PACIFIC JOURNAL OF MATHEMATICS
Vol. 61, No. 2, 1975

AN OBSTRUCTION TO LIFTING CYCLIC MODULES

MELVIN HOCHSTER

Let A be a ring (all ring are commutative, with identity), let $t \in A$ be a nonzerodivisor and not a unit, and let $B = A/At$. Let M be a B-module of finite type. We call an A-module E of finite type a *lifting* of M (or we say that "E lifts M" or "M lifts to E") if (1) t is not a zerodivisor on E and (2) $E/tE \cong M$. Grothendieck's lifting problem (GLP) is this: Suppose that (A, m) is a complete regular local ring and that $t \in m - m^2$, so that $B = A/tA$ is again regular. If M is a B-module of finite type, does M lift to A? A simple and completely elementary counterexample is given below for the case where M is cyclic.

One of the motivations for studying GLP is the possibility of settling Serre's conjecture [9] p. V-14 on multiplicities using lifting: this idea is discussed in [1], [2], and [3], where affirmative answers to the question are obtained in some cases. The terminology "Grothendieck's lifting problem" is taken from these papers. The first published discussion of the relevance of the lifting problem to multiplicities of which I am aware is in Nastold's paper [6], which was brought to my attention by the referee. (Of course, if A is *equicharacteristic* complete regular and $t \in m - m^2$ one can always lift, for then $A \cong B[[t]]$.)

In the general case, it seems that one can construct counterexamples by completing the example [8] of Serre of an unliftable variety along lines suggested by Laudal and Kleppe, but the example in [8] is complicated, hard to write down, and, so far as I know, the details of the proof that one can complete have not appeared.

In this note we give a different counterexample to lifting even cyclic modules in the context of GLP. (Later, we also propose a weaker lifting conjecture which would suffice for the multiplicities problem and to which I know no counterexamples.) The method for constructing counterexamples is totally elementary, and provides examples when B is a complete regular local ring of mixed characteristic as well as when B has positive characteristic. The obstruction to lifting we use is quite coarse: it comes out of the all but trivial (and, I assume, well known) lemma below. Much of the credit for focusing attention on this obstruction belongs to D. Ferrand.

Before stating the lemma we make some special conventions for the cyclic case. Suppose that (A, m) is a local ring and we wish to lift $M = A/J$, where $t \in J \subsetneq A$, a typical cyclic B-module, to A. By

Nakayama's lemma the lifting E must be a cyclic A-module A/I, and it is evident that the problem of lifting J is equivalent to the problem of finding an ideal $I \subset J$ such that $J = I + At$ and t is not a zerodivisor mod. I. Thus, if A is any ring, t a nonzerodivisor, $t \in J \subsetneqq A$, we call I a t-lifting (or, simply, a lifting) of J, etc. if $J = I + At$ and t is not a zerodivisor mod. I.

Let $C = A/J$ and let x be an indeterminate. We shall denote by ϕ the map of graded C-algebras $C[x] \longrightarrow gr_J A$ which maps x to the class $t + J^2$ of t in J/J^2.

LEMMA. *Let A be a ring, t a nonzerodivisor, and suppose $t \in J \subsetneqq A$. Let $C = A/J$ and ϕ be as above. If J has a t-lifting then there is a map $\psi\colon gr_J A \longrightarrow C[x]$ of graded C-algebras such that $\psi\phi = \mathrm{id}_{C[x]}$. In particular*:

(1) *ϕ is injective.*

(2) *For each n the map of C-modules $C \longrightarrow J^n/J^{n+1}$ which takes the class of 1 to the class of t^n embeds C in J^n/J^{n+1} as a direct summand.*

(3) *For every positive integer n, $J^{n+1}\colon At^n = J$.*

COROLLARY. *With the hypothesis of the Lemma, if J has a t-lifting then the map $h\colon C = A/J \longrightarrow J/J^2$ which takes $1 + J$ to $t + J^2$ embeds A/J as a direct summand of J/J^2. In particular, it is injective, and, hence, $J^2\colon At = J$.*

