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ON A CORRECTNESS CLASS OF THE BESSEL TYPE DIFFERENTIAL OPERATOR S_μ

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We have found a correctness class of the Bessel type differential operator $S_\mu = \partial^2/\partial x^2 - (4\mu^2 - 1)/4x^2$ among smooth functions using the Hankel transform and linear mapping theory. It is left as an open problem to find its correctness class among nonsmooth functions satisfying certain boundary conditions.

1. Preliminary. This is a continuation of our previous paper **[10]**. In **[10]** we investigated a uniqueness class of the Cauchy problem of the differential operator $S_\mu = \partial^2/\partial x^2 - (4\mu^2 - 1)/4x^2$, and left out a question on the genus of S_μ. In this paper we answer the question (Lemma 2.2); in fact the genus turned out to be less than that of the differential operator $i\partial/\partial x$, disproving our previous conjecture (**[10]**). We also find a correctness class of the same operator S_μ, using the same notations as in **[9]** or **[10]**. Let us start with the definitions of the spaces $B_{\mu,b}$, $H^\beta_{\mu,\alpha}$ and $Y^{2r}_{\mu,b}$.

DEFINITION 1.1. For any real numbers μ and $b > 0$, the space $B_{\mu,b}$ consists of smooth functions φ such that $\varphi(x) = 0$ for $x > b$ and satisfies the inequalities

$$\gamma^\mu_{b,k}(\varphi) = \sup_{0<x<\infty} |(x^{-1}D)^k(x^{-(\mu+1/2)}\varphi(x))| < \infty, \qquad k = 0, 1, 2, \cdots$$

DEFINITION 1.2. For any real numbers μ, $\alpha > 0$ and $\beta > 0$, the space $H^\beta_{\mu,\alpha}$ consists of smooth functions on $0 < x < \infty$ for which the inequalities

$$\delta^\mu_{k,q}(\varphi) = \sup_{0<x<\infty} |x^k(x^{-1}D)^q(x^{-(\mu+1/2)}\varphi(x))| \leqq CA^kB^qk^{k\alpha}q^{q\beta} \qquad k, q = 0, 1, 2, \cdots$$

are satisfied where $k^{k\alpha} = 1$, $q^{q\beta} = 1$ for $k, q = 0$ and the constants C, A and B depend on the testing function φ.

DEFINITION 1.3. For any real numbers μ, $r > 0$ and $b > 0$, $\Phi \in Y^{2r}_{\mu,b}$ if $z^{-(\mu+1/2)}\Phi(z)$ is an even entire function for which the inequalities

$$\nu_{b,k}^{\mu,2r}(\Phi) = \sup_{z=x+iy} |e^{-b|y|^{2r}}(z^{2k-\mu-1/2}\Phi(z)| < \infty \qquad k = 0, 1, 2, \cdots$$

are satisfied. Here $z^{-(\mu+1/2)}\Phi(z)$ is understood to be a principal value.

The topology of the spaces $B_{\mu,b}$, $H^{\beta}_{\mu,\alpha}$ and $Y^{2r}_{\mu,b}$ are generated by the seminorms $\{\gamma^{\mu}_{b,k}\}^{\infty}_{k=0}$, $\{\delta^{\mu}_{k,q}\}^{\infty}_{k,q=0}$ and $\{\nu^{\mu,2r}_{b,k}\}^{\infty}_{k=0}$ respectively. It is easy to see that all three spaces are Fréchet spaces.

From the definitions, we immediately have:

LEMMA 1.1. *If $b_1 < b_2$, $\alpha_1 < \alpha_2$ and $\beta_1 < \beta_2$, we have $B_{\mu,b_1} \subset B_{\mu,b_2}$, $H^{\beta 1}_{\mu,\alpha_1} \subset H^{\beta 2}_{\mu,\alpha_2}$ and $Y^{2r}_{\mu,b_1} \subset Y^{2r}_{\mu,b_2}$.*

Suppose φ belongs to the Zemanian space H_μ. Utilizing [**9**; p. 337], the space $H^{\beta}_{\mu,\alpha}$ is characterized by the following inequalities.

