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AVERAGING STRONGLY SUBADDITIVE SET FUNCTIONS IN UNIMODULAR AMENABLE GROUPS I

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Kieffer has considered the problem of averaging strongly subadditive, nonpositive, right invariant set functions S defined on the class $\mathscr{K}$ of precompact Borel subsets of a locally compact (unimodular) amenable group G as a means of defining entropy in a abstract probabilistic context. He shows if $\{A_\alpha\}$ is a net in $\mathscr{K}$ satisfying an appropriate growth condition then $\lambda(A_\alpha)^{-1}S(A_\alpha)$ has a limit depending only on S, where λ is right Haar measure on G. Here we prove a somewhat stronger result of the same type based on a Fundamental Inequality valid in any locally compact group G and for any set function S as described above:

$$\lambda(A)^{-1}S(KA) \leqq \lambda(K^{-1})^{-1}S(K)$$

for all sets A in $\mathscr{K}$ of positive measure and all open sets K in $\mathscr{K}$ which satisfy $\lambda(K) = \lambda(\bar{K})$, the so-called open continuity sets in $\mathscr{K}$.

This inequality when considered in a unimodular, amenable group G then yields the existence of the limit in question by standard amenable group arguments; this inequality also allows us to give a much more explicit formula for its value:

$$\inf\{\lambda(K)^{-1}S(K): K \text{ an open continuity set in } \mathscr{K}\}\,.$$

1. Definitions, notation, and other relevancies. Throughout the topological group G is assumed locally compact, noncompact, and equipped with a fixed right invariant Haar measure λ. Also $\mathscr{K} = \mathscr{K}(G)$ is the set of all precompact Borel subsets of G, $\mathscr{K}_0$ consists of all open sets in $\mathscr{K}$, and $\mathscr{K}_+$ consists of all sets in $\mathscr{K}$ of positive measure. Standard notation is generally in force, e.g. $\bar{A}$, A', A^0 represent the closure, complement, and interior of A (in G) respectively.

DEFINITION 1.1. For any subsets A and K of G,

$$[A]_K = \{g \in G: Kg \subseteq A\}\,.$$

DEFINITION 1.2. Let $\{A_\alpha\}$ be a net of sets from $\mathscr{K}_+$. Then $\{A_\alpha\}$ is:

(i) regular, $\{A_\alpha\} \in \mathscr{R}$, iff $\lim_\alpha \lambda(KA_\alpha)^{-1}\lambda([A_\alpha]_K) = 1$ for all $K \in \mathscr{K}_0$,

(ii) admissible, $\{A_\alpha\} \in \mathscr{A}$, iff $\lim_\alpha \lambda(\bar{A}_\alpha)^{-1}\lambda([A_\alpha]_K) = 1$ for all $K \in \mathscr{K}_0$,

(iii) full, $\{A_\alpha\} \in \mathscr{F}$, iff $\lim_\alpha \lambda(A_\alpha)^{-1}\lambda([A_\alpha]_K) = 1$ for all $K \in \mathscr{K}_0$,

(iv) almost closed, $\{A_\alpha\} \in \mathscr{C}$, iff $\lim_\alpha \lambda(\bar{A}_\alpha)^{-1}\lambda(A_\alpha) = 1$.

Note. The notation $[A]_K$ and the definition of $\mathscr{R}$ is essentially that of Kieffer [5]. It is trivial to verify that

$$[A]_K = \bigcap \{k^{-1}A \colon k \in K\} .$$

Observe also that the family $\mathscr{K}_0$ of (i)-(iii) could equivalently be replaced by any subfamily of $\mathscr{K}$ which is cofinal under inclusion. Finally, in case a set $B \subseteqq G$ is not Haar measurable, e.g. some $[A]_K$, $\lambda(B)$ denotes inner Haar measure. We next give some basic properties of these nets.

PROPOSITION 1.3.

