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SOME MAPPINGS WHICH DO NOT ADMIT AN AVERAGING OPERATOR

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The problem of determining for spaces X and Y necessary and sufficient conditions such that there exists a map ϕ of X onto Y which does not admit an averaging operator is considered. This corresponds to identifying the uncomplemented closed selfadjoint subalgebras of $C(X)$ which contain 1_X. Mappings ϕ of X onto Y are constructed which do not admit averaging operators, for example, when X is any uncountable compact metric space and Y is any countable product of intervals. Also, X can be any space containing an open set homeomorphic to a Banach space and $Y = X$. These results generalize earlier work by D. Amir and S. Ditor.

If ϕ is a mapping of X onto Y, the induced operator ϕ^0 from $C(Y)$ to $C(X)$ that takes $f \in C(Y)$ to $f \circ \phi \in C(X)$ is a multiplicative isometric isomorphism. In case ϕ is a quotient map (e.g., if X and Y are compact Hausdorff spaces) then $\phi^0(C(Y))$ consists of all functions in $C(X)$ which are constant on each point inverse of ϕ. We say *ϕ admits an averaging operator* if there is a projection of $C(X)$ onto $\phi^0(C(Y))$. It is easily seen that ϕ admits an averaging operator if and only if there exists a bounded linear operator u from $C(X)$ into $C(Y)$ such that $u\phi^0(f) = f$ for each $f \in C(Y)$ (see **[12]**, Cor. 3.2), and in this case u is called an *averaging operator* for ϕ.

Following the appearance of the monograph by A. Pelczynski on averaging and extension operators **[12]**, there has been much interest in the study of averaging operators (e.g., see **[2]**, **[3]**, **[4]**, **[5]**, **[6]**, **[15]**). A central problem in this study, known as the complemented subalgebra problem, is to determine necessary and sufficient conditions for a map ϕ from a compact Hausdorff space X onto a compact Hausdorff space Y to admit an averaging operator. Strong necessary conditions have been established in **[5]**. (Also, see **[2]** and **[3]**.) Two closely related problems are to determine for compact Hausdorff spaces X and Y necessary and sufficient conditions that there exists a map ϕ of X onto Y which (1. admits; 2. does not admit) an averaging operator. Since this corresponds to determining the complemented and uncomplemented closed selfadjoint subalgebras of $C(X)$ which contain 1_X by Stone's Theorem **[14**, p. 122], results of this type yield information about the structure of $C(X)$.

In 1968, S. Ditor established that there is a map ϕ of $[0, 1]$ onto itself which does not admit an averaging operator (see [6] and also [5]). In [3], it was shown that if a topological space X contains an open 0-dimensional compact metric space K with $K^{(\omega)}$ nonempty, then there is a map ϕ of X onto itself which does not admit an averaging operator. The same result was also established if K is a first-countable compact subset of X and $\mathrm{Int}(K)^{(n)}$ contains an isolated point for each integer n. It has recently been shown [4] that if X and Y are compact metric spaces with $|X^{(\alpha)}| \geqq |Y^{(\alpha)}|$ for each ordinal number α, X is 0-dimensional, and $Y^{(\omega)}$ is nonempty, there is a map ϕ of X onto Y which does not admit an averaging operator. (Also, see [4] for other related results.)

All of the preceding results except the one by Ditor require the space X to be 0-dimensional. In this paper, we continue this study by considering Hausdorff spaces X and Y which are not necessarily 0-dimensional and establishing sufficient conditions such that there will exist a map ϕ of X onto Y which does not admit an averaging operator. For example, we show that if X is locally a Banach space at some point, then there is a map ϕ of X onto itself which does not admit an averaging operator (Theorem 2). The same conclusion holds if $X = I^\alpha$ for any cardinal number $\alpha \geqq 1$ (Corollary 1.1). Another corollary is that if X is any nondispersed compact Hausdorff space and Y is any cube I^α, $1 \leqq \alpha \leqq \aleph_0$, then there exists a map ϕ of X onto Y which does not admit an averaging operator (Corollary 3.1). These results generalize the previously mentioned result by Ditor and the well-known theorem by D. Amir [1] that $C[0, 1]$ contains an uncomplemented subspace isometrically isomorphic to $C[0, 1]$.

The terminology used herein is standard and follows that in Dunford and Schwartz's *Linear Operators I* [9] and Dugundji's *Topology* [7]. We let $I = [0, 1]$.

Let S be a topological space. The *cone* K over S is the quotient space $(I \times S)/R$ where R is the equivalence relation $(0, x) \sim (0, x')$ for all $x, x' \in S$ (see [7, p. 126]). The vertex of this cone is $v = \{0\} \times S$ and S is identified with the base $\{1\} \times S$. Let $Y = I \times K$ and $\dot{Y} = (\{0, 1\} \times K) \cup (I \times S)$. Frequently, $\dot{Y}$ is the boundary of Y. The preceding assumptions about $\dot{Y}$ are satisfied by many topological spaces. For example, the closed unit ball $K = \{x \in B \mid \|x\| \leqq 1\}$ in a Banach space B is the cone on the unit sphere $S = \{x \in B \mid \|x\| = 1\}$ and the cone on the cube I^α for $\alpha \geqq 0$ is homeomorphic to $I^{\alpha+1}$ (α finite) or I^α (α infinite).

