PACIFIC JOURNAL OF MATHEMATICS
Vol. 64, No 1, 1976

# CERTAIN CONGRUENCES ON ORTHODOX SEMIGROUPS

JANET E. MILLS

Letting $\kappa$ be the minimum unitary-congruence on a regular semigroup $S$ and $\xi$ be the minimum congruence such that $S/\xi$ is a semilattice of groups, it is the purpose of this paper to characterize all regular semigroups for which $\kappa \cap \xi$ is the identity relation. That is, we describe all regular semigroups which are subdirect products of a unitary semigroup and a semilattice of groups. In the process of doing this, a description of $\xi$ is given for any orthodox semigroup.

In A. H. Clifford's paper on radicals in semigroups, **[2]**, a diagram was given presenting the relationship between various classes of regular semigroups and certain minimum congruences. Two questions were left open. The first was to find all subdirect products of a band and a semilattice of groups, that is, all semigroups for which $\beta \cap \xi$ is the identity, where $\beta$ is the minimum band-congruence. This was solved by Schein in **[14]** and also by Petrich in Theorem 3.2 of **[11]**. The second question involves finding all subdirect products of a semilattice of groups and a regular semigroup whose set of idempotents is unitary. In this paper we find that any such semigroup can be described as a semilattice of unitary semigroups on which $\mathcal{H} \cap \sigma$ is a unitary-congruence, where $\sigma$ is the minimum group-congruence. A description will also be given in terms of restrictions on the structure homomorphisms. In order to accomplish this, we first give an explicit characterization of $\xi$, the minimum semilattice of groups-congruence, on any orthodox semigroup.

## 1. Preliminary results.

For a regular semigroup $S$, $E_S$ denotes the set of idempotents of $S$. If $E_s$ is a subsemigroup then $S$ is said to be orthodox.

PROPOSITION 1.1. **[11**; Proposition 2.5] *On a regular semigroup $S$, the following are equivalent: for $s, t \in S$,*

(i) $e, es \in E_S$ *implies* $s \in E_S$;
(ii) $e, se \in E_S$ *implies* $s \in E_S$;
(iii) $e, ese \in E_S$ *implies* $s \in E_S$;
(iv) $e, set \in E_S$ *implies* $st \in E_S$;
(v) $ese = e \in E_S$ *implies* $s \in E_S$.

If any one of these five conditions holds, then $E_S$ is said to be *unitary*. For brevity, we shall call $S$ unitary if $E_S$ is a unitary subset of $S$. It is easily seen that any unitary semigroup is an orthodox semigroup. For inverse semigroups, those whose idempotents satisfy condition (v) are called proper by McAlister in **[8]** and **[9]**, and he has given a description of all such inverse semigroups in terms of partially ordered sets and groups **[9]**. Unitary semigroups are not closed under homomorphisms; in fact, as is shown in **[8]**, every inverse semigroup is an idempotent-separating homomorphic image of a unitary inverse semigroup.

A congruence on a semigroup is an equivalence relation which is compatible with multiplication. For two congruences $\rho$, $\rho'$ on a semigroup $S$, $\rho \subseteq \rho'$ if $a\rho b$ implies $a\,\rho'\,b$. The identity relation on $S$ will be denoted by $\iota$, or $\iota_S$, if emphasis is needed. For the basic properties of congruences, the reader is referred to [**3**; §1.4, §1.5].

For a class $\mathscr{C}$ of semigroups, a congruence $\rho$ is called a $\mathscr{C}$-congruence on $S$ if $S/\rho$ is in $\mathscr{C}$. For a regular semigroup, the following notation will be used:

$\kappa$ = the minimum unitary-congruence,
$\beta$ = the minimum band-congruence,
$\eta$ = the minimum semilattice-congruence,
$\mathscr{Y}$ = the minimum inverse-congruence,
$\xi$ = the minimum semilattice of groups-congruence,
$\sigma$ = the minimum group-congruence,
$\mu$ = the maximum idempotent-separating congruence.

The Green relations will be noted as usual, and for brevity, a semilattice of groups-congruence will be called a $SG$-congruence. That each of the above minimum congruences exists is explained in **[5]**, and also noted there are some of the following relationships which will be useful here:

$$\mu \subseteq \mathscr{H} \subseteq \beta \subseteq \eta; \qquad \kappa \subseteq \beta \cap \sigma; \qquad \xi \subseteq \eta \cap \sigma.$$

The following result will be needed for later work.

