PACIFIC JOURNAL OF MATHEMATICS
Vol. 62, No. 2, 1976

UNIFORM ALGEBRAS SPANNED BY HARTOGS SERIES

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Let A be a uniform algebra on a compact space X, and let R be an upper semi-continuous function from X to $[0, \infty)$. Let

$$Y = \{(x, \zeta) \in X \times \mathbf{C} : |\zeta| \leqq R(x)\},$$

and let B be the uniform algebra on Y generated by polynomials in ζ with coefficients in A. The maximal ideal space M_B of B then has the form

$$M_B = \{(\varphi, \zeta) \in M_A \times \mathbf{C} : |\zeta| \leqq \tilde{R}(\varphi)\}$$

for some function $\tilde{R}$ on M_A. We will give several characterizations of $\tilde{R}$ in terms of R. One description involves Hartogs functions, another involves Jensen measures. We will also treat the problem of characterizing the continuous functions on M_B which lie in the algebra B.

1. Introduction. Let D be a domain of holomorphy in $\mathbf{C}^n$, and let R be a lower semi-continuous function from D to $(0, +\infty]$. Let G be the domain in $\mathbf{C}^{n+1}$ described by

$$G = \{(z, \zeta): z \in D, |\zeta| < R(z)\}.$$

A holomorphic function f on G can be expanded in a Hartogs series

$$f(z, \zeta) = \sum_{j=0}^{\infty} f_j(z)\zeta^j,$$

where each f_j is a holomorphic function on D. For fixed z, the radius of convergence $R_f(z)$ of the series is given by

$$-\log R_f(z) = \limsup_{j \to \infty} \frac{\log |f_j(z)|}{j}.$$

The upper semi-continuous regularization S_f of $-\log R_f$ is a plurisubharmonic function on D which is dominated by $-\log R$. The domain

$$G_f = \{(z, \zeta): z \in D, |\zeta| < e^{-S_f(z)}\}$$

is then a domain of holomorphy containing G, and f extends holomorphically to G_f. It is not hard to conclude that the interior $\tilde{G}$ of the intersection of the G_f, f holomorphic on G, is the envelope of holomorphy of G; that is, $\tilde{G}$ is a domain of holomorphy to which all holomorphic functions on G extend analytically. Furthermore, if S is the lower plurisubharmonic envelope of $-\log R$, then

$$\tilde{G} = \{(z, \zeta): z \in D, |\zeta| < e^{-S(z)}\}.$$

For details, see **[9]**.

There is a roughly equivalent theorem which describes the polynomial hull of a compact subset of $\mathbf{C}^{n+1}$ which is "circled" in one of the variables. Indeed, let K be a compact subset of $\mathbf{C}^{n+1} = \mathbf{C}^n \times \mathbf{C}$ with the property that if $(z, \zeta_0) \in K$, then $(z, \zeta) \in K$ for all $\zeta \in \mathbf{C}$ satisfying $|\zeta| = |\zeta_0|$. Let J be the projection of K on $\mathbf{C}^n$, and define the upper semi-continuous function R on J by

$$R(z) = \sup\{|\zeta|: (z, \zeta) \in K\}, \qquad z \in J.$$

Let $\hat{J}$ and $\hat{K}$ be the polynomial hulls of J and K in $\mathbf{C}^n$ and $\mathbf{C}^{n+1}$ respectively. Evidently $\hat{K}$ includes $\hat{J} \times \{0\}$, while $\hat{J}$ coincides with the projection of $\hat{K}$ into $\mathbf{C}^n$. Furthermore, it is clear that $\hat{K}$ has the form

$$\hat{K} = \{(z, \zeta): z \in \hat{J}, |\zeta| \leqq \tilde{R}(z)\}$$

for some upper semi-continuous function $\tilde{R}$ on $\hat{J}$. The problem is to describe $\tilde{R}$ in terms of R. The solution can be obtained easily from the results described above concerning envelopes of holomorphy of Hartogs domains. *The function* $-\log \tilde{R}$ *is the upper envelope of the family of real-valued functions on* $\hat{J}$ *which extend plurisubharmonically to some neighborhood of* $\hat{J}$, *and which are dominated by* $-\log R$ *on* J. We wish to extend this latter result to the context of uniform algebras. Section 2 is devoted to the formulation of the basic result. The proof is given in §§3 and 4.

The remainder of the paper is aimed at obtaining a Mergelyan-type theorem for compact subsets of $\mathbf{C}^2$ which are circled in one variable. In §§5 and 6, the problem is reduced to a weighted approximation problem for analytic functions of one complex variable. In §7, the weighted approximation problem is solved in certain special cases.

Notations and conventions. We rely on **[5]** for standard notation and background material. The set of real numbers is denoted by $\mathbf{R}$, the complex numbers by $\mathbf{C}$, and the (strictly) positive integers by Z_+. The space of continuous complex-valued functions on a topological space S is denoted by $C(S)$, while the subspace of real-valued continuous functions on S is denoted by $C_R(S)$. The supremum norm over S is denoted by

$$\|f\|_S = \sup\{|f(s)|: s \in S\}.$$

All measures are regular complex Borel measures.

In §2 through 4 it will be convenient to focus on the function Q related to R by

$$Q = -\log R, \qquad R = e^{-Q},$$

it being understood that $-\log 0 = +\infty$ and $e^{-\infty} = 0$. The function corresponding to $\tilde{R}$ is given by

$$\tilde{Q} = -\log \tilde{R}, \qquad \tilde{R} = e^{-\tilde{Q}}.$$

2. Formulation of the theorem. Let X be a compact space, and let A be a uniform algebra on X. The maximal ideal space M_A of A is a compact space which contains X as a closed subspace. The functions in A will be regarded as continuous functions on M_A.