(i) *$\mathscr{R} \neq \emptyset$ iff $\mathscr{A} \neq \emptyset$ iff $\mathscr{F} \neq \emptyset$ iff G is unimodular and amenable,*

(ii) *If G is discrete $\mathscr{R} = \mathscr{A} = \mathscr{F}$, otherwise $\mathscr{R} \subset \mathscr{A} \subset \mathscr{F}$ with proper inclusions* (if $\mathscr{F} \neq \emptyset$),

(iii) $\{A_\alpha\} \in \mathscr{R}, \mathscr{A}, \mathscr{F}$ *implies* $\{A_\alpha^0\} \in \mathscr{R}, \mathscr{A}, \mathscr{F}$ *respectively,*

(iv) $\mathscr{A} = \mathscr{F} \cap \mathscr{C}$,

(v) $\{A_\alpha\} \in \mathscr{A}$ *and* $\{B_\alpha\}$ *subsets of* $\mathscr{K}$ *such that* $\lambda(\bar{A}_\alpha)^{-1}\lambda(\bar{B}_\alpha\} \to 0$ *implies* $\{C_\alpha\} \in \mathscr{A}$ *where* $C_\alpha = A_\alpha \cup B_\alpha$,

(vi) $\{A_\alpha\} \in \mathscr{F}$ *and* $\{B_\alpha\}$ *subsets of* $\mathscr{K}$ *such that* $\lambda(A_\alpha)^{-1}\lambda(B_\alpha) \to 0$ *implies* $\{C_\alpha\} \in \mathscr{F}$ *where* $C_\alpha = A_\alpha \cup B_\alpha$.

Note. (v) and (vi) show that $\mathscr{A}$ and $\mathscr{F}$ are closed under "small" enlargement as indicated whereas $\mathscr{R}$ is much more sensitive and apparently does not admit any obvious perturbations (for nondiscrete G).

Proof. For $K \in \mathscr{K}_0$, $Kg \subseteqq A$ iff $Kg \subseteqq A^0$ and thus $[A]_K = [A^0]_K$ and (iii) follows. Assume moreover that $K \in \mathscr{K}_0$ always contains the identity e of G (such sets are cofinal) to guarantee that $[A]_K \subseteqq A$ and upon writing

$$\lambda(\bar{A})^{-1}\lambda([A]_K) = (\lambda(\bar{A})^{-1}\lambda(A))(\lambda(A)^{-1}\lambda([A]_K)) ,$$

where both factors on the right are $\leqq 1$, (iv) follows. Next (v) and (vi) follow readily from the inclusion

$$[A]_K \subseteqq [A \cup B]_K \text{ and } \lambda(\bar{C}_\alpha)^{-1}\lambda(\bar{A}_\alpha) \to 1 \text{ and } \lambda(C_\alpha)^{-1}\lambda(A_\alpha) \to 1 ,$$

respectively. Turning to (ii), equality for discrete G follows from known arguments, e.g. [2, Theorem 1], and the inclusions are clear in general since $A_\alpha \subseteqq \bar{A}_\alpha \subseteqq KA_\alpha$ if $e \in K$ (which we may always assume by the note following Definition 1.2). On the other hand to see that they are proper we utilize (vi) and (v). First, we observe that any nondiscrete locally compact σ-compact group G contains a Borel null set N such that $\bar{N} = G$ (the same is true for arbitrary G if one replaces null by locally null):

For let K be any compact normal subgroup of G with $\lambda(K) = 0$ and for which G/K is second countable (e.g. as follows from [4, Theorem A.9] upon choosing $\lambda(W_i) \to 0$) and let C be any countable dense subset of G/K. Then the full inverse image of C in G, say N, is the union of a countable family of cosets of K and is clearly a Borel null set in G with $\bar{N} = G$.

