeomorphic to the one-point compactification of a discrete space of power bigger than m_k .

Proof. Let $z \in \beta N - N$, let \mathcal{U} be its neighbourhood. Then clearly \mathcal{U} has an open-and-closed subspace, homeomorphic to $r_M(\varphi(1))$ (recall that $X_n = r_M(\varphi(1))$ with one exception). Thus, it suffices to notice that $r_M(\varphi(1))$ has a subspace with the mentioned properties. On the other hand, if $z \notin \beta N - N$, then $z \in X_n \times \{n\}$ for some n and it is easy to see, that there exists a neighbourhood of z which has no subspace of the considered form.

Therefore clearly $h(\beta N - N) = \beta N - N$. Now apply Lemma 2: there exist finite sets $A, B \subset N$ with $1 \in A \cap B$ and a bijection $\psi: N - A \rightarrow N - B$ with $h(X_n) = X_{\psi(n)}$ for all $n \in N - A$. Then $h(\bigcup_{n \in A} (X_n \times \{n\})) = \bigcup_{m \in B} (X_m \times \{m\})$, which is a contradiction, since then $r_M(\varphi(x+a-1))$ is homeomorphic to $r_M(\varphi(x'+b-1))$, where $a = \text{card } A$, $b = \text{card } B$, and so $x-x'$ is an integer.

(e) Let $\{p_i; i \in I\}$ be a collection of positive primes. For each $i \in I$ denote by L_i the set of all primes bigger than p_i . Let $\{m_{i,k}; i \in I, k \in L_i\}$ be a collection of pairwise distinct infinite cardinals. Denote $M_i = \{m_{i,k}; k \in L_i\}$. Let q_{M_i} be the representation of $C_{p_i^\infty}$ constructed in (d). We construct a representation r of $C = \prod_{i \in I} C_{p_i^\infty}$. If $\bar{x} = \{\bar{x}_i; i \in I\} \in C$, i.e. $\bar{x}_i \in C_{p_i^\infty}$ for each i , put $r(\bar{x}) = s\{\{q_{M_i}(x_i); i \in I\}\}$. The proof that r is really a representation is analogous to (b).

(f) For any Abelian group G there exists a collection $\{p_i; i \in I\}$ of primes such that G is isomorphic to a subgroup of $\prod_{i \in I} C_{p_i^\infty}$. This follows from Theorem 5.1 in [9].

II. Applications. First, since each compact Hausdorff space

induces a unique uniform and proximity structure, the above result yields: every Abelian group has a representation by sums of

- (1) uniform spaces;
- (2) proximity spaces.

Second, since the category of Boolean spaces is dual to the category of Boolean algebras, each Abelian group has a representation by products in it.

Now, let X be a compact Hausdorff space. Denote by $C(X)$ the set of all real continuous functions on X . $C(X)$ may be structured in a lot of ways, for example as a ring or a lattice etc. Thus, C may be considered as a contravariant functor of the category of compact Hausdorff spaces into the category of rings or lattices and so on. We can apply the result about representation of any Abelian groups by sums of Boolean spaces and obtain results about representation by products if $C(X)$ is structured such that

- (a) C turns sums into products;
- (b) if X and X' are nonhomeomorphic, then $C(X)$ and $C(X')$ are nonisomorphic.

The condition (a) is usually satisfied evidently; the condition (b) are often classical theorems.

We show some examples:

(1) commutative rings (with unit), (a) is satisfied, evidently; (b) is the classical Gelfand-Kolmogoroff theorem;

(2) commutative monoids. Here we consider the multiplicative semigroup of the real continuous functions, (a) is evident, (b) see [8];

(3) lattices, (a) is evident, (b) follows from the Kaplansky theorem;

(4) linear lattices (and linear homomorphism), (a) is evident, (b) follows also from the Kaplansky theorem;

(5) Banach spaces. Here, we consider linear contractions as morphisms (i.e. bounded linear operators with norm ≤ 1). Then (a) is evident, (b) follows from the classical Banach-Stone theorem;

(6) Banach algebras (morphisms are bounded homomorphisms with norm ≤ 1). (a) is evident, (b) follows also from the Banach-Stone theorem.

On the other hand (b) is not satisfied if we consider $C(X)$ only as a linear space. It is very easy to verify that no nontrivial group has a representation by products in the category of linear spaces.

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