Recall that a probability measure σ on X is a *Jensen measure* for $\varphi \in M_A$ if

$$\log|f(\varphi)| \leqq \int \log|f|\, d\sigma, \qquad f \in A.$$

The existence of Jensen measures for each $\varphi \in M_A$ was first established by E. Bishop **[1]**. The Jensen measures for φ form a convex, weak-star compact set of probability measures on X. The Jensen measures on X for φ can be characterized as those measures σ on X such that $\sigma(X) = 1$, and $\int \log|f|\, d\sigma \geqq 0$ for all $f \in A$ satisfying $|f(\varphi)| = 1$.

The family of *Hartogs functions* on M_A is the smallest family $\mathcal{H}$ of functions from M_A to $[-\infty, +\infty)$ with the following two properties:

(2.1) $\dfrac{\log|f|}{m} \in \mathcal{H}$ for all $m \in Z_+$ and all $f \in A$.

(2.2) If $\{w_n\}_{n=1}^{\infty}$ is a sequence in $\mathcal{H}$ which is bounded above, and $w = \limsup_{n\to\infty} w_n$, then $w \in \mathcal{H}$.

Each Hartogs function is a Borel function which is bounded above.

As an example, let X be a compact subset of $\mathbf{C}^n$, and let $A = P(X)$ be the uniform algebra on X generated by the analytic polynomials. A theorem of H. Bremermann **[2, 7]** then implies that any function on X which extends to be plurisubharmonic in a neighborhood of the polynomial hull $\hat{X}$ of X is a Hartogs function. For related material on Hartogs functions, there is the work of N. Sibony [7, 8]. The following lemma, in the case that X is a compact subset of $\mathbf{C}$ and w extends subharmonically to a neighborhood of $\hat{X}$, has been proved by B. Cole [3].

LEMMA 2.1. *If w is a Hartogs function on M_A, and if σ is a Jensen measure on X for φ, then* $w(\varphi) \leqq \int w\, d\sigma$.

Proof. On account of the definition of the family of Hartogs functions, it suffices to show that if $\{w_n\}_{n=1}^{\infty}$ is a sequence of Borel functions on M_A such that $w_n \leqq b < \infty$ while $w_n(\varphi) \leqq \int w_n\, d\sigma$, then the function $w = \limsup w_n$ satisfies $w(\varphi) \leqq \int w\, d\sigma$. The latter assertion follows immediately from Fatou's Lemma.

Now let Q be a lower semi-continuous function from X to $(-\infty, +\infty]$. Let Y be the compact subset of $X \times \mathbf{C}$ defined by

$$Y = \{(x, \zeta) \in X \times \mathbf{C} : |\zeta| \leqq e^{-Q(x)}\}. \tag{2.3}$$

Let B be the uniform closure in $C(Y)$ of the functions of the form

$$F(x, \zeta) = \sum_{j=0}^{m} f_j(x)\zeta^j, \qquad (x, \zeta) \in Y, \tag{2.4}$$

where $f_0, \cdots, f_m \in A$. Every complex-valued homomorphism Φ of B is then determined uniquely by its action on A and its value on the coordinate function ζ. In other words, M_B can be identified with certain pairs $(\varphi, \zeta_0) \in M_A \times \mathbf{C}$, where φ is the restriction of Φ to A, and $\zeta_0 = \Phi(\zeta)$. In fact, M_B coincides with the B_0-convex hull of Y in $M_A \times \mathbf{C}$, where B_0 is the algebra of polynomials of the form

$$f(\varphi, \zeta) = \sum_{j=0}^{m} f_j(\varphi)\zeta^j, \quad (\varphi, \zeta) \in M_A \times \mathbf{C}, \quad f_0, \cdots, f_m \in A. \tag{2.5}$$

Since Y is invariant under the rotations $(x, \zeta) \to (x, e^{i\theta}\zeta)$, $0 \leqq \theta \leqq 2\pi$, the space M_B is also invariant under the rotations. Consequently M_B includes the entire circle $\{(\varphi, e^{i\theta}\zeta_0); 0 \leqq \theta \leqq 2\pi\}$ just as soon as it contains one point (φ, ζ_0) of the circle. Now the polynomials (2.5) depend analytically on the parameter ζ of the disc $\{(\varphi, \zeta): |\zeta| < |\zeta_0|\}$. From the maximum modulus principle, we conclude that the entire disc belongs to the B_0-convex hull of Y just as soon as one point of its boundary circle lies in the B_0-convex hull. It follows that M_B has the form

(2.6) $$M_B = \{(\varphi, \zeta) \in M_A \times \mathbf{C} : |\zeta| \leqq e^{-\tilde{Q}(\varphi)}\}$$

for some lower semi-continuous function $\tilde{Q}$ from M_A to $(-\infty, +\infty]$. A description of $\tilde{Q}$ in terms of Q is given by the following theorem.

THEOREM 2.2. *Let Q be a lower semi-continuous function from X to $(-\infty, +\infty]$. Define Y, B and $\tilde{Q}$ as above, so that M_B is given by (2.6). Then for $\varphi \in M_A$, $\tilde{Q}(\varphi)$ is equal to each of the following.*

(2.7) $$\sup\left\{\frac{\log|f(\varphi)|}{m} : m \in Z_+, f \in A, \frac{\log|f|}{m} < Q \text{ on } X\right\}$$

(2.8) $$\sup\{w(\varphi) : w \text{ is a Hartogs function on } M_A,\ w < Q \text{ on } X\}$$

(2.9) $$\inf\left\{\int Q\,d\sigma : \sigma \text{ is a Jensen measure on } X \text{ for } \varphi\right\}.$$

In particular, the quantities defined by (2.7), (2.8) and (2.9) are equal. The equality of (2.7) and (2.9) in the case that Q is continuous has been established by D. A. Edwards **[4]**.