Now fix any $\{A_\alpha\} \in \mathscr{F}$ and let N be a Borel null set for which $\lambda(\bar{N}) = +\infty$ (if G is not σ-compact, let $G_1 \subseteqq G$ be a σ-compact, non-compact, open subgroup of G and choose N as above with $\bar{N} = G_1$). Then for any $N_\alpha \subseteqq N$ and $N_\alpha \in \mathscr{K}$, $\{C_\alpha\} \in \mathscr{F}$ where $C_\alpha = A_\alpha \cup N_\alpha$ by (vi) (taking $B_\alpha = N_\alpha$). But $\bar{N}_\alpha \subseteqq \bar{C}_\alpha$ which may be chosen to satisfy $\lambda(\bar{N}_\alpha) > 2\lambda(A_\alpha)$ (since $\lambda(\bar{N}) = +\infty$), and for such a choice of $\{N_\alpha\}$ the net $\{C_\alpha\} \notin \mathscr{A}$. Similarly (in light of (i)), let $\{A_\alpha\} \in \mathscr{A}$. Then trivially by (v), if F_α is chosen to be any finite subset of G for each α we have $\{C_\alpha\} \in \mathscr{A}$ where $C_\alpha = A_\alpha \cup F_\alpha$. Next fix any $K_0 \in \mathscr{K}_0$ and choose F_α to satisfy $\lambda(K_0 F_\alpha) > 2\lambda(A_\alpha)$ additionally, which is possible since G is not compact. Then clearly the net $\{C_\alpha\} \notin \mathscr{R}$ since it violates the defining property for $K = K_0$. Finally to prove (i): that G unimodular and amenable implies $\mathscr{R} \neq \varnothing$ is straightforward and shown in [5]. On the other hand, since $\mathscr{R} \subseteqq \mathscr{A} \subseteqq \mathscr{F}$ trivially, it suffices to show $\mathscr{F} \neq \varnothing$ implies G unimodular and amenable to complete the proof of the equivalence. If G were not unimodular, let $K \in \mathscr{K}_0$ be chosen such that $\Delta(k) \geqq 2$ for some $k \in K$, where Δ is the modular function of G. Then since $K[A]_K \subseteqq A$ for any $A \subseteqq G$ we have

$$\lambda(A) \geqq \lambda(K[A]_K) \geqq \lambda(k[A]_K) = \Delta(k)\lambda([A]_K) \geqq 2\lambda([A]_K)\,,$$

showing that $\mathscr{F} = \varnothing$. Therefore G is unimodular, and if $\{A_\alpha\} \in \mathscr{F} \neq \varnothing$ and $K \in \mathscr{K}_0$ we have

$$\lambda(K[A_\alpha]_K)^{-1}\lambda([A_\alpha]_K) \longrightarrow 1 \text{ since } K[A_\alpha]_K \subseteqq A_\alpha\,,$$

and consequently by inversion invariance in G,

$$\lambda([A_\alpha]_K^{-1}K^{-1})^{-1}\lambda([A_\alpha]_K^{-1}) \longrightarrow 1$$

and the net $\{[A_\alpha]_K^{-1}\}$ eventually satisfies condition (A) of [3] for any $\varepsilon > 0$ and K^{-1} which is equivalent to amenability.

We now define and investigate a special family of subsets of G

which is crucial to our analysis.

DEFINITION 1.4. $\mathscr{K}_c = \{K \in \mathscr{K} : \lambda(\bar{K}) = \lambda(K)\}$ is called the class of (precompact) weak continuity sets of G.

Note. This terminology is so chosen because a set $K \in \mathscr{K}$ satisfying the stronger condition $\lambda(\bar{K}) = \lambda(K^0)$ is called a continuity set of G [6, p. 174], i.e. iff χ_K is Riemann integrable. Of course for $K \in \mathscr{K}_0$ the conditions coincide. Also observe $K \in \mathscr{K}_c$ iff $K^{-1} \in \mathscr{K}_c$.

The following result gives an equivalent characterization of $\mathscr{K}_c$ which is used in the sequel.