Proof. To construct ψ we give explicitly its n^{th} graded piece $\psi_n\colon J^n/J^{n+1} \longrightarrow A/J$ (we identity A/J with $(A/J)x^n$, by sending $1 \mapsto x^n$). We want $\psi_n(t^n + J^{n+1}) = 1 + J$. Now $J^n = (I + At)^n \subset I + At^n$. We have a composite map: $J^n \hookrightarrow \mathrm{I} + At^n \twoheadrightarrow (I + At^n)/I \cong (A/I)t^n \overset{\alpha}{\cong} A/I \twoheadrightarrow A/J \cong (A/J)x^n$ where the maps are the obvious inclusions, quotient maps, and identifications, except for α: α is the inverse of the map $A/I \longrightarrow (A/I)t^n$ which sends $1 \mapsto t^n$, which is an isomorphism precisely because t is not a zerodivisor modulo I. This $\alpha(t^n) = 1$, and the composite map $J^n \longrightarrow A/J$ sends t^n to $1 + J$. We let ψ_n be the induced map $J^n/J^{n+1} \longrightarrow A/J \cong (A/J)x^n$. It is clear that ψ_n is a C-module retraction. The ψ_n fit together to give a homomorphism of graded algebras, for if $u \in J^m$, $v \in J^n$, $u + J^{m+1}$ maps to $a + J$ in A/J, and $v \in J^{n+1}$ maps to $b + J$ in A/J, then $uv \in J^{m+n}$ represents an element of J^{m+n}/J^{m+n+1}; moreover, if $u = i_1 + (a + j_1)t^m$ and $v = i_2 + (b + j_2)t^n$, where $i_1, i_2 \in I$, $j_1, j_2 \in J$ then $uv = i_3 + (ab + j_3)t^{m+n}$, clearly, where $i_3 \in I$, $j_3 \in J$, whence $\psi_{m+n}(uv + J^{m+n+1}) = \psi_m(u + J^{m+1})\psi_n(v + J^{n+1})$.

Now, (1) and (2) are clear, while (3) is readily seen to be a restatement of the fact that ϕ_n is injective. The Corollary is just the special case $n = 1$.

EXAMPLE 1. We now observe that even the condition $J^2: At = J$ obstructs lifting in the context of GLP. Let V be a complete discrete valuation ring with maximal ideal generated by 2, let $K = V/2V$, let x_{ij}, $1 \leqq i \leqq 3$, $1 \leqq j \leqq 2$ be six analytic indeterminates over V, let $A = V[[X_{i,j}]]_{i,j}$, let $t = 2$, let $q = \sum_{i=1}^{3} x_{i1}x_{i2}$ and let $J = (2, q, \{x_{i,j}^2\})_{i,j}A$. Then J has no t-lifting I: in fact $J^2: 2A \not\subset J$. To see this let $D = \sum_{i=1}^{3} x_{i1}x_{i2}x_{i+1,1}x_{i+1,2}$ (where the first subscript is taken mod. 3). Then $2D = q^2 - \sum_{i=1}^{3}(x_{i1}^2)(x_{i2}^2) \in J^2$ and $D \in J^2: 2A$, but $D \notin J$. To prove that $D \notin J$ we work mod. $H = (2, \{x_{i,j}^2\}_{i,j})A$. Let $'$ denote residues mod. H. A/H is a graded finite dimensional K-algebra and any degree four element of $J(A/H) = q'(A/H)$ is in the K-span of the products $q' \cdot (E' \cup F')$ where $E = \{x_{i1}x_{i2}: 1 \leqq i \leqq 3\}$ and $F = \{x_{ij}x_{i^*j^*}: 1 \leqq i < i^* \leqq 3, 1 \leqq j \leqq 2, 1 \leqq j^* \leqq 2\}$. ($E' \cup F'$ gives all square-free terms.) Each element of $q' \cdot E'$ is the sum of two of the three monomials in D' while if $u = x_{ij}x_{i^*j^*} \in F$, then, mod. H, $u \equiv x_{ij}x_{i^*j^*}x_{i''1}x_{i''2'}$ where i'' is the unique element of $\{1, 2, 3\} - \{i, i^*\}$, and these twelve distinct monomials do not occur in D' at all. Thus, D' is not in the K-span of $q' \cdot (E' \cup E')$ and $D \notin J$.

EXAMPLE 2. Let V, K, A, q, J and D be precisely the same as in Example 1, and let $u \in J^2$. Let $t_u = 2 + u$. Then $Dt_u = 2D + uD \in J^2$, and $D \notin J$. Thus, we have $J^2: At_u \not\subset J$ again, but now $B = A/At_u$ is a regular local ring of mixed characteristic (rather than characteristic 2) if we choose u properly, e.g. $u = x_{11}^4 + x_{12}^2 + x_{11}^2$, and the ramification of 2 cannot be absorbed into the coefficient ring of B, i.e. B is precisely the kind of regular local ring for which Serre's conjecture is not known.