LEMMA 1.2. *$\varphi \in H^{\beta}_{\mu,\alpha}$ if and only if φ satisfies the following inequalities.*

$$|D^q(x^{-\mu-1/2}\varphi(x))| \leqq C_q B^q q^{q\beta} \exp(-a' x^{1/2}), \qquad q = 0, 1, 2, \cdots$$

where a' is a positive constant less than $a = \alpha e^{-1} A^{1/\alpha}$.

Let $\boldsymbol{h}_\mu$ for $\mu \geqq -1/2$ be the conventional Hankel transform defined by ([**17**; p. 561])

$$[\boldsymbol{h}_\mu \varphi(x)](y) = \int_0^\infty \varphi(x)\sqrt{xy}\, J_\mu(xy)dx$$

where $J_\mu(x)$ is the Bessel function of the first kind. Then we restate the following theorem from [**10**]:

THEOREM 1.1. *For any real numbers $\mu \geqq -1/2$ and $r > 0$, the Hankel transform $\boldsymbol{h}_\mu$ is an isomorphism from the space $B_{\mu,b}$ onto the space $Y^{2r}_{\mu,b}$.*

2. Parabolic systems. I. M. Gelfand and G. E. Shidov ([**3**; pp. 105–164]) investigated a correctness class of parabolic equations of the differential operator $i\,\partial/\partial x$ such that

$$\frac{\partial u(x,t)}{\partial t} = P\left(i\frac{\partial}{\partial x}\right) u(x,t) \tag{2.1}$$

$$u_0(x) = u(x,0) \tag{2.2}$$

where $u(x, t)$ is an $m \times 1$ column vector, $P(\xi)$ is an $m \times m$ matrix in ξ with constant coefficients, all the eigenvalues of $P(\xi)$ are even powers of ξ, $x = (x_1, \cdots, x_m) \in R^m$ and $t \in R$. The Fourier transform of the system (2.1)–(2.2) yields

$$\frac{\partial \hat{u}(\xi, t)}{\partial t} = P(\xi)\hat{u}(\xi, t) \tag{2.1$'$}$$

$$\hat{u}_0(\xi) = \hat{u}(\xi, 0) \tag{2.2$'$}$$

where $\xi = \sigma + i\eta$. The formal solution of the system (2.1)$'$–(2.2)$'$ is given by

$$\hat{u}(\xi, t) = \exp(tP(\xi)\hat{u}_0(\xi).$$

Let p_0 and h be the reduced order and the parabolicity index respectively of the system (2.1)$'$–(2.2)$'$. Then according to [**3**; p. 41 and p. 114], the following inequalities hold:

$$\|\exp(tP(\xi))\| \leqq C_1 \exp(at\,|\,\xi\,|^{p_0}) \tag{2.3}$$

$$\|\exp(tP(\sigma))\| \leqq C_2 \exp(-\,bt\,|\,\sigma\,|^{h}) \tag{2.4}$$

where C_1, C_2, a and b are positive constants. Let ρ be the genus of the given system. Then in a domain defined by

$$|\,\eta\,| \leqq B_1(1 + |\,\sigma\,|)^{\rho}, \qquad \rho \geqq 1 - (p_0 - h) \tag{2.5}$$

the inequality

$$\|\exp(tP(\xi))\| \leqq C_3 \exp(-\,b't\,|\,\sigma\,|^{h}) \tag{2.6}$$

is satisfied where b' is arbitrarily close to b. Utilizing [**2**; p. 217], the inequalities (2.3) and (2.6) lead us to

$$\begin{aligned} \|\exp(tP(\xi))\| &\leqq C_4 \exp(-\,bt\,|\,\sigma\,|^{h} + a't\,|\,\eta\,|^{p_0/\rho}) \\ &\leqq C_4' \exp(a't\,|\,\eta\,|^{p_0/\rho}) \end{aligned} \tag{2.7}$$

where a' is a constant $\leqq B_1(a + b)$.

Let S_μ for $\mu \geqq -1/2$ be the Bessel type differential operator defined by $S_\mu = \partial^2/\partial x^2 - (4\mu^2 - 1)/4x^2$, and consider the Cauchy problem of the system (2.1)–(2.2) with $i\partial/\partial x$ replaced by S_μ:

$$\frac{\partial u(x, t)}{\partial t} = P(S_\mu)u(x, t) \tag{2.8}$$

$$u_0(x) = u(x,0). \tag{2.9}$$

Since P is an even polynomial, the system (2.1)–(2.2) is parabolic if and only if the system (2.8)–(2.9) is parabolic.