LEMMA 1.5. *$K \in \mathscr{K}_c$ iff for every $\varepsilon > 0$ there corresponds a symmetric open neighborhood $O = O(\varepsilon)$ of e such that $\lambda(O^2K^{-1}) - \lambda(K^{-1}) < \varepsilon$.*

Proof. $K \in \mathscr{K}_0$ iff $K^{-1} \in \mathscr{K}_c$ iff $\lambda(\overline{K^{-1}}) = \lambda(K^{-1})$. Now for any neighborhood O of e we have $\overline{K^{-1}} \subseteqq O^2K^{-1}$ and consequently

$$\lambda(\overline{K^{-1}}) - \lambda(K^{-1}) \leqq \lambda(O^2K^{-1}) - \lambda(K^{-1}) < \varepsilon$$

implying $\lambda(\overline{K^{-1}}) = \lambda(K^{-1})$, i.e. $K^{-1} \in \mathscr{K}_c$, if the second condition holds for every $\varepsilon > 0$. Conversely, assume $K \in \mathscr{K}_c$. By the regularity of λ, given any $\varepsilon > 0$ there corresponds an open $U = U(\varepsilon) \supseteqq \overline{K^{-1}}$ such that $\lambda(U) - \lambda(\overline{K^{-1}}) < \varepsilon$. Now U' is closed and $\bar{K}$ compact implying $U'\bar{K}$ is closed in G. Moreover, since $\bar{K}^{-1} = \overline{K^{-1}} \subseteqq U$, we must have $e \notin U'\bar{K}$ and thus by the continuity of group operations there exists an open symmetric neighborhood O of e such that O^2 is disjoint from $U'\bar{K}$, i.e. $O^2\bar{K}^{-1} = O^2\overline{K^{-1}} \subseteqq U$, and consequently

$$\begin{aligned}\lambda(O^2K^{-1}) - \lambda(K^{-1}) &= \lambda(O^2K^{-1}) - \lambda(\overline{K^{-1}}) \text{ (since } K^{-1} \in \mathscr{K}_c)\\ &\leqq \lambda(O^2\overline{K^{-1}}) - \lambda(\overline{K^{-1}}) \leqq \lambda(U) - \lambda(\overline{K^{-1}}) < \varepsilon\end{aligned}$$

as was to be shown.

The following results show how pervasive the open (weak) continuity sets are in $\mathscr{K}_0$.

LEMMA 1.6. *If $C \subseteqq U$ are subsets of G with C compact and $U \in \mathscr{K}_0$ then there is an $O \in \mathscr{K}_0 \cap \mathscr{K}_c$ such that $C \subseteqq O \subseteqq \bar{O} \subseteqq U$.*

Proof. Since G is normal as a topological space, by Urysohn's lemma there is a continuous $f: G \to [0, 1]$ such that $f(C) = 0$ and $f(U') = 1$. Now for each t in $(0, 1)$ set $O_t \doteqdot f^{-1}([0, t))$ and observe that O_t is always open and $C \subseteqq O_t \subseteqq \bar{O}_t \subseteqq U$ since $\bar{O}_t \subseteqq f^{-1}([0, t])$. But $f^{-1}([0, t]) = O_t \cup f^{-1}(\{t\})$, and since

$$\bigcup \{f^{-1}(\{t\}) \colon t \in (0, 1)\} \subseteqq U \text{ and } \lambda(U) < +\infty ,$$

the disjointness of the $f^{-1}(\{t\})$ implies $\lambda(f^{-1}\{t\}) = 0$ for all but at most countably many $t \in (0, 1)$. Any such nonexceptional $t \in (0, 1)$ yields an $O_t \in \mathscr{K}_c$ as desired.

COROLLARY 1.7. *For any* $K \in \mathscr{K}$,
(i) $\sup \{\lambda(O) \colon \bar{O} \subseteqq K, O \in \mathscr{K}_0 \cap \mathscr{K}_c\} = \lambda(K^0)$,
(ii) $\inf \{\lambda(O) \colon K \subseteqq O, O \in \mathscr{K}_0 \cap \mathscr{K}_c\} = \lambda(\bar{K})$.