REMARK 1. Similar examples undoubtedly exist for every positive prime characteristic p.

REMARK 2. One can consider more general lifting problems, which were open for a while. Let $A \to B$ be a homomorphism, and let M be a B-module of finite type. Call an A-module E of finite type a *lifting* of M if (1) $\mathrm{Tor}_i^A(B, E) = 0$ if $i \geqq 1$ and (2) $B \otimes_A E \cong M$. Suppose that $B = A/At$, where t is a nonzerodivisor. Then condition 1) is equivalent to the assertion that t is not a zerodivisor on B. One can ask, if $pd_B M < \infty$, does M lift to A? The technique of Peskine-Szpiro [7], Ch. I, §2 gives counterexamples even if (A, m) is a complete regular local ring of equicharacteristic 0, $B = A/At$, and $pd_B M = 2$. However, in their examples $t \in m^2$ and B is not regular: in fact, their obstruction is that a certain B-module has infinite projective dimension when it should have finite projective dimension, and so cannot provide examples with B regular.

REMARK 3. The obstruction indicated in the Corollary, i.e. that the map $h: A/J \to J/J^2$ which takes $1 + J$ to $t + J^2$ split, is precisely the obstruction to lifting A/J "as far as" A/t^2A. For if $s: J/J^2 \to A/J$ is a splitting (left inverse for h) then if $I_1 = \mathrm{Ker}\,(J/J^2 \xrightarrow{s} A/J)$, A/I_1 "lifts" A/J "as far as" A/t^2A. If one has a good criterion for lifting from "mod. t" to "mod. t^2", then one can iterate it to lift, successively, mod. t^4, mod. t^8, etc., and, utlimately, given t-adic completeness, back to the original ring. This idea motivated some of the more general lifting conjectures.

REMARK 4. It is clear that in some sense the problem with lifting from B to A in the context of GLP is p-torsion: in the examples here, even when $p \neq 0$ in the regular ring $B = A/At$, there is p-torsion on A/J. I believe it would be too naive to conjecture that simply because there is no p-torsion on A/J that one can lift; on the other hand, I feel there is almost certainly a good result along the following lines: "If there is no p-torsion on the following modules $\cdots$ (canonically associated with J, t), then J lifts".

For example, we have the following:

PROPOSITION. *Let (A, m) be a regular local ring of mixed characteristic, let $t \in m - m^2$, suppose $t \in J \subsetneqq A$, and let $C = A/J$. Let $p = \mathrm{char}\,(A/m)$. If p is not a zerodivisor on C and also not a zerodivisor on $\mathrm{Ext}^1_C\,(J/(At + J^2), C) = E$, then the map $h: C \to J/J^2$ which takes $1 + J$ to $t + J^2$ splits.*

Proof. If h does not split them either (i) h is not injective or (ii) h is injective and the exact sequence

$$0 \longrightarrow C \xrightarrow{h} J/J^2 \longrightarrow J/(At + J^2) \longrightarrow 0$$

is not split, i.e. represents a nonzero element of $E = \mathrm{Ext}^1_C(J/(At+J^2), C)$. Since p is not a zerodivisor on C, E, both situations (i) and (ii) are preserved upon localizing at p: hence there is a prime $Q \supset J$ such that $p \notin Q$ and the splitting fails after localizing at Q (or, equivalently, at Q/J). But then if we localize at Q and complete, the splitting will still fail. This is impossible: $\hat{A}_Q$ is complete and equicharacteristic 0, so that $J\hat{A}_Q$ *does* have a t-lifting.

For the rest of this note we restrict our attention to the following situation:

(*) Let V be a complete discrete valuation ring with residue class field K of characteristic $p > 0$, and suppose $0 \neq p$ in V generates the maximal ideal. Let A be $V[[x_1, \cdots, x_n]]$ and let t be a regular parameter in A such that $p \neq 0$ in the regular local ring $B = A/At$. Let $t \in J \subsetneqq A$; and suppose that J is *prime*.

We want to make some observations about this situation.

