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COMMUTANTS OF MULTIPLIERS AND TRANSLATION OPERATORS

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This note discusses a method for the determination of the commutant of a set of translation operators on the ring of bounded functions in $C(X)$, X locally compact.

1. Introduction. If X is a locally compact group, then X can be represented and studied as a class of operators on $C_b(X)$. Indeed, if $x \in X$ and $f \in C_b(X)$, then one defines an operator U_x on $C_b(X)$ by $U_x f(y) = f(y+x)$, $\forall f \in C_b(X)$. The commutant of $\{U_x : x \in X\}$ was studied in **[3]** and **[4]**.

The general method of §3 for the determination of the commutant of a set of translation operators is applied to the study of the commutant of a class of operators on bounded analytic functions. In §4 some directly obtainable results are given on commutants of multiplier operators on bounded analytic functions.

2. Definitions. Let X denote a locally compact Hausdorff space. Let $C(X)$ denote the algebra of continuous complex valued functions on X and let $C_b(X)$ denote the subalgebra of $C(X)$ consisting of bounded continuous functions. One can study $C_b(X)$ in the topologies κ, β, or σ; respectively, uniform convergence on compact subsets of X, the strict topology or uniform convergence on X. The topology σ is a norm topology with $\|f\| = \sup_{x \in X} |f(x)|$. The strict topology was introduced in **[6]** and its properties may be found in **[5]**, **[6]**, **[8]** and **[9]**. Particularly relevant here are that the σ and β bounded subsets of $C_b(X)$ coincide; on β bounded subsets of $C_b(X)$, the strict topology coincides with κ; and a sequence $\{f_n\}$ converges β to zero if and only if it is σ bounded and κ convergent **[6]**.

Denote by $(C(X), \text{c.o.})$ the space $C(X)$ endowed with the compact open topology or equivalently with the topology of uniform convergence on compact subsets. The space $(C(X \to X), \text{c.o.})$ is similarly defined.

Let B denote the bounded analytic functions on the open unit disc $D = \{z : |z| < 1\}$ in the complex plane. Since B is a κ closed subspace of $C_b(D)$, one may consider B endowed with any one of the topologies k, β or σ, denoted respectively by (B, κ), (B, β) and (B, σ). The algebra $[\beta : \beta]$ of all continuous linear operators from (B, β) into (B, β) has been studied in **[1]** and **[2]**, and it is closely related to an algebra of operators studied in **[4]**. In particular $[\beta : \beta]$ is a norm closed subalgebra of $[\sigma : \sigma]$

the algebra of all continuous linear operators from (B, σ) into (B, σ) and $[\beta: \beta]$ seems more tractable than the larger algebra $[\sigma: \sigma]$.

We will be concerned with translation operators and the commutant in various spaces of operators of a given set of operators.

DEFINITION. For ϕ a continuous map of X into X, the translation operator U_ϕ is defined on $C_b(X)$ by

$$U_\phi f(x) = f(\phi(x)).$$

DEFINITION. Let G be a set of operators in $[\beta: \beta]$. Then the commutant of G in $[\beta: \beta]$ is

$$\text{Comm}(G) = \{V \in [\beta: \beta]: TV = VT, \forall\, T \in G\}.$$

3. Commutants of operators on $C_b(X)$ or on B. The first result in this section is a general method for determining the commutant of a set of translation operators on $C_b(X)$. We associate with each operator in the commutant a linear functional on $C_b(X)$.

Now assume that to each $x \in X$ is associated one and only one continuous map ϕ_x from X into X. In the special case when X is a group, then ϕ_x might be defined by $\phi_x(a) = a + x$ and X is then represented as a group of operators on $C_b(X)$. In general we define the operator $U_{\Phi(x)}$ on $C_b(X)$ by

$$U_{\Phi(x)} f(y) = f(\phi_x(y)) \quad \text{where} \quad \Phi(x) = \phi_x.$$

The topology σ is not appropriate for our purposes because the map $x \to U_x f$ will not in general be continuous (see **[5]**). This difficulty is overcome by using the strict topology on $C_b(X)$ (see Lemma 2). The following three lemmas follow from the definitions and known properties of the compact open and strict topologies and also from the characterization of continuity via nets.