THEOREM 1. *There exists a map ϕ of Y onto itself such that $\phi(y) = y$ for each $y \in \dot{Y}$ and ϕ does not admit an averaging operator.*

Proof. Let $\phi_0 : I \to I$ be a monotone map such that $\phi_0(0) = 0$ and $\phi_0(1) = 1$. Define a map $\tilde{\phi}$ from $I \times I \times S$ onto itself by

$$\tilde{\phi}(t, t', s) = (tt' + (1 - t')\phi_0(t),\ t', s)$$

and let

$$\phi : I \times K \to I \times K$$

be the map induced by $\tilde{\phi}$ on the quotient space. We claim that ϕ maps $I \times (K - \{v\})$ bijectively to itself and that $\phi \mid \dot{Y}$ is the identity. The second statement is obvious. For the first, suppose (t_1, t_1', s_1) and (t_2, t_2', s_2) are two points of $I \times I \times S$ with $t_1' > 0$ such that

$$\tilde{\phi}(t_1, t_1', s_1) = \tilde{\phi}(t_2, t_2', s_2).$$

Then $t_1' = t_2'$, $s_1 = s_2$, and

$$t_1 - t_2 = \frac{1 - t_1'}{t_1'}[\phi_0(t_2) - \phi_0(t_1)].$$

Thus, $t_1 \neq t_2$ implies $\phi_0(t_1) \neq \phi_0(t_2)$. The claim now follows from the fact that ϕ_0 is monotone, for if $t_1 < t_2$, then $\phi_0(t_1) < \phi_0(t_2)$; hence,

$$t_1 t_1' + (1 - t_1')\phi_0(t_1) < t_2 t_2' + (1 - t_2')\phi_0(t_2)$$

and $\tilde{\phi}(t_1, t_1', s_1) \neq \tilde{\phi}(t_2, t_2', s_2)$, a contradiction.

Next, define $E : C(I) \to C(Y)$ by

$$Ef(t, x) = f(t)$$

for $(t, x) \in I \times K$. Then E is a linear operator with $\|E\| = 1$ and RE is the identity operator on $C(I)$ where $R : C(Y) \to C(I)$ is the restriction operator with $Rf(t) = f(t, v)$. Moreover, since the nondegenerate point inverses of ϕ all lie in $I \times v$ (where they are of the form $\phi_0^{-1}(t) \times v$) it is clear that if $f \in C(I)$ and f is constant on each $\phi_0^{-1}(t)$ for each $t \in I$, then $E(f)$ is constant on each $\phi^{-1}(t, x)$ for $(t, x) \in I \times K$. Equivalently, $E(\phi_0^0[C(I)]) \subset \phi^0[C(Y)]$.

Let ϕ_0 be a map such that $\phi_0^0[C(I)]$ is uncomplemented in $C(I)$. For example, if ψ is the Cantor map from the Cantor set $\mathscr{C}$ onto I defined by $\psi(\Sigma_{i=1}^{\infty} 2\xi_i/3^i) = \Sigma_{i=1}^{\infty} \xi_i/2^i$, then ϕ_0 can be selected to be the map of I onto itself which extends ψ and is constant on the disjoint intervals of $I - \mathscr{C}$ (see [**5**, Cor. 5.8]). Then either by Corollary 5.5 in [**5**] or Corollary 1.4 in [**2**], ϕ_0 does not admit an averaging operator.

Suppose P is a bounded projection of $C(Y)$ onto $\phi^0[C(Y)]$. Define $P_0 : C(I) \to C(I)$ by $P_0 = RPE$. Then P_0 is a bounded linear operator and

$$P_0[C(I)] = RPE[C(I)] \subset R\phi^0[C(Y)] \subset \phi_0^0[C(I)].$$

Moreover, if $f \in \phi_0^0[C(I)]$, then $Ef \in \phi^0[C(Y)]$ and $P_0(f) = RPE(f) = RE(f) = f$; hence, $P_0^2 = P_0$ and P_0 is a projection of $C(I)$ onto $\phi_0^0[C(I)]$, which is a contradiction.

COROLLARY 1.1. *Suppose $X = I^\alpha$ for some cardinal $\alpha \geqq 1$. Then there exists a map ϕ of X onto itself which does not admit an averaging operator.*

Proof. $I^\alpha = I \times K$ where K is always a cone except when $\alpha = 1$, in which case the above-mentioned result of Ditor applies.