Note that the inequality (2.7) $\leqq$ (2.8) follows from the definition of Hartogs function, while the inequaltiy (2.8) $\leqq$ (2.9) follows from Lemma 2.1. In §3 we will establish the equality of (2.7) and (2.9) in the abstract setting considered by Edwards. The proof of Theorem 2.2 will be given in §4.

Finally, observe that if X is a compact subset of $\mathbf{C}^n$, and $A = P(X)$, then (2.7)–(2.9) can be replaced by the supremum of $w(\varphi)$, over all functions w which extend to be plurisubharmonic in a neighborhood of the polynomial hull $\hat{X}$ of X, and which satisfy $w < Q$ on X.

3. The envelope condition for $\mathcal{W}$-measures. Let $\mathcal{W}$ be a family of upper semi-continuous functions from X to $[-\infty, +\infty)$, which has the following properties:

(3.1) If $m \in Z_+$ and $v, w \in \mathcal{W}$, then $(v + w)/m \in \mathcal{W}$.

(3.2) $0 \in \mathcal{W}$

(3.3) If $w \in \mathcal{W}$, then $\sup\{w(x) : x \in X\} \geqq 0$.

A probability measure σ on X is a $\mathcal{W}$-measure if

$$\int w\,d\sigma \geqq 0, \qquad \text{all } w \in \mathcal{W}.$$

Let $\mathcal{Q}$ be the family of real-valued continuous functions q on X which satisfy $q \geqq w$ for some $w \in \mathcal{W}$. Then

(3.1)′ $\mathscr{Q}$ is a convex cone,

(3.2)′ $\mathscr{Q}$ includes the positive functions, and

(3.3)′ $-1 \notin \mathscr{Q}$.

Since every $w \in \mathscr{W}$ is the lower envelope of functions in $\mathscr{Q}$, a probability measure σ on X is a $\mathscr{W}$-measure if and only if $\int q\, d\sigma \geqq 0$ for all $q \in \mathscr{Q}$. Consequently the set of $\mathscr{W}$-measures is a weak-star compact convex subset of the space of measures on X.

The following theorem has been proved, in the case that Q is continuous, by Edwards **[4]**. In particular, the Edwards Theorem shows that there exist $\mathscr{W}$-measures on X.

THEOREM 3.1. *Let Q be a lower semi-continuous function from X to $(-\infty, +\infty]$. Then the following quantities are equal*:

$$\sup\{c \in \mathbf{R}\colon w + c < Q \textit{ for some } w \in \mathscr{W}\} \tag{3.4}$$

$$\inf\left\{\int Q\, d\sigma \colon \sigma \textit{ is a } \mathscr{W}\textit{-measure}\right\}. \tag{3.5}$$

Proof. Note first that $(3.4) \leqq (3.5)$ for all Q. Indeed, if $w \in \mathscr{W}$ satisfies $w + c < Q$, and if σ is a $\mathscr{W}$-measure, then $c \leqq \int (w + c) d\sigma \leqq \int Q\, d\sigma$.

To establish the reverse inequality, we suppose first that Q is continuous. Rather than invoke the Edwards Theorem, we give a proof which is based on the standard proof of the existence of Jensen measures.

Suppose that $c < (3.5)$, that is, that

$$c < \int Q\, d\sigma \tag{3.6}$$

for all $\mathscr{W}$-measures σ. Let τ be any nonzero measure on X such that $\int q\, d\tau \geqq 0$ for all $q \in \mathscr{Q}$. Then (3.2)′ shows that $\tau \geqq 0$, so that $\sigma = \tau/\tau(X)$ is a $\mathscr{W}$-measure, and by (3.6) $\int (Q - c) d\tau > 0$. This shows that there is no half-space in $C_R(X)$ which contains both $\mathscr{Q}$ and $c - Q$. The cone generated by $\mathscr{Q}$ and $c - Q$ must then coincide with $C_R(X)$. In particular, there exist $a \geqq 0$ and $q \in \mathscr{Q}$ such that $-1 = a(c - Q) + q$. Since $-1 \notin \mathscr{Q}$, $a > 0$. Hence $Q - c = (1 + q)/a$ belongs to $\mathscr{Q}$, and $Q > w + c$ for some $w \in \mathscr{W}$. That shows that $c \leqq (3.4)$. If follows that $(3.4) = (3.5)$.

Next suppose that Q is lower semi-continuous and bounded. Then (3.4) coincides with the quantity

$$\sup_{v \in C_R, v<Q} [\sup\{c: w + c < v \text{ for some } w \in \mathcal{W}\}]$$

Using Edwards' Theorem, we find that this coincides with

$$\sup_{v \in C_R, v<Q} \left[\inf\left\{\int v\, d\sigma: \sigma \text{ is a } \mathcal{W}\text{-measure}\right\}\right] \tag{3.7}$$

The boundedness of Q allows us to apply the version of the Minimax Theorem given in [**5**, p. 40]. Interchanging the orders of the supremum and infimum in (3.7), we find that (3.7) is equal to (3.5). That proves the theorem in this case.