Observation 1. In order to prove Serre's conjecture on multiplicities it would suffice to prove that J has a t-lifting in this situation. To emphasize that J is prime, we write "P" instead of "J" i.e. $P = J$.

Observation 2. Under the hypothesis (*) $h: A/J \rightarrow J/J^2$ (or $A/P \rightarrow P/P^2$) is automatically injective, i.e. $p^2: At = P$. For t is a regular parameter $\Rightarrow A/tA$ is regular $\Rightarrow (A/tA)_{P/tA} \cong A_P/tA_P$ is regular $\Rightarrow t$ is a regular parameter in the regular local ring $A_P \Rightarrow P^2A_P: tA_P = PA_P$ and $P^2: At \subset (P^2A_P: tA_P) \cap A = PA_P \cap A = P$. However it is completely unclear why there should be any reason for h to split.

Observation 3. Under the hypothesis (*), if $h: A/P \rightarrow P/P^2$ splits, then the map $\phi: (A/P)[x] \rightarrow gr_P A$ of graded C-algebras (where $C = A/P$) splits. For if s splits (is a left inverse for) h, s induces a graded C-algebra map $\varepsilon: \operatorname{symm}_C P/P^2 \rightarrow \operatorname{symm}_C C \cong C[x]$. If we localize at P, since A_P is regular we have an isomorphism: $\operatorname{symm}_C (P/P^2)_P \xrightarrow{\beta} gr_{PC_P} A_P$ (both are polynomial rings in height P variables over the field $C_P = (A/P)_{P'}$ and the map is induced by an isomorphism of their first graded pieces). Now, β is induced by localization from the natural map $\delta: \operatorname{symm}_C (P/P^2) \rightarrow gr_P A$ (which in turn is induced from the inclusion of P/P^2 as the one-forms of $gr_P A$). Hence, we have a commutative diagram:

$$\begin{tikzcd} C[x] \arrow[d, hook, "\delta"] & \operatorname{symm}_C (P/P^2) \arrow[l, two heads, "\varepsilon"'] \arrow[r, two heads, "\delta"] \arrow[d] & gr_P A \arrow[d, "\alpha"] \\ C_P[x] & \operatorname{symm}_C (P/P^2)_P \arrow[l, "\gamma"] \arrow[r, "\cong"', "\beta"'] & gr_{PC_P} A_P . \end{tikzcd}$$

Since β is an isomorphism, we have a homomorphism $\psi = \gamma\beta^{-1}\alpha: gr_P A \rightarrow C_P[x]$. Since δ is surjective, $\operatorname{Im} \psi = \operatorname{Im} \psi\delta = \operatorname{Im} \delta\varepsilon = C[x] \subset C_P[x]$. Thus, ψ, with its range restricted, is the (unique) graded C-algebra homomorphism which makes the diagram

$$\begin{tikzcd} \operatorname{symm}_C (P/P^2) \arrow[r, two heads, "\delta"] \arrow[dr, dashed, "\varepsilon"'] & gr_P A \arrow[d, dashed, "\psi"] \\ & C[x] \end{tikzcd}$$

commute, and it follows from the definition of ε that ψ is a left inverse for ϕ.

This shows that under the hypothesis (*), the "main case", the entire obstruction to lifting presented by the Lemma is no worse than the obstruction presented by the Corollary. The graded algebra map splits if and only if h splits.

However, it appears that even if the graded algebra map splits, we have merely taken a feeble first step towards lifting J.

Observation 4. It is useful to put the obstruction to splitting given by the Corollary in a more concrete form. We therefore note:

PROPOSITION. *Under the hypothesis* (*) (so that $J = P$ is prime), *the map* $h\colon C = A/J \to J/J^2$ *splits if and only if for every* (irreducible) *A-ideal* $L \supset J$ (primary to the maximal ideal m of A),

$$LJ\colon At = L\ .$$

Either or both of the parenthetical phrases may be omitted without affecting the validity of this statement.

In fact, $A/J \to J/J^2$ makes A/J a direct summand if and only if A/J is a pure submodule, and since A/J is a complete local domain, this holds if and only if for every ideal L/J of A/J, $h^{-1}(L(J/J^2)) = (A/J)$ (i.e. $A/J \to J/J^2$ is cyclically pure: see [4]), and it suffices to know this for (m/J)-primary irreducible ideals L/J (again, see [4]). But $h^{-1}(L(J/J^2)) = L(A/J)$ if and only if $LJ\colon At = L$.