The Hankel transform of the system (2.8)–(2.9) gives ([**20**; p. 139])

$$\frac{\partial U(\xi, t)}{\partial t} = P(-\xi^2)U(\xi, t) \tag{2.8$'$}$$

$$U_0(\xi) = U(\xi, 0) \tag{2.9$'$}$$

where $U(\xi, t) = (\boldsymbol{h}_\mu u(x, t))(\xi)$. Obviously the formal solution of (2.8)′–(2.9)′ is given by

$$U(\xi, t) = \exp(tP(-\xi^2))U_0(\xi). \tag{2.10}$$

An inspection of (2.3), (2.4), (2.5) and (2.10) show us

LEMMA 2.1. *The reduced order and the parabolicity index of the system* (2.8)–(2.9) *are* $2p_0$ *and* $2h$ *respectively.*

Let ρ' be the genus of the system (2.8)′–(2.9)′. Then from the inequalities (2.5) and (2.6), we get

LEMMA 2.2. $1 - 2(p_0 - h) \leqq \rho' < \rho \leqq 1$.

Since $p_0 \geqq h$ ([**3**; p. 116]), it follows that $\rho' > 0$ if and only if $p_0 - h < 1/2$. Now we shall prove the main theorem.

THEOREM 2.1. *For any real numbers* $\mu \geqq -1/2$ *and* $d > 0$, *let* $u_0(x)$ *belong to the space* $B_{\mu, d}$. *If* P *is an even polynomial, and if the genus* ρ' *of the system* (2.8)′–(2.9)′ *is positive, then the correctness class of the system* (2.8)–(2.9) *belongs to the space* $B_{\mu, d+a_0}$ *for any* $0 \leqq t \leqq T$, *where* $a_0 = a'T$, *and* a' *is a positive constant to be determined.*

Proof. Since the formal solution of the Hankel transformed system (2.8)′–(2.9)′ of (2.8)–(2.9) is given according to (2.10) by

$$U(\xi, t) = \exp(tP(-\xi^2))U_0(\xi),$$

let us first estimate the norm of $\exp(tP(-\xi^2))$.

Since the reduced order and the parabolicity index of the system (2.8)′–(2.9)′ is $2p_0$ and $2h$ respectively from Lemma 2.1, we get from the inequalities (2.3) and (2.4) that

$$\|\exp(tP(-\xi^2))\| \leqq C\exp(at|\xi|^{2p_0}) \tag{2.11}$$

$$\|\exp(tP(-\sigma^2))\| \leqq C_1\exp(-bt|\sigma|^{2h}). \tag{2.12}$$

Since the genus ρ' of the system (2.8)′–(2.9)′ is positive, an application of [**3**; p. 115] and the inequality (2.6) reveal that

$$\|\exp(tP(-\xi^2))\| \leqq C_2\exp(-b't|\sigma|^{2h}) \tag{2.13}$$

where b' differs from b by an arbitrarily small number. Utilizing [**2**; p. 217], the inequalities (2.11), (2.12) and (2.13) lead us to

$$\begin{aligned} \|\exp(tP(-\xi^2))\| &\leqq C_3\exp(-b't|\sigma|^{2h} + a't|\eta|^{2p_0/\rho'}) \\ &\leqq C_3'\exp(a't|\eta|^{2p_0/\rho'}) \\ &\leqq C_3'\exp(a'T|\eta|^{2p_0/\rho'}) \end{aligned} \tag{2.14}$$