Since the next theorem is applicable to a space X which contains an open set homeomorphic to Euclidean n-space for $n \geqq 1$, it generalizes the previously mentioned results of Amir and Ditor.

THEOREM 2. *Suppose X contains an open set homeomorphic to some* (nonzero) *Banach space. Then there exists a map ϕ of X onto itself which does not admit an averaging operator.*

Proof. If B is a Banach space of dimension greater than one, then $B = R \times B_1$ where R is the real line and B_1 is a Banach space. Let K be the unit ball in B_1. By Theorem 1, there exists a map ψ of $Y = I \times K$ onto itself such that $\psi^0[C(Y)]$ is uncomplemented in $C(Y)$ and ψ is the identity on $\dot{Y}$. Since B may be identified with an open set in X, we define $\phi : X \to X$ to be ψ on B and the identity otherwise. (If $B = R$, we simply extend the Cantor function $\psi : I \to I$ used by Ditor to $\phi : X \to X$.)

Suppose P is a projection of $C(X)$ onto $\phi^0[C(X)]$. Since Y is bounded in B, there is a closed neighborhood V of Y in B. Let $Z = Y \cup (V - \operatorname{Int} V)$ and define $T : C(Y) \to C(Z)$ by $Tf(x) = f(x)$ for $x \in Y$ and $Tf(x) = 0$ otherwise. Then T is a linear operator with $\|Tf\| = \|f\|$ (i.e., T is a simultaneous extension operator). By the Borsuk-Dugundji Simultaneous Extension Theorem (see [**8**, p. 360] or [**13**, p. 37]), there is a linear operator $E : C(Z) \to C(V)$ with $\|Ef\| = \|f\|$ and $Ef(x) = f(x)$ for $x \in Z$. Let $M = \{f \in C(V) | f(x) = 0$ for $x \in (V - \operatorname{Int} V)\}$ and define $L : M \to C(X)$ by $Lf(x) = f(x)$ for $x \in V$ and $Lf(x) = 0$ otherwise. Clearly, L is a simultaneous extension operator with $\|Lf\| = \|f\|$. Let R be the restriction operator from $C(X)$ onto $C(Y)$ and define $P_0 = \text{RPLET}$. Clearly, P_0 is a linear operator on $C(Y)$ with $\|P_0\| = \|P\|$. Moreover,

$$P_0[C(Y)] \subset RP[C(X)] \subset R\phi^0[C(X)] \subset \psi^0[C(Y)]$$

and if $f \in \psi^0[C(Y)]$, then $LET(f) \in \phi^0[C(X)]$ and $P_0(f) = \text{RPLET}(f) =$

RLET$(f)=f$. Therefore P_0 is a projection of $C(Y)$ onto $\psi^0[C(Y)]$, which is a contradiction.

In the next theorem, we suppose S *is a locally connected compact* metric space, K is the cone over S, and $Y = I \times K$. Recall that a topological space X is called *dispersed* if X contains no perfect subsets.

THEOREM 3. *Suppose S is a locally connected compact metric space and X is a nondispersed compact Hausdorff space* (e.g., an uncountable compact metric space). *Then there exists a map ϕ of X onto Y which does not admit an averaging operator.*

Proof. Since Y is a nonempty locally connected continuum, it follows by the Hahn-Mazurkiewicz-Sierpinski Theorem [**10,** p. 256] that there is a map ν of I onto Y. Since X is nondispersed, there is a map ψ of X onto I [**11.** Thm. 1]. By Theorem 1, there is a map π of Y onto itself such that $\pi^0[C(Y)]$ is uncomplemented in $C(Y)$. Let $\phi = \pi\nu\psi$. We show $\phi^0[C(Y)]$ is uncomplemented in $C(X)$. Suppose P is a projection of $C(X)$ onto $\phi^0[C(Y)]$. If $\lambda = \nu\psi$ and $P_0 = (\lambda^0)^{-1}P\lambda^0$, then P_0 is a linear operator from $C(Y)$ into $\pi^0[C(Y)]$. Moreover, if $f \in \pi^0[C(Y)]$, $f = \pi^0 g$ for some $g \in C(Y)$ and $P_0(f) = (\lambda^0)^{-1}P\lambda^0(\pi^0 g) = (\lambda^0)^{-1}P\phi^0(g) = (\lambda^0)^{-1}\phi^0(g) = (\lambda^0)^{-1}\lambda^0(\pi^0 g) = f$. Thus P_0 is projection of $C(Y)$ onto $\pi^0[C(Y)]$, a contradiction.

Since the continuous image of a dispersed space is dispersed [**11**], we obtain the following characterization.

COROLLARY 3.1. *Let $1 \leqq n \leqq \aleph_0$. If X is a compact Hausdorff space, then there is a map of X onto I^n which does not admit an averaging operator if and only if X is not dispersed.*

In particular, if $1 \leqq m, n \leqq \aleph_0$, then there is a map of I^m onto I^n which does not admit an averaging operator.

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