Note. (ii) says that $K \in \mathscr{K}$ is a weak continuity set iff K may be covered arbitrarily close in measure by open (weak) continuity sets.

Proof. $O \in \mathscr{K}_0$ and $\bar{O} \subseteqq K$ imply $O \subseteqq K^0$ and $\lambda(O) \leqq \lambda(K^0)$ so $\leqq$ is clear in (i). Also $K \subseteqq O$ implies $\bar{K} \subseteqq \bar{O}$, and since $O \in \mathscr{K}_c$ we have $\lambda(\bar{K}) \leqq \lambda(\bar{O}) = \lambda(O)$ so $\geqq$ is clear in (ii). Conversely, given any $\varepsilon > 0$ the regularity of λ yields a compact $C \subseteqq K^0$ satisfying $\lambda(K^0) - \varepsilon < \lambda(C)$. But by Lemma 1.6 (with $U = K^0$) we may find an $O \in \mathscr{K}_0 \cap \mathscr{K}_c$ such that $C \subseteqq O \subseteqq \bar{O} \subseteqq K^0 \subseteqq K$ implying $\lambda(O) \geqq \lambda(C) > \lambda(K^0) - \varepsilon$ implying $\geqq$ also holds in (i) and equality follows. Finally, regularity of λ means we may find a $U \in \mathscr{K}_0$ with $\bar{K} \subseteqq U$ and $\lambda(U) - \lambda(\bar{K}) < \varepsilon$. Thus by Lemma 1.6 again (with $C = \bar{K}$) we may find an $O \in \mathscr{K}_0 \cap \mathscr{K}_c$ such that $K \subseteqq \bar{K} \subseteqq O \subseteqq \bar{O} \subseteqq U$ and consequently

$$\lambda(O) - \lambda(\bar{K}) \leqq \lambda(U) - \lambda(\bar{K}) < \varepsilon ,$$

i.e. $\lambda(O) < \lambda(\bar{K}) + \varepsilon$, and $\leqq$ is true in (ii) and the proof of equality is complete.

We conclude this section with a definition of the set functions which we shall consider as well as a fundamental "rearrangement" result needed in the next section.

DEFINITION 1.8. A set function $S \colon \mathscr{K} \to R$ is said to be regular iff

(i) $S \leqq 0$ always, $S(\varnothing) = 0$,
(ii) $S(A \cup B) + S(A \cap B) \leqq S(A) + S(B)$ for $A, B \in \mathscr{K}$
(iii) $S(Ag) = S(A)$ for $A \in \mathscr{K}$ and $g \in G$,

i.e. a nonpositive, normalized, strongly subadditive, and (right) translation invariant function on the precompact Borel subsets of G.

Note. These are the same set functions considered in [5] where their probabilistic genesis is also described.

We conclude this section with the discrete rearrangement theorem for strongly subadditive set functions.

PROPOSITION 1.9. *Let* $S: \mathscr{K} \to R$ *satisfy*

$$S(K \cup T) + S(K \cap T) \leqq S(K) + S(T)$$

for all $K, T \in \mathscr{K}$. *Then*:

(i) *if* $K_i \in \mathscr{K}$, $1 \leqq i \leqq n$, *then* $\sum_{i=1}^n S(I_i) \leqq \sum_{i=1}^n S(K_i)$, *where* $I_j = \{g \in G: \sum_{i=1}^n \chi_{K_i}(g) \geqq j\}$, *or equivalently the union of all intersections of* j *distinct* K_i.