We conclude with the following point: Serre's conjecture on multiplicities over a regular local ring $B = A/At$ is known except in the situation of (*), and we may further suppose that t is an Eisenstein polynomial. To deduce the result for B from the result for A, it would suffice to know the following:

Weak lifting conjecture. Under the hypothesis (*), *let Q be a prime of B such that $p \notin Q$. Then some B-module M whose support is defined by Q lifts to A.*

Proof that the weak lifting conjecture implies Serre's conjecture. It suffices to show (cf. [8] Ch. V, §4) that for each pair (Q, Q_1) of primes of B with $Q + Q_1$ of coheight 0, $e_B(B/Q, B/Q_1) \geqq 0$, with inequality $\Leftrightarrow \dim(B/Q) + \dim(B/Q_1) = \dim B$. (Here $e_B(M, N) = \sum_{i=0}^{\dim B}(-1)^i l(\mathrm{Tor}_i^B(M, N))$.) Choose a counterexample with $\dim(B/Q) + \dim(B/Q_1)$ as small as possible. If $p \in Q$ and $p \in Q_1$, then the result is known ([5], Prop. 4). Now suppose, say, $p \notin Q$. Choose M such that Supp $M = \{P \in \mathrm{Spec}(B): P \supset Q\}$ and M has a lifting E to A. Then $\dim A - \dim E - \dim(B/Q_1) = \dim B - \dim M - \dim(B/Q_1)$ and $\dim M = \dim(B/Q)$. Thus $e_A(E, B/Q_1) \geqq 0$ with inequality if and only if $\dim(B/Q) + \dim(B/Q_1) = \dim B$. Each $\mathrm{Tor}_i^A(E, B/Q_1)$ is isomorphic with $\mathrm{Tor}_i^B(M, B/Q_1)$, and M has a prime filtration involving a positive number λ of copies of B/Q and certain other primes Q_j^* such that $Q_j^* \supsetneqq Q$. Hence $e_A(E, B/Q_1) = e_B(M, B/Q_1) =$ (for suitable $\mu_j)\lambda e_B(B/Q, B/Q_1) + \sum_j \mu_j e_B(B/Q_j^*, B/Q_1) = \lambda e_B(B/Q, B/Q_1)$ because $\dim(B/Q_j^*) + \dim(B/Q_1) < \dim(B/Q) + \dim(B/Q_1)$ (so that the result holds for (Q_j^*, Q_1), and $\dim(B/Q_j^*) + \dim(B/Q_1) < \dim B \Rightarrow e_B(B/Q_j^*, B/Q_j) = 0$ for each j). Thus $e_B(B/Q, B/Q_1) = (1/\lambda)e_A(E, B/Q_1) \geqq 0$, with inequality if and only if $\dim(B/Q) + \dim(B/Q_1) = \dim B$.

I know no counterexamples to this weak lifting conjecture.

References

1. D. A. Buchsbaum and D. Eisenbud, *Lifting modules and a theorem on finite free resolutions*, Ring Theory, Academic Press, New York, (1972), 63–74.
2. ———, *Remarks on ideals and resolutions*, Symposia Math., **XI** (1973), 193–204.
3. ———, *Some structure theorems for finite free resolutions*, Advances in Math., **12** (1974), 84–139.
4. M. Hochster, *Cyclic purity versus purity in excellent Noetherian rings*, preprint.
5. Mme M.-P. Malliavin-Brameret, *Structure d'anneaux réguliers et caractéristiques d'Euler-Poincaré*, *Bull.* Sc. Math. 2e *série*, **97** (1973), 81–88.
6. H.-J. Nastold, *Zur Serreschen Multiplizitätstheorie in der arithmetischen Geometrie*, Math. Annalen, **143** (1961), 333–343.
7. C. Peskine and L. Szpiro, *Dimension projective finite et cohomologie locale*, Publ. I. H. E. S. Paris, No. **42**, (1973), 323–395.
8. J.-P. Serre, *Exemples de variétés projectives en caractéristique p non relèvables en caractéristique zéro*, Proc. Nat. Acad. Sci., **47** (1961), 108–109.
9. ———, *Algèbre locale. Multiplicités*, Springer-Verlag Lecture Notes in Math. No. 11, New York, 1965.

Received May 13, 1975. Research supported in part by a grant from the National Science Foundation.

PURDUE UNIVERSITY