For the general case, define $Q_a = \min(Q, a)$, so that Q_a is bounded, and Q_a increases to Q as $a \to +\infty$. Define a function $\hat{Q}_a$ on the set of $\mathcal{W}$-measures by

$$\hat{Q}_a(\sigma) = \int Q_a\, d\sigma, \qquad \sigma \text{ a } \mathcal{W}\text{-measure.}$$

Since Q_a is a lower semi-continuous function on X, $\hat{Q}_a$ is a lower semi-continuous function on the compact space of $\mathcal{W}$-measures. Furthermore, $\hat{Q}_a(\sigma)$ increases to $\hat{Q}(\sigma) = \int Q\, d\sigma$ as $a \to +\infty$. Now a variant of Dini's Theorem asserts that if $\{h_n\}$ is any monotone increasing sequence of lower semi-continuous functions on a compact space, and $h = \lim h_n$, then $\inf h_n$ increases to $\inf h$. Applying this theorem to the $\hat{Q}_a$, we conclude that

$$\inf\{\hat{Q}_a(\sigma): \sigma \text{ a } \mathcal{W}\text{-measure}\} \tag{3.8}$$

increases to (3.5). According to the result already established, (3.8) coincides with

$$\sup\{c: w + c < Q_a \text{ for some } w \in \mathcal{W}\}. \tag{3.9}$$

Since (3.9) increases to (3.4) as $a \to +\infty$, (3.4) = (3.5). That completes the proof of Theorem 3.1.

Now return to the uniform algebra A on X, and fix $\varphi \in M_A$. Consider the family $\mathcal{W}$ of upper semi-continuous functions defined by

$$\mathscr{W} = \left\{ \frac{\log|f|}{m} : m \in Z_+, f \in A, |f(\varphi)| = 1 \right\}.$$

If $\log|f_1|/m_1$ and $\log|f_2|/m_2$ belong to $\mathscr{W}$, and $m \in Z_+$, then

$$\frac{1}{m}\left[\frac{\log|f_1|}{m_1} + \frac{\log|f_2|}{m_2}\right] = \frac{\log|f_1^{m_2} f_2^{m_1}|}{m m_1 m_2}$$

also belongs to $\mathscr{W}$. Hence (3.1) is valid, and (3.2) and (3.3) are also easy to check. In this case, the condition that σ be a $\mathscr{W}$-measure is equivalent to the condition that σ be a Jensen measure for φ. Theorem 3.1 then shows that (2.9) coincides with

$$(3.10)\ \sup\left\{c : \exists\, m \in Z_+ \text{ and } g \in A \text{ with } |g(\varphi)| = 1 \text{ and } \frac{\log|g|}{m} + c < Q\right\}.$$

If g is as in (3.10), then the function $f = ge^{c/m}$ satisfies $\log|f(\varphi)|/m = c$ and

$$\frac{\log|f|}{m} = \frac{\log|g|}{m} + c < Q.$$

We conclude that (3.10) coincides with (2.7), so that (2.7) is equal to (2.9). The equality of (2.7), (2.8) and (2.9) is thereby established.

4. Completion of the proof of Theorem 2.2. The first lemma shows that $(2.9) \leqq \tilde{Q}(\varphi)$.

LEMMA 4.1. *There exists a Jensen measure σ on X for φ such that* $\int Q\, d\sigma \leqq \tilde{Q}(\varphi)$.

Proof. Let τ be a Jensen measure on Y for the point $(\varphi, e^{-\tilde{Q}(\varphi)}) \in M_B$ with respect to the algebra B. Let π denote the natural projection of M_B onto M_A, and let $\sigma = \pi^*\tau$ denote the projection of τ onto X. Then σ is a Jensen measure on X for φ with respect to the algebra A. Applying Jensen's inquality to the function $\zeta \in B$, we obtain

$$\begin{aligned}
-\tilde{Q}(\varphi) &\leqq \int \log|\zeta|\, d\tau(x,\zeta) \\
&\leqq -\int Q(x) d\tau(x,\zeta) \\
&= -\int Q(x) d\sigma(x).
\end{aligned}$$

That does it.

To complete the proof of Theorem 2.2, it suffices to show that $\tilde{Q}(\varphi) \leqq (2.8)$. We will establish this inequality in the case that Q is bounded, and then we will approach the unbounded case by means of a limit argument.

Suppose then that Q is bounded. Choose $b > 0$ such that $e^{-Q} \geqq b > 0$ on X. Then M_B includes $X \times \Delta_b$, where Δ_b is the open disc $\{|\zeta| < b\}$.

Let $F \in B$. For each $x \in X$, $F(x, \zeta)$ depends analytically on $\zeta \in \Delta_b$. Consequently F has a Hartogs expansion

$$F(x, \zeta) = \sum_{j=0}^{\infty} f_j(x)\zeta^j, \tag{4.1}$$

where

$$|f_j(x)| \leqq b^{-j} \|F\|. \tag{4.2}$$

This estimate shows that the coefficient functions f_j depend continuously on $F \in B$ in the norm of uniform convergence. Since polynomials in ζ with coefficients in A are dense in B, all of the f_j lie in A.