LEMMA 1. *Let $S \subseteq X$. The map $\Phi: S \to (C(X \to X), \text{c.o.})$ is continuous if and only if the map ψ is continuous where*

$$\psi: SxX \to X, \quad \textit{with} \quad \psi(x, y) = \phi_x(y).$$

LEMMA 2. *Assume that the map $\Phi: X \to (C(X \to X), \text{c.o.})$ is continuous. Then the map $\psi: X \to (C_b(X), \beta)$ is continuous where f is fixed in $C_b(X)$ and $\psi(X) = U_{\Phi(x)} f$.*

LEMMA 3. *Let $F = \{f_\alpha : \alpha \in A\}$ where $f_\alpha \in C(X \to X)$. Assume that*

F is compact in $(C(X \to X)$, c.o.) and let K be a compact set in X. Then $F(K) = U_\alpha f_\alpha(K)$ is compact in X.

Let $\mathcal{M}$ denote the continuous linear functionals on $(C_b(X), \beta)$ and let $\mathcal{B}$ denote the continuous linear operators from $(C_b(X), \beta)$ into $(C_b(X), \beta)$. Let G be a collection of continuous maps from X into X and for $x_0 \in X$, let $G(x_0) = \{g(x_0): g \in G\}$.

DEFINITION. The commutant in $\mathcal{B}$ of the operators $\{U_g: g \in G\}$ is $\{T \in \mathcal{B}: TU_g = U_gT, \forall\, g \in G\}$. Denote this commutant by $\mathrm{Comm}(U_G)$. The following Theorem generalizes results in **[4]** and **[5]**.

THEOREM 1. *Let G be a semi-group with identity of continuous maps of X into X. Assume that $\exists\, x_0 \in X$ such that $G(x_0) = X$ and specify one map ϕ_x with $\phi_x(x_0) = x$. Assume that the map $\Phi: x \to \phi_x$ is continuous from X into $(C(X \to X)$, c.o.). Let*

$$\mathcal{N} = \{L \in \mathcal{M}: L(U_\phi f) = L(U_\psi f), \forall\, \phi, \psi \in G \ni \phi(x_0) = \psi(x_0), \forall\, f \in C_b(X)\}.$$

Then there exists a one-to-one norm preserving correspondence between $\mathrm{Comm}(U_G)$ and $\mathcal{N}$ given by

(1) $\quad Tf(x) = L(U_{\phi_x} f)$, for some $\phi_x \in G$ with $\phi_x(x_0) = x$.

Proof. Given $L \in \mathcal{N}$, define T by $Tf(x) = L(U_{\phi_x} f)$. Certainly T is well defined since if $\phi(x_0) = \psi(x_0)$, then $L(U_\phi f) = L(U_\psi f)$ because L is in $\mathcal{N}$. The function $Tf(x)$ is continuous since the maps $x \to U_{\Phi(x)} f \to L(U_{\Phi(x)} f) = Tf(x)$ are continuous as maps $X \to (C_b(X), \beta) \to$ the complex numbers. Also Tf is bounded and $\|T\| \leqq \|L\|$ since $|Tf(x)| \leqq \|L\|\,\|U_{\Phi(x)} f\| = \|L\|\,\|f\|$. Now let $\phi \in G$. Then T commutes with U_ϕ since

$$(TU_\phi)f(x) = T(U_\phi f)(x) = L(U_{\Phi(x)}(U_\phi f))$$

and, letting $(\phi \circ \phi_x)(y) = \phi(\phi_x(y))$, we have

$$\begin{aligned}(U_\phi T)f(x) &= Tf(\phi(x)) = Tf(\phi(\phi_x(x_0)))\\ &= L(U_{\phi \circ \Phi(x)} f) = L(U_{\Phi(x)} U_\phi f).\end{aligned}$$

To show that T is in $\mathcal{B}$ it suffices to prove that T is β continuous on β bounded sets **[7]**. Let S be a bounded set of functions in $C_b(X)$. Let $\epsilon > 0$ and assume K is a compact set in X. Choose K_1 and δ such that $\|g\|_{K_1} < \delta$ implies $|L(g)| < \epsilon$. Let $K_2 = \{\phi_x(y): x \in K, y \in K_1\}$. Then the map $\Phi\colon x \to \phi_x$ being continuous implies $\{\phi_x : x \in K\}$ is compact in $C(X \to X, \text{c.o.})$. Hence by Lemma 3, K_2 is compact.