where $a' > a$ depends on b, C and C_2. Let $a_0 = a'T$ and let $r > p_0/\rho'$. Then the inequality (2.14) shows that $\exp(tP(-\xi^2))$ belongs to the space $\xi^{-(\mu+1/2)}Y^{2r}_{\mu,a_0}$. Since every testing function in $Y^{2r}_{\mu,d}$ is of a form $\xi^{\mu+1/2}\Phi(\xi)$, where Φ is an even entire function from Definition 1.3, and since $U_0(\xi) = \boldsymbol{h}_\mu u_0(x)$ belongs to the space $Y^{2r}_{\mu,d}$ if $u_0(x) \in B_{\mu,d}$ from Theorem 1.1, it follows that $U(\xi,t) = \exp(tP(-\xi^2)U_0(\xi))$ belongs to the space $Y^{2r}_{\mu,d+a_0}$ from Lemma 1.1. Consequently, $u(x,t) = \boldsymbol{h}^{-1}_\mu U(\xi,t)$ belongs to the space $B_{\mu,d+a_0}$ by virtue of Theorem 1.1. Continuous dependence of the solution on the initial function $u_0(x)$ follows from the fact of uniform convergence of the inverse Hankel transform $\boldsymbol{h}^{-1}_\mu U(\xi,t)$ with respect to t. This proves the theorem.

REMARK. Since T is any positive real number, the correctness class of the system (2.8)–(2.9) approaches to the Zemanian space H_μ as $T \to \infty$, because the space $B_{\mu,d}$ approaches to the Zemanian space H_μ as an inductive limit space ([**19**] and [**20**]) as $d \to \infty$.

Suppose the system (2.8)–(2.9) is petrowsky-parabolic such that real parts of the eigenvalues of P are bounded by a negative constant for $|\sigma| = 1$. Then $p_0 = p = h$ ([**3**; pp. 112–113]) where p is the order of the polynomial P, and so $\rho' = 1$. Consequently, the inequality (2.14) is satisfied for any $a' > d$. Thus we have

THEOREM 2.2. *For the petrowsky-parabolic system* (2.8)–(2.9), *suppose the initial function* $u_0(x)$ *belongs to the space* $B_{\mu,d}$. *Then for any small* $\epsilon > 0$, *the correctness class of the system* (2.8)–(2.9) *belongs to the space* $B_{\mu,d(1+\epsilon T)}$ *for any* $T > 0$.

Now let us consider for the case of negative genus. In this case according to [**2**; p. 210] and [**3**; p. 123], $\partial^m/\partial\sigma^m \exp(tp(-\sigma^2))$ is majorized for Re $\xi = \sigma$ by

$$\left\| \frac{\partial^k}{\partial\sigma^k} \exp(tP(-\sigma^2)) \right\| \leqq CB^k k^{k(1-\sigma'/2h)} \exp(-b't|\sigma|^{2h}). \tag{2.15}$$

Since every testing function in the space $H^{\beta}_{\mu,\alpha}$ is of a form $x^{\mu+1/2}\varphi_1(x)$, where φ_1 is a smooth function on $0 < x < \infty$, Lemma 1.2 and the inequality (2.15) show us that $\exp(tP(-\sigma^2)) \in x^{-(\mu+1/2)} H^{(1-\rho'/2h)}_{\mu,1/2h}$. Let Φ belong to the space $Y^{2r}_{\mu,b}|_{z=x}$ (restriction to the real axis). Then

$$\begin{aligned}&\sup_{-\infty<x<\infty} |x^{2k}(\exp(tP(-\sigma^2)))(x^{-(\mu+1/2)}\Phi(x))| \\ &\qquad \leqq \sup_{-\infty<\sigma<\infty} |\exp(tP(-\sigma^2))| \sup_{-\infty<x<\infty} |x^{2k}(x^{-(\mu+1/2)}\Phi(x))| < \infty,\end{aligned}$$

where the last inequality is a consequence of (2.15) and Definition 1.3. It follows that $\exp(tP(-\sigma^2))$ is a multiplier in the space $Y^{2r}_{\mu,b}|_{\xi=\sigma}$ for any $r > 0$, $b > 0$. Consequently, if the initial function $u_0(x)$ belongs to the space $B_{\mu,b}$ for any $b > 0$, then $U(\sigma, t) = \exp(tP(-\sigma^2))U_0(\sigma)$ belongs to the space $Y^{2r}_{\mu,b}|_{\xi=\sigma}$ and therefore $u(x,t) = \boldsymbol{h}^{-1}_{\mu}U(\sigma,t)$ belongs to the same space $B_{\mu,b}$ as $u_0(x)$ does. Continuous dependence of the solution on the initial function $u_0(x)$ is evident since $\boldsymbol{h}^{-1}_{\mu}U(\sigma,t)$ is uniformly convergent with respect to t. Thus we have proved