For $\varphi \in M_A$, let $R_F(\varphi)$ be the radius of convergence of the Hartogs series

$$F(\varphi, \zeta) = \sum_{j=0}^{\infty} f_j(\varphi)\zeta^j. \tag{4.3}$$

The analyticity of F in the disc $\{(\varphi, \zeta) : |\zeta| \leqq e^{-\tilde{Q}(\varphi)}\}$, implies $R_F(\varphi) \geqq e^{-\tilde{Q}(\varphi)}$, so that

$$-\log R_F(\varphi) \leqq \tilde{Q}(\varphi), \qquad \varphi \in M_A.$$

On the other hand, if $-\log R_F(\varphi) \leqq a$ for all $F \in B$, then each $F \in B$ has an analytic continuation to the disc $\{(\varphi, \zeta) : |\zeta| < e^{-a}\}$, and $\tilde{Q}(\varphi) \geqq a$. We conclude that

$$\tilde{Q}(\varphi) = \sup\{-\log R_F(\varphi) : F \in B\}. \tag{4.4}$$

Now the expression for the radius of convergence of (4.3) is

$$-\log R_F(\varphi) = \limsup_{j \to \infty} \frac{\log |f_j(\varphi)|}{j}. \tag{4.5}$$

The estimate (4.2) shows that the functions $\log |f_j|/j$ are uniformly bounded above on M_A. Consequently (4.5) shows that $-\log R_F$ is a

Hartogs function on M_A. Furthermore, since the functions in B are analytic on the discs $\{x\}\times\{|\zeta|<e^{-Q(x)}\}$, $x\in X$, we have

$$-\log R_F(x)\leqq Q(x), \qquad x\in X.$$

These facts, together with (4.4), show that $\tilde{Q}(\varphi)\leqq(2.8)$. We conclude that Theorem 2.2 is valid whenever Q is bounded.

Now let Q be an arbitrary lower semi-continuous function on X. For $k\in Z_+$ define $Q_k=\min(Q,k)$. Let Y_k be the compact subset of $X\times\mathbf{C}$ determined by Q_k, let B_k be the associated uniform algebra on Y_k, and let $\tilde{Q}_k$ be the associated function on M_A determined by Q_k.

Suppose that $(\varphi,\zeta)\in M_{B_k}$ for all k. Then $|F(\varphi,\zeta)|\leqq\|F\|_{Y_k}$ for all polynomials F in ζ with coefficients in A. Since the Y_k decrease to Y, we obtain $|F(\varphi,\zeta)|\leqq\|F\|_Y$ for all such F. Hence $(\varphi,\zeta)\in M_B$. It follows that M_{B_k} decreases to M_B as $k\to\infty$, and consequently $\tilde{Q}_k$ increases to $\tilde{Q}$ as $k\to\infty$.

By the result already obtained, $\tilde{Q}_k(\varphi)$ is equal to

$$(4.6)\qquad \sup\left\{\frac{\log|f(\varphi)|}{m}: m\in Z_+, f\in A, \frac{\log|f|}{m}<Q_k \text{ on } X\right\}.$$

Now (4.6) increases to (2.7) as $k\to+\infty$. It follows that $\tilde{Q}(\varphi)=(2.7)$. That completes the proof of Theorem 2.2.

5. The Hartogs series expansion. Suppose $F\in C(Y)$ is such that $F(x,\zeta)$ depends analytically on ζ in each disc

$$\Delta_x=\{(x,\zeta):|\zeta|<R(x)\}, \qquad x\in X.$$

The *Hartogs series expansion* of F is the power series

$$F(x,\zeta)\sim\sum_{j=0}^{\infty}f_j(x)\zeta^j,$$

where the coefficients f_j are functions on $\{R>0\}$ given by

$$(5.1)\qquad f_j(x)=\frac{1}{2\pi i}\int_{|\zeta|=R(x)}\frac{F(x,\zeta)}{\zeta^{j+1}}\,d\zeta, \qquad j\geqq 0.$$

Define $F_0(x,\zeta)=F(x,0)$, and for $j\geqq 1$ define

$$(5.2)\qquad F_j(x,\zeta)=\begin{cases}\zeta^j f_j(x), & R(x)>0\\ 0, & R(x)=0.\end{cases}$$

The Hartogs series for F then takes the form

$$F \sim \sum_{j=0}^{\infty} F_j.$$

LEMMA 5.1. *Let F and the F_j be as above. Then each F_j is continuous on Y, and*

$$\|F_j\|_Y \leqq \|F\|_Y. \tag{5.3}$$

Proof. The statement is evidently true for F_0, so we assume that $j \geqq 1$. From (5.1) we obtain

$$|f_j(x)| \leqq R(x)^{-j} \|F\|_{\Delta_x}.$$

Since $|\zeta| \leqq R(x)$,

$$|F_j(x, \zeta)| \leqq \|F\|_{\Delta_x} \leqq \|F\|,$$

this for all $(x, \zeta) \in Y$ satisfying $R(x) > 0$. The estimate is trivial if $R(x) = 0$. That establishes (5.3). Furthermore, the expression

$$f_j(x) = \frac{1}{2\pi i} \int_{|\zeta| = R(x)} \frac{F(x, \zeta) - F(x, 0)}{\zeta^{j+1}}\, d\zeta,$$

yields the estimate

$$|F_j(x, \zeta)| \leqq \|F - F_0\|_{\Delta_x}, \qquad x \in X. \tag{5.4}$$

Now suppose (x_α, ζ_α) is a net in Y which converges to (x_0, ζ_0). There are various cases to consider.

Suppose first that $\zeta_0 \neq 0$. Then there is $c > 0$ such that $|\zeta_\alpha| \geqq c > 0$ eventually. Then also $R(x_\alpha) \geqq c > 0$. In view of the continuity of F, the expression

$$F_j(x_\alpha, \zeta_\alpha) = \frac{\zeta_\alpha^j}{2\pi i} \int_{|\xi| = c} \frac{F(x_\alpha, \xi)}{\xi^{j+1}}\, d\xi, \tag{5.5}$$

shows that $F_j(x_\alpha, \zeta_\alpha)$ converges to $F_j(x_0, \zeta_0)$. That proves the continuity of F_j at any point $(x_0, \zeta_0) \in Y$ for which $\zeta_0 \neq 0$.