Therefore if $\|f\|_{K_2} < \delta$, then we have

$$\| U_{\Phi(x)} f \|_{K_1} = \sup_{g \in K_1} |f(\phi_k(y))| < \delta, \ \forall\, x \in K$$

and

$$\| Tf \|_K = \sup_{x \in K} |Tf(x)| = \sup\{|L(U_{\Phi(x)} f)| : x \in K,\ \phi(x_0) = x\} < \epsilon.$$

Now assume $T \in \operatorname{Comm}(U_G)$ and define L by

$$Lf = Tf(x_0). \tag{2}$$

To see that L is in $\mathcal{M}$, let S be a bounded set in $(C_b(X), \beta)$. Let K be compact in X, $x_0 \in K$, and $\epsilon > 0$. Choose K_1 compact in $(C_b(X), \beta)$ and $\delta(\epsilon) > 0$ such that if $\|f\|_{K_1} < \delta$ and $f \in S$, then $\|T(f)\|_K < \epsilon$. Then for f in S and $\|f\|_{K_1} < \delta$, we have $|L(f)| = |Tf(x_0)| \leqq \|Tf\|_K < \epsilon$. Thus L is in $\mathcal{M}$. Now assume that ϕ and ψ are in G with $\phi(x_0) = \psi(x_0)$. Let f be in $C_b(X)$. Then

$$L(U_\phi f) = T(U_\phi f)(x_0) = U_\phi Tf(x_0) = Tf(\phi(x_0))$$

and similarly $L(U_\psi f) = Tf(\psi(x_0))$. Thus $L(U_\phi f) = L(U_\psi f)$ and $L \in \mathcal{N}$.

To show that the maps are inverse let L be in $\mathcal{N}$ and apply (1) to obtain $T \in \operatorname{Comm}(U_G)$ and then apply (2) to obtain $L' \in \mathcal{N}$. Then

$$L'(f) = Tf(x_0) = L(U_{\Phi(x_0)} f) = Lf.$$

Now given $T \in \operatorname{Comm}(U_G)$ apply (2) to obtain $L \in \mathcal{N}$ and apply (1) to L to obtain $T' \in \operatorname{Comm}(U_G)$. Then

$$T'f(x) = L(U_{\Phi(x)} f) = T(U_{\Phi(x)} f)(x_0) = U_{\Phi(x)} Tf(x_0) = Tf(x).$$

Thus the maps (1) and (2) are inverse. Notice also that under (1) we have $\|T\| \leqq \|L\|$ and under (2) we have $\|L\| \leqq \|T\|$.

We obtain as a corollary the following result in **[5]** where X had a group structure which was not necessarily abelian. For ease of exposition we assume X is abelian. Then X has a representation as linear operators (translation operators) on $C_b(X)$. For a in X, and f in $C_b(X)$ we define

$$U_a f(x) = f(a + x).$$

Let $G = \{\phi_a : a \in X\}$ where $\phi_a(x) = x + a$. Notice that $\phi_a(0) = a$ and we are thus associating the map ϕ_a with $a \in X$ as in Theorem 1.

COROLLARY 1 **[4]**. *Let $\mathcal{B}$ denote the continuous linear operators from $(C_b(X), \beta)$ into itself where X is a locally compact Hausdorff abelian group.*

Then there exists a one-to-one norm preserving correspondence between $\mathrm{Comm}(U_G)$ *and $\mathcal{M}$ given by*

$$Tf(x) = L(U_x f), \quad x \in X, \quad f \in C_b(X).$$

Proof. We apply Theorem 1 with $x_0 = 0$ the identity in X. Then $G(0) = X$ and $\mathcal{N} = \mathcal{M}$. It only remains to be verified that the map $X \times X$ into X given by (x, y) into $\phi_x(y)$ is continuous. But if $\{(x_\alpha, y_\alpha): \alpha \in A\}$ is a net in $X \times X$ converging to (x, y), then since ϕ is continuous $\{\phi_{x_\alpha}(y_\alpha) = x_\alpha + y_\alpha : \alpha \in A\}$ converges to $\phi_x(y) = x + y$.