THEOREM 2.3. *Suppose the genus ρ' of the system* (2.8)′–(2.9)′ *is negative. If the initial function $u_0(x)$ belongs to the space $B_{\mu,b}$ for any $b > 0$, then the correctness class of the system* (2.8)–(2.9) *belongs to the same space $B_{\mu,b}$ for $0 \leqq t \leqq T$, where $T = (2h/b'e)A^{1/2h}$.*

3. Hyperbolic systems. Consider a system of partial differential equations with constant coefficients

$$\frac{\partial u(x,t)}{\partial t} = P(S_\mu)u(x,t) \tag{3.1}$$

$$u_0(x) = u(x,0) \tag{3.2}$$

where $u(x,t)$ is an $m \times 1$ column vector, $S_\mu = D_x^2 - (4\mu^2 - 1)/4x^2$ as before, and $P(\xi) = (P_{jk}(\xi))$ is an $m \times m$ matrix whose eigenvalues $\lambda_j(\xi)$ $(j = 1, 2, \cdots, s)$ are even powers of ξ. Let the system (3.1) be hyperbolic with S_μ replaced by $i\,\partial/\partial x$. Then for $\Lambda(\xi) = \max_j \operatorname{Re} \lambda_j(\xi)$ $(\xi = \sigma + i\tau)$,

$$\Lambda(\xi) \leqq a|\xi| + b$$
$$\Lambda(\sigma) \leqq C.$$

The Hankel transform of the system (3.1)–(3.2) yields

$$\frac{\partial U(\xi, t)}{\partial t} = P(-\xi^2)U(\xi, t) \tag{3.1$'$}$$

$$U_0(\xi) = U(\xi, 0). \tag{3.2$'$}$$

Let $\lambda'_j(\xi)$ $(j = 1, 2, \cdots, s')$ be the eigenvalues of $P(-\xi^2)$ and let $\Lambda'(\xi) = \max_{1 \leqq j \leqq s'} \operatorname{Re} \lambda'_j(\xi)$. Then

$$\Lambda'(\xi) \leqq a'|\xi|^2 + b'$$
$$\Lambda'(\sigma) \leqq C'.$$

Thus a proof similar to that of Theorems 2.1 or 2.2 leads us to

THEOREM 3.1. *For $\mu \geqq -1/2$, let the initial data $u_0(x)$ belong to the space $B_{\mu,b}$ for any $b > 0$. Then the correctness class of the hyperbolic system* (3.1)–(3.2) *belongs to the same space $B_{\mu,b}$ for $0 \leqq t \leqq T$ where T is any positive real number.*

4. Petrowsky-correct system. Consider a system of differential equations

$$\frac{\partial u(x, t)}{\partial t} = P(S_\mu)u(x, t) \tag{4.1}$$

$$u_0(x) = u(x, 0) \tag{4.2}$$

where u, P and S_μ are given as in §3. Suppose the system (4.1)–(4.2) is Petrowsky-correct with S_μ replaced by $i\partial/\partial x$. Let Λ and Λ' be defined as before. Then

$$\Lambda(\sigma) \leqq C, \qquad \operatorname{Re} \xi = \sigma$$

and

$$\Lambda'(\sigma) \leqq C', \qquad \operatorname{Re} \xi = \sigma.$$

An argument similar to the proof of Theorem 2.1 yields

THEOREM 4.1. *For any real numbers $\mu \geqq -1/2$ and $b > 0$, let the initial data $u_0(x)$ belong to the space $B_{\mu,b}$. If the genus ρ' of the system*

(3.1)′–(3.2)′ is positive, then for all $0 \leqq t \leqq T$, $T > 0$, the correctness class of the system (4.1)–(4.2) belongs to the space $B_{\mu, b+b'T}$ where b' is arbitrarily near to b.

REMARK. Since $\|\exp(tP(-\xi^2))\| \leqq B(1+|\sigma|^2)^{h'}\exp(b't|\tau|^{2p_0/\rho'})$ where h' is the correctness exponent of (3.1)′–(3.2)′ and p_0 is the reduced order of $P(\xi)$, we have to choose $r > p_0/\rho'$ so that $U_0(\xi) \mapsto \exp(tP(-\xi^2))U_0(\xi)$ be a continuous linear mapping from $Y^{2r}_{\mu,b}$ into $Y^{2r}_{\mu,b+b'T}$.