Now suppose $\zeta_0 = 0$. Then $F_j(x_0, \zeta_0) = 0$, so we must show that

$$\lim_\alpha F_j(x_\alpha, \zeta_\alpha) = 0. \tag{5.6}$$

Passing to a subnet, we can assume that one of the following conditions holds:

$$R(x_\alpha) = 0 \quad \text{for all} \quad \alpha, \tag{5.7}$$

$$R(x_\alpha) > 0, \quad \text{and} \quad \lim R(x_\alpha) = 0, \tag{5.8}$$

$$R(x_\alpha) \geqq c > 0 \quad \text{for all} \quad \alpha. \tag{5.9}$$

If (5.7) is valid, then $F_j(x_\alpha, \zeta_\alpha) = 0$ by definition, so that (5.6) is true. If (5.8) is valid, then the continuity of F implies that the oscillation of F on the disc $\{(x_\alpha, \zeta): |\zeta| \leqq R(x_\alpha)\}$ tends to zero. The estimate (5.4) then yields (5.6). If (5.9) is valid, then (5.5) and the continuity of F show that $\lim_\alpha F_j(x_\alpha, \zeta_\alpha) = F_j(x_0, 0) = 0$. In any event, we obtain (5.6), as required.

Now we seek conditions on F which place F in the algebra B spanned by the monomials $\zeta^j f_j$, where $f_j \in A$. The first observation is as follows.

THEOREM 5.2. *Suppose $F \in C(Y)$ is such that for each $x \in X$, $F(x, \zeta)$ depends analytically on ζ in the disc $\{|\zeta| < R(x)\}$. Let $F \sim \Sigma F_j$ be the Hartogs expansion of F. Then $F \in B$ if and only if $F_j \in B$ for all $j \geqq 0$.*

Proof. Suppose $F_j \in B$ for all $j \geqq 0$. Let $\{G_n\}$ be the sequence of Česaro means of the partial sums of the series ΣF_j. Then $G_n \in B$, $\|G_n\|_Y \leqq \|F\|_Y$, and G_n converges uniformly to F on each slice $\{(x, \zeta): |\zeta| \leqq R(x)\}$ of Y. By Lebesgue's Theorem, $\int G_n d\eta \to \int F d\eta$ for each measure η on Y. In particular, if η is a measure on Y orthogonal to B, then η is also orthogonal to F. By the Hahn–Banach Theorem, $F \in B$.

The converse follows immediately from Lemma 5.1, which shows that the terms F_j of the Hartogs expansion of F depend continuously on F, in the norm of uniform convergence. A reformulation of this is as follows.

THEOREM 5.3. *Fix $j \geqq 0$. Suppose $F \in C(Y)$ has the form $F(x, \zeta) = \zeta^j f(x)$ for some (not necessarily continuous) function f on X. Then $F \in B$ if and only if there is a sequence $\{f_n\}$ in A such that $|f_n(x) - f(x)| R(x)^j$ converges to zero uniformly on X as $n \to \infty$.*

Proof. Suppose $F \in B$. Let $\{F_n\}$ be a sequence of polynomials in ζ with coefficients in A such that $\|F_n - F\|_Y \to 0$. Let $f_n \in A$ be the coefficient of ζ^j in the expression for F_n. By Lemma 5.1, $\|\zeta^j f_n - F\|_Y \to 0$, and consequently $|f_n(x)-f(x)|R(x)^j$ converges uniformly on X to zero. The converse is trivial.

6. A problem concerning weighted approximation. Theorem 5.3 leads to the following weighted approximation problem. Let g be a complex-valued function such that

(6.1) g is defined on the subset $\{R>0\}$ of X,

(6.2) for each $\epsilon>0$, the restriction of g to the compact set $\{R \geqq \epsilon\}$ is continuous, and

(6.3) $g(x)R(x)\to 0$ as $R(x)\to 0$.

It is easy to see that the conditions (6.1), (6.2) and (6.3) are necessary and sufficient in order that ζg extend continuously to Y. The question of whether ζg belongs to B can be rephrased as follows. *Which functions g satisfying* (6.1), (6.2) *and* (6.3) *are such that gR can be approximated uniformly on X by functions of the form fR, $f \in A$?*

A reasonable description of functions in B would lead to a solution of the weighted approximation problem. Conversely, suppose that one could solve the approximation problem for all admissible weight functions. Applying the solutions to the weights R^j, $j \geqq 1$, one would obtain a description of each term F_j of the Hartogs expansion of a function F in B, and hence a characterization of the functions in B.

As an example, let X be a compact subset of the complex plane $\mathbf{C}$ which does not separate $\mathbf{C}$, and let A be the algebra $P(X)$ of uniform limits on X of analytic polynomials. Then Y is a compact subset of $\mathbf{C}^2$, and $B=P(Y)$. The polynomial hull $\hat{Y}$ of Y is then given by

$$\hat{Y}=\{(z,\zeta)\colon z\in X,\ |\zeta|\leqq \tilde{R}(z)\},$$

where $-\log\tilde{R}$ is the upper envelope of the functions subharmonic in a neighborhood of X which are dominated by $-\log R$ on X.