We now apply Theorem 1 to the case when G consists of analytic maps. A uniqueness set for b is a subset S of D such that if $f, g \in B$ and $f = g$ on S, then $f = g$ on D. In order to apply Theorem 1 it is sufficient if $G(z_0)$ determines the functions in B uniquely. Let B^* denote the continuous linear functionals on (B, β).

COROLLARY 2. *Let G be a collection of analytic maps of D into D. Assume that $G(z_0)$ is a uniqueness set for B.*

Then there exists a one-to-one correspondence between $\mathrm{Comm}(U_G)$ and a subset of B^*.

Proof. Given T in $\mathrm{Comm}(U_G)$, define L by $Lf = Tf(z_0)$. Then L is well defined and L is in B^*. Indeed, if the sequence $\{f_n\}$ in B converges β to zero, then $\{Tf_n\}$ converges β to zero and so $\{Tf_n(z_0)\}$ converges to zero. It suffices to consider sequences since a subset of B is β closed if and only if it is β sequentially closed (**[1]**, **[8]**).

If T_1 and T_2 are in $\mathrm{Comm}(U_G)$ and they both map to L, then for any f in B, $T_1 f(z_0) = T_2 f(z_0)$. For any ϕ_x in G with $\phi_x(z_0) = x$ we have, letting $g = \phi_x$,

$$L(U_g f) = T_1(U_g f)(z_0) = U_g T_1 f(z_0) = T_1 f(x)$$

and

$$L(U_g f) = T_2(U_g f)(z_0) = U_g T_2 f(z_0) = T_2 f(x).$$

Thus $T_1 f(x) = T_2 f(x)$ for all x in $G(z_0)$. Thus $T_1 f = T_2 f$ and $T_1 = T_2$.

One can recover T from L by defining for $x \in G(z_0)$

$$Tf(x) = L(U_g f) \tag{3}$$

where $g = \phi_x$ and $\phi_x(z_0) = x$.

COROLLARY 3. *Let G be a collection of holomorphic maps of D into D such that $G(z_0)$ is a uniqueness set for B. Denote by ϕ_x a distinguished element of G mapping z_0 to x, $x \in G(z_0)$. Assume that $\phi_x(z)$ is analytic in x, for each $x \in G(z_0)$.*

Then there exists a one-to-one correspondence between $\mathrm{Comm}(U_G)$ and

$$\mathcal{N} = \{L \in \mathcal{M}: L(U_g f) = L(U_h f)\ \forall\ g, h \in G \ni g(z_0) = h(z_0), \forall\ f \in B\},$$

given by

$$Tf(x) = L(U_{\Phi(x)} f), \qquad f \in B.$$

Proof. The map from T to L maps $\mathrm{Comm}(U_G)$ into $\mathcal{N}$. We have to verify that any L in $\mathcal{N}$ maps into $\mathrm{Comm}(U_G)$ and this only requires verification that $Tf(x)$ is analytic in x. This follows by using Morera's Theorem, the characterization of $\mathcal{M}$ as the Radon measures on D and the analyticity of $\phi_x(z)$ in x.

We consider the maps ϕ_x from D into D given by $\phi_x(z) = g(x)z$ where g is a fixed analytic nonconstant function in B. A special case is $g(x) = x$. Denote the map $z \to g(x)z$ by $\phi_{g(x)}$. Thus $\phi_{g(x)}(z) = g(x)z$. Observe that $\phi_{g(x)}(z)$ is analytic in x and $G(z_0) = \{g(x)z_0: x \in D\}$ is a set of uniqueness for B for any $z_0 \neq 0$ in D because $g(D)$ is an open set.