Suppose now the genus ρ' is nonpositive. Then for Re $\xi = \sigma$,

$$\left\| \frac{\partial^k}{\partial x^k} \exp(tP(-\sigma^2)) \right\| \leqq C_k(1+|\sigma|^2)^{r_k}$$

where $r_k \leqq h' + \rho' k$, h' is given as before. Then the following theorem is a direct consequence of Theorem 2.3.

THEOREM 4.2. *For any real numbers $\mu \geqq -1/2$ and $b > 0$, let the initial data $u_0(x)$ belong to the space $B_{\mu,b}$. If the genus of the system (3.1)′–(3.2)′ is nonpositive, the correctness class of the system (4.1)–(4.2) belongs to the same space $B_{\mu,b}$ for all $0 \leqq t \leqq T$ where T is any positive real number.*

5. Concluding remarks. So far we showed that for given smooth initial data with compact support a correctness class of the differential operator S_μ with constant coefficients consists of smooth functions having compact support for any given $0 \leqq t \leqq T$. In the case of the differential operator $i\ \partial/\partial x$, Gelfand and Shilov ([3]) found a correctness class among nonsmooth functions via Fourier transforms and convolutions. Ineed their correctness class for small t consists of ordinary functions of exponential type for parabolic and Petrowsky-correct system or k times differentiable functions without any conditions at infinity for hyperbolic system. It is thus an open problem to find a correctness class of the differential operator S_μ among nonsmooth functions. The problem is equivalent to find conditions on the space of Hankel transformable functions with sutiable boundary conditions so that convolutions be allowable. As an example we were not able to identify the space of the convolution

$$\boldsymbol{h}_\mu(x^{-(\mu+1/2)}) * \boldsymbol{h}_\mu(H^{1-\rho'/(2h)}_{\mu,1/(2h)}) \quad \text{for} \quad x > 0$$

in the proof of Theorem 2.2.

Motivated by Schoenberg ([11]), Hirschman, Jr. ([6]) first introduced $\#$-convolution defined by

$$(f \# g)(t) = \int_0^\infty \int_0^\infty f(x)g(y)D(t,x,y)d\nu(x)d\nu(y)$$

where f and g are $L^1(0,\infty)$-functions with respect to the Radon measure $d\nu(x) = x^{2\gamma}[2^{\gamma-1/2}\Gamma(\gamma+1/2)]^{-1}dx$, $\gamma > 0$,

$$D(t,x,y) = 2^{3\gamma-5/2}[\Gamma(\gamma+1/2)]^2(txy)^{-2\gamma+1}[\Gamma(\gamma)\pi^{1/2}]^{-1}[\Delta(t,x,y)]^{2\gamma-2}$$

and $\Delta(t,x,y)$ is the area of the triangle whose sides are t, x and y if such a triangle exists otherwise is zero. With his definition of Hankel transform $\wedge$ given by

$$f^\wedge(t) = \int_0^\infty f(x)g(xt)d\nu(x)$$

where

$$g(x) = 2^{\gamma-1/2}\Gamma(\gamma+1/2)x^{1/2-\gamma}J_{\gamma-1/2}(x)$$

he then showed that the $\#$-convolution is a counterpart of the $*$-convolution for the Fourier transform. Further he proved that if f, $f^\wedge \in L^1(0,\infty)$, f and $f^\wedge$ are inverse to each other under the Hankel transform (5.1). Later on Haimo ([5]) showed that the space of $\#$-convolutionable functions is an algebra with L^1-norm and thus the assumption that $f^\wedge \in L^1(0,\infty)$ is superfluous. Unfortunately their theories may not be applicable to our case since $x^{-(\mu+1/2)}$ does not belong to $L^1(0,\infty)$ with respect to $d\nu$. It is not known that every function in $\boldsymbol{h}_\mu(H^{1-p'/(2h)}_{\mu,1/(2h)})$ is L^1-integrable with respect to $d\nu$.

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