(ii) *if* (A, Σ, μ) *is a finite measure space and* $a \rightsquigarrow K_a$ *is any* (measurable) *simple function from* A *into* $\mathscr{K}$ *then*

$$\int_0^\infty S(I_t)dt \leqq \int_A S(K_a)d\mu(a)$$

where

$$I_t = \left\{g \in G: \int_A \chi_{K_a}(g)d\mu(a) \geqq t\right\} \text{ for } t \geqq 0 .$$

Note. (ii) is written in integral form for the purposes of application even though it is an inequality of finite sums formally derived from (i). Of course $I_t = \varnothing$ for $t > \mu(A)$.

Proof. (i) is well known and readily proved by induction on n. Note that strong subadditivity itself is the case $n = 2$. To prove (ii) we first partition A into finitely many measurable sets, say $A = \bigcup \{A_i: 1 \leqq i \leqq r\}$, such that K_a is constant on each A_i. Then

$$\int_A S(K_a)d\mu(a) = \sum_{i=1}^r S(K^i)\mu(A_i) , \tag{1}$$

where K^i is the constant value of K_a on A_i. Note

$$\int_A \chi_{K_a}(g)d\mu(a) = \Sigma \{\mu(A_i): 1 \leqq i \leqq r \text{ and } g \in K^i\} . \tag{2}$$

Now first assume all $\mu(A_i)$ are rational, say $\mu(A_i) = n_i/d$, $1 \leqq i \leqq r$. Then

$$\sum_{i=1}^r S(K^i)\mu(A_i) = \frac{1}{d}\sum_{i=1}^r n_i S(K^i) = \frac{1}{d}\sum_{j=1}^n S(K_j) \tag{3}$$

where $n = \sum_{i=1}^r n_i$ and K_j, $1 \leqq j \leqq n$, is simply any listing of the K^i where each K^i occurs n_i times for $1 \leqq i \leqq r$. Now by (i),

$$\sum_{j=1}^n S(I'_j) \leqq \sum_{j=1}^n S(K_j) , \tag{4}$$

where $I'_j = \{g \in G: \sum_{i=1}^n \chi_{K_i}(g) \geqq j\}$ for any $j \geqq 0$ (not necessarily integral). Since $I_t = I_{[t+1]}$, a.e. $\sum_{j=1}^n S(I'_j) = \int_0^\infty S(I'_t)dt$ in "integral" form. Furthermore, from equations (2) and (3) we see that $g \in I'_t$ iff $g \in I_{t/d}$ for any $t \geqq 0$, i.e. $I'_t = I_{t/d}$, and consequently by (1), (3) and (4),

$$\int_0^\infty S(I_t)dt = \frac{1}{d}\int_0^\infty S(I_{t/d})dt = \frac{1}{d}\int_0^\infty S(I'_t)dt \leqq \frac{1}{d}\sum_{j=1}^n S(K_j)$$
$$= \sum_{i=1}^r S(K^i)\mu(A_i) = \int_A S(K_a)d\mu(a) ,$$

and (ii) is proved if $\{\mu(A_i)\}$ are all rational. The general result follows upon passage to the limit since both sides of the inequality in (ii) are in fact finite sums which depend continuously on the $\{\mu(A_i)\}$ in light of (1) and (2).

2. The average of a regular set function. We now define the "average" which we are to analyze in the remainder of the paper.

DEFINITION 2.1. For any (real-valued) set function S on $\mathscr{K}$, let $M_S(K) = \lambda(K)^{-1}S(K)$ for $K \in \mathscr{K}_+$.

The main result of this paper is:

THEOREM 2.2. *If S is any regular set function on $\mathscr{K}$ and $\{A_\alpha\} \in \mathscr{A}$, then*

$$\lim_\alpha M_S(A_\alpha) = \inf\{M_S(K): K \in \mathscr{K}_c \cap \mathscr{K}_0\}$$
$$= \inf\{M_S(K): K \in \mathscr{K}_c \cap \mathscr{K}_+\} .$$

Note. The second equality follows readily from Corollary 1.7 (ii) and the monotonicity of S. This result is to be compared with the basic result of [5] which establishes the existence of the limit on the left for $\{A_\alpha\} \in \mathscr{R}$.