We say that a linear operator T from B to B is a multiplier on B if there exists a sequence $\{c_n\}_{n=0}^{\infty}$ of complex numbers such that if $f(z) = \Sigma\, a_n z^n \in B$, then $Tf(z) = \Sigma\, a_n c_n z^n$. Let Δ denote the class of all such multipliers. It is known that $\Delta \subseteq [\beta: \beta]$, and the sequences $\{c_n\}$ associated with operators in Δ have been characterized **[1]**. In particular if $\{c_n\}_{n=0}^{\infty}$ is a sequence of complex numbers such that $\limsup |c_n|^{1/n} < 1$, then an operator T defined on B by $T(\Sigma\, a_n z^n) = \Sigma\, a_n c_n z^n$ for $\Sigma\, a_n z^n \in B$ is a multiplier in $[\beta: \beta]$.

4. Commutants of multipliers. Clearly Δ is a commutative algebra. Furthermore if $T \in [\beta: \beta]$ commutes with every operator in Δ, then $T \in \Delta$. In fact a stronger result (Proposition 1) holds. First recall that an eigenvalue for a linear operator is called simple if the corresponding eigenspace has dimension one. One can easily verify that if $T \in \Delta$ has associated sequence $\{c_n\}$, then the eigenvalues of T are simple if and only if the sequence $\{c_n\}$ has no repeated terms. Notice that Δ does actually contain operators with only simple eigenvalues. For example, let $c_n = (1/2)^n$. Proposition 1 follows readily.

PROPOSITION 1. *If $T \in [\beta : \beta]$ commutes with an operator $V \in \Delta$ whose eigenvalues are all simple, then $T \in \Delta$.*

COROLLARY 4. *Let $g(x)$ be a fixed nonconstant analytic function mapping D into D. Let $G = \{\phi_{g(x)} : x \in D\}$. Then* $\mathrm{Comm}(U_G) = \Delta$.

This follows because for any such function $g(x)$, $|g(x)| < 1$ for $x \in D$ and hence $U_{\phi_{g(x)}} \in \Delta$ and has simple eigenvalues.

PROPOSITION 2. *Let $T \in [\beta : \beta]$. Then* $\mathrm{Comm}(T) = \Delta$ *if and only if $T \in \Delta$ and the eigenvalues of T are all simple.*

DEFINITION. For $|a| \leqq 1$, define the translation operator U_a by $U_a f(z) = f(az)$.

The operator U_a is in Δ for $|a| \leqq 1$, and it has associated sequence $\{a^n\}$. Thus if $a^n \neq 1$, $\forall\, n$, then $\mathrm{Comm}(U_a) = \Delta$. For example, this holds for an operator U_a with $|a| < 1$. If $|a| = 1$ and $a^{n_0} = 1$, then Δ is a proper subset of $\mathrm{Comm}(U_a)$, moreover:

PROPOSITION 3. *Let $T \in [\beta : \beta]$ and $a^{n_0} = 1$. Then $T \in \mathrm{Comm}(U_a)$ if and only if $u_n^{(m)} = 0$ whenever $n - m \neq n_0 s$ for s an integer, where $T(z^n) = u_n(z)$.*

It is not known which operators with these sorts of gaps in the sequence $\{c_n\}$ are in $[\beta : \beta]$. However, there certainly are some in $[\beta : \beta]$ which are not in Δ. For example, for a fixed n_0, define the operator T by

$$T(z^n) = \begin{cases} 0 & n \neq 1 \\ z + z^{1+n_0} & n = 1. \end{cases}$$

One expects in a given situation, that if G is "large" enough then $\mathrm{Comm}(G)$ will consist only of constant multiples of the identity operator I.

PROPOSITION 4. *Let $G \subseteq [\beta : \beta]$ and assume*

(i) *G contains at least one multiplier all of whose eigenvalues are simple and*

(ii) *G contains one linear fractional transformation U_ϕ where $\phi(z) = (z - a)/(1 - \bar{a}z)$, with $0 < |a| < 1$.*

Then Comm $G = \{cI\colon c \text{ complex}\}$.

Finally we observe that for a given operator $V \in [\beta\colon \beta]$, Comm$(V)$ will contain all power series in V which converge to an element in $[\beta\colon \beta]$. But, in particular, for the operator U_a, for $|a| < 1$, it can be shown that not all operators in Comm(U_a) can be obtained by power series.

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