The natural analogue of Mergelyan's Theorem would assert that $P(\hat{Y})$ coincides with the algebra of continuous functions F on $\hat{Y}$ such that F is analytic on the interior of $\hat{Y}$, and such that for each $z\in X$, $F(z,\zeta)$ depends analytically on ζ on the disc $\{|\zeta|<\tilde{R}(z)\}$. The problem in weighted approximation which is equivalent to this Mergelyan-type theorem is the following. If g satisfies the conditions (6.1), (6.2) and (6.3) with respect to the weight $\tilde{R}$, and if g is analytic on the interior of the set $\{\tilde{R}>0\}$, then can gR be approximated uniformly on X by functions of the form fR, when f is an analytic polynomial?

The answer to these problems is affirmative, for weights R which are zero on the interior of X. Rather than treat the problem in this generality, we restrict our attention in the next section to a special case.

7. The case $X = \partial\Delta$, $A = P(\bar{\Delta})$. Let Δ be the open unit disc in the complex plane. In this section, we consider the case in which X is the unit circle $\partial\Delta$, and A is the disc algebra $P(\bar{\Delta})$. Then $M_A = \bar{\Delta}$, while $-\log \tilde{R}$ is the upper envelope of the functions on $\bar{\Delta}$ which extend to be subharmonic in a neighborhood of $\bar{\Delta}$. Since every boundary point of Δ is regular, and since $-\log R$ is lower semi-continuous, we obtain

$$-\log \tilde{R}(z) = \begin{cases} -\log R(z), & z \in \partial\Delta \\ -\displaystyle\int \log R(\xi)\, d\mu_z(\xi), & z \in \Delta, \end{cases} \tag{7.1}$$

where μ_z is the Poisson measure on $\partial\Delta$ for $z \in \Delta$. If $\log R \notin L^1(d\theta)$, we interpret $\tilde{R}(z) = 0$. The polynomial hull $\hat{Y}$ of Y in $\mathbf{C}^2$ is then given by

$$\hat{Y} = \{(z, \zeta) \in \mathbf{C}^2 : |z| \leqq 1, |\zeta| \leqq \tilde{R}(z)\}. \tag{7.2}$$

We wish to describe the algebra $B = P(Y)$. We begin by considering the space $B^\perp$ of measures on Y which are orthogonal to B.

Let Π denote the projection of Y onto $\partial\Delta$ given by $\Pi(z, \zeta) = z$. If η is a measure on Y, then $\Pi^*\eta$ is the measure on $\partial\Delta$ defined on functions h on $\partial\Delta$ by

$$\int h\, d\Pi^*\eta = \int (h \cdot \Pi)\, d\eta.$$

LEMMA 7.1. *Let η be a measure on Y which is an extreme point of the unit ball of $B^\perp$. Then either $\Pi^*(|\eta|) \ll d\theta$, or else there exists $z_0 \in \partial\Delta$ such that η is supported on the set $\Pi^{-1}(\{z_0\})$.*

Proof. Let E be a subset of $\partial\Delta$ of zero length. By the Rudin–Carleson Theorem, E is a peak interpolation set for A. Consequently $\Pi^{-1}(E)$ is a peak set for E. By Glicksberg's Theorem, the restriction of η to $\Pi^{-1}(E)$ lies in $B^\perp$. Since the measure η is assumed to be an extreme point of the unit ball of $B^\perp$, we conclude that η is carried either by $\Pi^{-1}(E)$ or by $Y\backslash\Pi^{-1}(E)$. It follows that either η has zero mass on $\Pi^{-1}(E)$ for all closed subsets E of $\partial\Delta$ which have zero length, or there is a point $z_0 \in \partial\Delta$ such that η is carried by $\Pi^{-1}(\{z_0\})$. In the former case, $\Pi^*(|\eta|) \ll d\theta$. This proves the lemma.

LEMMA 7.2. *Let $j \geqq 1$, and let g be a function defined on $\{R > 0\}$, such that $\zeta^j g \in C(Y)$. Suppose that there is a sequence $\{g_n\}$ in $P(\bar{\Delta})$ such that $g_n R^j$ converges to gR^j in $L^2(d\theta)$. Then $\zeta^j g \in B$.*

Proof. For simplicity, we assume that $j = 1$. Also, we can assume that $|gR| \leqq 1$, and that $g_m R$ converges a.e. $(d\theta)$ to gR. From the inequality

$$\log^+|a| \leqq |a-b|, \qquad a, b \in \mathbf{C}, |b| \leqq 1,$$

we obtain

$$\int \log^+|g_n R|\, d\theta \leqq \int |g_n R - gR|\, d\theta \to 0.$$

Since the functions $\log^+|g_n R|$ are upper semi-continuous, there exist smooth functions u_n on $\partial\Delta$ such that

$$u_n > \log^+|g_n R|, \tag{7.3}$$

and

$$\int u_n\, d\theta \to 0.$$

Let $^*u_n$ be the conjugate harmonic function of u_n, normalized to vanish at 0, and set

$$f_n = \exp[-(u_n + i\,^*u_n)] \in P(\bar{\Delta}). \tag{7.4}$$

Since u_n converges to 0 in $L^1(d\theta)$, $^*u_n$ converges to 0 in $L^p(d\theta)$ for $0 < p < 1$. Passing to a subsequence, we can therefore assume that f_n converges to 1 a.e. $(d\theta)$.

Now consider the functions $\zeta f_n g_n \in B$. From (7.3) and (7.4) we obtain $|f_n g_n| R \leqq 1$ on $\partial\Delta$, so that

$$\|\zeta f_n g_n\|_Y \leqq 1. \tag{7.5}$$

Furthermore,

$$f_n g_n R \to gR \quad \text{a.e.} \quad (d\theta). \tag{7.6}$$

Let η be an extreme point of the unit ball of $B^\perp$. If $\Pi^*(|\eta|) \ll d\theta$, then (7.6) shows that $\zeta f_n g_n$ converges to ζg a.e. $(d\eta)$. Furthermore, the convergence is bounded, by (7.5). Since $\int \zeta f_n g_n\, d\eta = 0$, also $\int \zeta g\, d\eta = 0$.