The following proposition is at the center of our proof:

PROPOSITION 2.3 (The Fundamental Inequality). *If $A \in \mathscr{K}_+$ and $\mathscr{K} \in \mathscr{K}_c \cap \mathscr{K}_0$, then*

$$\lambda(A)^{-1}S(KA) \leqq \lambda(K^{-1})^{-1}S(K) .$$

Note. that K is assumed open is merely for technical convenience —to assure that $KA \in \mathscr{K}_+$, but the restriction to $\mathscr{K}_c$ is crucial.

Proof. Fix $K \in \mathscr{K}_c \cap \mathscr{K}_0$ and $\varepsilon > 0$ and choose O as in Lemma 1.5, for this K and ε. Then cover A by $\bigcup\{Oa: a \in A\}$. and extract a finite subcover $\bigcup\{Oa_i: 1 \leqq i \leqq n\}$ (since the compact set $\bar{A} \subseteqq \bigcup\{Oa: a \in A\}$ also). Define the finite partition of A into n sets

$$A_1 \div Oa_1 \cap A, A_i = (Oa_i - Oa_1 - \cdots - Oa_{i-1}) \cap A, 1 < i \leqq n ,$$

and the (measurable) simple function from A into $\mathscr{K}$ by $a \rightsquigarrow K_a \doteqdot KOa_k \cap KA$ for $a \in A_k \subseteqq Oa_k$. Since O is symmetric $a \in Oa_k$ implies $a_k \in Oa$ and consequently for all $a \in A$,

$$(1) \qquad Ka \subseteqq KOa_k \cap KA = K_a \subseteqq KO^2a \cap KA \subseteqq KA\ .$$

The monotonicity and translation invariance of S give

$$(2) \qquad \int_A S(K_a)d\lambda(a) \leqq \int_A S(Ka)d\lambda(a) = \lambda(A)S(K)\ .$$

Now by Proposition 1.9 (ii),

$$(3) \qquad \int_0^\infty S(I_t)dt \leqq \int_A S(K_a)d\lambda(a)\ ,$$

where

$$I_t = \left\{g \in G \colon \int_A \chi_{K_a}(g)d\lambda(a) \geqq t\right\}\ .$$

Moreover, in light of (1) we have for $t > 0$,

$$I_t \subseteqq I_t^* \doteqdot \left\{g \in G : \int_A \chi_{KO^2a \cap KA}(g)d\lambda(a) \geqq t\right\} \subseteqq KA$$

and consequently by (2), (3), and the monotonicity of S,

$$(4) \qquad \int_0^\infty S(I_t^*)dt \leqq \lambda(A)S(K)\ .$$

Finally for $g \in KA$, $g \in KO^2a \cap KA$ iff $a \in O^2K^{-1}g$ and therefore

$$\int_A \chi_{KO^2a \cap KA}(g)d\lambda(a) = \begin{cases} \lambda(O^2K^{-1}g \cap A)\ , & g \in KA \\ 0 & , \ g \notin KA\ , \end{cases}$$

Consequently $I_t^* = \varnothing$ and $S(I_t^*) = 0$ for $t > \lambda(O^2K^{-1})$ and thus

$$(5) \qquad \begin{aligned} \int_0^\infty S(I_t^*)dt &= \int_0^{\lambda(O^2K^{-1})} S(I_t^*)dt \\ &\geqq \int_0^{\lambda(O^2K^{-1})} S(KA)dt = \lambda(O^2K^{-1})S(KA)\ , \end{aligned}$$

which in conjunction with (4) yields

$$\lambda(O^2K^{-1})S(KA) \leqq \lambda(A)S(K)$$

or

$$\lambda(A)^{-1}S(KA) \leqq \lambda(O^2K^{-1})^{-1}S(K)\ ,$$

and the proposition follows upon letting $\varepsilon \downarrow 0$.

We derive the following important corollary on our way to proving Theorem 2.2.