On the other hand, if η is carried by a set of the form $\Pi^{-1}(\{z_0\})$, then $\int \zeta f\, d\eta = 0$, becuase ζg is a constant multiple of ζ on $\Pi^{-1}(\{z_0\})$.

By Lemma 7.1, $\int \zeta f\, d\eta = 0$ for all extreme points η of the unit ball of $B^\perp$. It follows that $\zeta g \in B$. This concludes the proof.

THEOREM 7.3. *Suppose that* $\log R \notin L^1(d\theta)$. *Then the polynomial hull of* Y *is given by*

$$\hat{Y} = Y \cup \{(z, 0) \colon z \in \Delta\}.$$

Furthermore, $P(\hat{Y})$ *consists of precisely those functions* $F \in C(\hat{Y})$ *such that* $F(z, 0)$ *depends analytically on* $z \in \Delta$, *while for each* $z \in \partial\Delta$, $F(z, \zeta)$ *depends analytically on* ζ *in the disc* $\{|\zeta| < R(z)\}$.

Proof. The description of $\hat{Y}$ follows from (7.1) and (7.2). Furthermore, each function in $P(\hat{Y})$ has the specified properties of analyticity.

Conversely, suppose $F \in C(\hat{Y})$ is analytic on the discs described above. Let $F \sim \Sigma F_j$ be the Hartogs expansion of F. Then each F_j has the same properties of analyticity as does F. By Theorem 5.2, it suffices to show that each F_j belongs to B. Now $F_0(z, \zeta) = F(z, 0)$ lies in the disc algebra $P(\bar{\Delta})$, so we can restrict our attention to the F_j for $j \geqq 1$.

Fix $j \geqq 1$. Write $F_j(z, \zeta) = \zeta^j g(z)$, where g is defined on $\{R > 0\}$. By a result related to Beurling's Theorem, the nonintergrability of $\log R$ implies that the subspace of $L^2(d\theta)$ generated by $z^n R^j$, $n \geqq 0$, has the form $\chi_E L^2(d\theta)$ where χ_E is the characteristic function of the set $\{R > 0\}$. In particular, there exist analytic polynomials g_n such that $g_n R^j$ converges to gR^j in $L^2(d\theta)$. By Lemma 7.2, $\zeta^j g \in R$. The proof is complete.

THEOREM 7.4. *Suppose that* $\log R \in L^1(d\theta)$. *Then the polynomial hull* $\hat{Y}$ *of* Y *is given by* (7.1) *and* (7.2). *Furthermore,* $P(\hat{Y})$ *consists of precisely those functions* $F \in C(\hat{Y})$ *such that* F *is analytic on the interior of* $\hat{Y}$, *while for each* $z \in \partial\Delta$, $F(z, \zeta)$ *depends analytically on* ζ *in the disc* $\{|\zeta| < R(z)\}$.

Proof. Again the problem boils down to showing that if $F \in C(\hat{Y})$ has the form $F(z, \zeta) = \zeta^j g(z)$, where $j \geqq 1$ and g is analytic on Δ, then $F \in P(\hat{Y})$. For simplicity, we treat only the case $j = 1$.

Suppose then that $F \in C(\hat{Y})$ is of the form

$$F(z, \zeta) = \zeta g(z),$$

where g is defined on the set $\{\tilde{R} > 0\}$, and g is analytic on Δ. We can assume that

$$|F(z, \zeta)| \leqq 1, \qquad (z, \xi) \in \hat{Y}.$$

Let h be an outer function in $H^\infty(\Delta) \cong H^\infty(d\theta)$ which satisfies

$$|h(e^{i\theta})| = R(e^{i\theta}) \qquad \text{a.e.} \qquad (d\theta).$$

Then

$$|h(z)| = \tilde{R}(z), \qquad z \in \Delta.$$

Consequently

$$|h(z)g(z)| \leqq 1, \qquad z \in \Delta,$$

so that $hg \in H^\infty(\Delta)$.

We claim that the radial boundary values of $h(z)g(z)$ coincide with $h(e^{i\theta})g(e^{i\theta})$ for almost all $e^{i\theta} \in \partial\Delta$. Indeed, suppose $e^{i\theta} \in \partial\Delta$ is such that h has a radial boundary value $h(e^{i\theta})$ at $e^{i\theta}$, and $h(e^{i\theta}) \neq 0$. Then $\tilde{R}(re^{i\theta}) = |h(re^{i\theta})| \geqq \epsilon > 0$ for r sufficiently near 1. Since the restriction of g to the set $\{\tilde{R} \geqq \epsilon\}$ is continuous, we conclude that $g(re^{i\theta})$ tends to $g(e^{i\theta})$ as r increase to 1. Hence $h(e^{i\theta})g(e^{i\theta})$ is the radial boundary value of hg at $e^{i\theta}$, and the claim is established.

Now $hg \in H^\infty(d\theta)$. By Beurling's Theorem, there is a sequence of analytic polynomials $\{g_n\}$ such that $g_n h$ converges to hg in $L^2(d\theta)$. By Lemma 7.2, $\zeta g \in B$. The proof is complete.

As mentioned earlier, the results and proofs in this section carry over to the case in which $X = \partial K$, where K is a compact subset of the complex plane with connected complement, and $A = P(K)$.

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Received May 21, 1975. This work was supported under NSF Grant No. MPS–74–7035.

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