COROLLARY 2.4. *If S is any regular set function on $\mathscr{K}$ and $\{A_\alpha\} \in \mathscr{F}$, then*

$$\inf_{K \in \mathscr{K}_+} M_S(K) = \inf_{K \in \mathscr{K}_0} M_S(K) \leqq \varliminf_\alpha M_S(A_\alpha) \leqq \varlimsup_\alpha M_S(A_\alpha)$$

$$\begin{aligned} &\leqq \inf \{M_S(K)\colon K \in \mathscr{K}_c \cap \mathscr{K}_0\} \\ &= \inf \{M_S(K)\colon K \in \mathscr{K}_c \cap \mathscr{K}_+\} . \end{aligned} \tag{6}$$

Proof. Only the last inequality requires a proof as all the other relations are straightforward, e.g. the note after Theorem 2.2. Fix $K \in \mathscr{K}_c \cap \mathscr{K}_0$ and note that $\lambda(K^{-1}) = \lambda(K)$ since G must be unimodular. Let $A'_\alpha \doteqdot [A_\alpha]_K$ implying $KA'_\alpha \subseteqq A_\alpha$ and $S(A_\alpha) \leqq S(KA'_\alpha)$ and consequently (for $\lambda(A'_\alpha) > 0$),

$$\begin{aligned} M_S(A_\alpha) = \lambda(A_\alpha)^{-1}S(A_\alpha) &\leqq \lambda(A_\alpha)^{-1}\lambda(A'_\alpha)\{\lambda(A'_\alpha)^{-1}S(KA'_\alpha)\} \\ &\leqq \lambda(A_\alpha)^{-1}\lambda(A'_\alpha)M_S(K) , \quad \text{by Proposition 2.3} . \end{aligned} \tag{7}$$

Thus since

$$\{A_\alpha\} \in \mathscr{F}, \quad \varlimsup_\alpha M_S(A_\alpha) \leqq M_S(K)$$

and the last inequality of (6) follows immediately.

The following simple lemma is independent of the preceding argument and needed for Theorem 2.2.

LEMMA 2.5. *If $\{A_\alpha\} \in \mathscr{C}$ then*

$$\inf \{M_S(K)\colon K \in \mathscr{K}_c \cap \mathscr{K}_0\} \leqq \varliminf_\alpha M_S(A_\alpha) .$$

Proof. Since $\{A_\alpha\} \in \mathscr{C}$ upon utilizing Corollary 1.7 (ii) we obtain a net $\{O_\alpha\}$ with $\bar{A}_\alpha \subseteqq O_\alpha \in \mathscr{K}_0 \cap \mathscr{K}_c$ such that $\lambda(O_\alpha)^{-1}\lambda(A_\alpha) \to 1$. Consequently, since $S(O_\alpha) \leqq S(A_\alpha)$ by monotonicity, we have

$$\inf \{M_S(K)\colon K \in \mathscr{K}_c \cap \mathscr{K}_0\} \leqq \varliminf_\alpha M_S(O_\alpha) \leqq \varliminf_\alpha M_S(A_\alpha)$$

as desired.

Theorem 2.2 follows immediately from Corollary 2.4 and Lemma 2.5 since $\mathscr{A} = \mathscr{F} \cap \mathscr{C}$ by 1.3 (iv).

3. Concluding comments. The question arises as to whether Theorem 2.2 remains valid with $\mathscr{A}$ replaced by $\mathscr{F}$. This is in fact not the case for general regular S, but is permissible if S satisfies a mild "continuity" condition. See [2] for a discussion and proof of this and related questions such as when Proposition 2.3 is valid for general $K \in \mathscr{K}_0$, when the inf of $M_S(K)$ on $\mathscr{K}_0$ is the same as on $\mathscr{K}_0 \cap \mathscr{K}_c$, and the utility of the standard (Følner) summing nets in this context. An analogous theory for nonunimodular amenable groups is to be desired but this author has had no success.

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