LEMMA 1.2. *Let $\mathscr{B}$ be a class of regular semigroups and $\mathscr{C}$ be a subclass of $\mathscr{B}$ such that for any $S \in \mathscr{B}$, the minimum $\mathscr{C}$-congruence on $S$, $\rho$, exists. If $\tau$ is any congruence defined on all semigroups in $\mathscr{B}$ such that $\tau$ is the identity on any $\mathscr{C}$-semigroup, then $\tau \subseteq \rho$ in $\mathscr{B}$.*

*Proof.* Let $\tau$ be such a congruence, $S \in \mathscr{B}$. Then $\tau \vee \rho$ is a congruence and $\tau \vee \rho = \iota$ on any $\mathscr{C}$-semigroup. Now $S/\rho$ is a $\mathscr{C}$-semigroup and $\rho \subseteq \tau \vee \rho$, so $(S/\rho)/(\tau \vee \rho)/\rho = S/\rho/\iota = S/\rho$. On the other hand, $(S/\rho)/(\tau \vee \rho)/\rho = S/(\tau \vee \rho)$. Hence $S/\rho \simeq S/(\tau \vee \rho)$. Therefore $\rho = \tau \vee \rho$ and $\tau \subseteq \rho$.

Let $\{S_\alpha\}_{\alpha \in A}$ be a family of semigroups and $T$ be a subsemigroup of the direct product $\Pi_{\alpha \in A} S_\alpha$. For each $\alpha \in A$, $\pi_\alpha$ is the natural projection of $T$ into $S_\alpha$. A semigroup $S$ is a subdirect product of $S_\alpha$, $\alpha \in A$, if $S$ is isomorphic to a subsemigroup $T$ of $\Pi_{\alpha \in A} S_\alpha$ such that $T\pi_\alpha = S_\alpha$ for all $\alpha$ in $A$. For the particular case we are interested in, the relationship between congruences and subdirect products is as follows (see [**10**; II.1.4]). For congruences $\lambda$, $\rho$ on a semigroup $S$, $S$ is a subdirect product of $S/\lambda$ and $S/\rho$ if and only if $\lambda \cap \rho = \iota_S$. The aim of this paper is to describe all subdirect products of a unitary semigroup and a semilattice of groups. It is evident that this is equivalent to finding all semigroups for which $\kappa \cap \xi$ is the identity congruence. It is the latter attack that we shall make. From now on, we will assume that all semigroups are regular.

**2. The minimum semilattice of groups-congruence.** It was shown in [**5**] that $\eta \cap \sigma$ is the smallest congruence $\rho$ such that $S/\rho$ is a semilattice of groups and is unitary. Thus, in general, $\xi$ is strictly contained in $\eta \cap \sigma$. Recall [**7**; Theorem 3.1] that on an orthodox semigroup $S$,

$$a \sigma b \leftrightarrow eae = ebe \text{ for some } e \text{ in } E_S.$$

To find $\xi$ on any orthodox semigroup, we first describe $\xi$ on any inverse semigroup and extend it to an orthodox semigroup via the method developed in Theorem 3.1 of [**7**].

THEOREM 2.1. *Let $S$ be an inverse semigroup. The minimum SG-congruence $\xi$ on $S$ can be defined as follows*:

$$a\xi b \leftrightarrow a\eta b \quad \textit{and} \quad ea = eb \quad \textit{for some} \quad e^2 = e\eta a.$$

*Proof.* Let $a\tau b$ if and only if $a\eta b$ and $ea = eb$ for some $e^2 = e\eta a$. It is easily seen that $\tau$ is an equivalence relation on $S$. Let $a\tau b$ and $x$ be in $S$. Then $a\eta b$ and $ea = eb$ for some $e^2 = e\eta a$. Since $\eta$ is a congruence, $ax\eta bx$. Let $f$ be any idempotent such that $f\eta x$. Then $ef\eta ax$ and

$$(ef)(ax) = f(ea)x = f(eb)x = (fe)(bx) = (ef)(bx);$$

therefore, $ax\tau bx$. On the other hand, $xa\eta xb$ and $xe\eta xa$. Thus, since $\eta$ is a semilattice congruence, $xex^{-1}\eta xa$. In addition,

$$(xex^{-1})(xa) = xe(x^{-1}x)a = xea = xeb = (xex^{-1})(xb);$$

that is, $xa\tau xb$.

To see that $S/\tau$ is a semilattice of groups, it is sufficient to show that $aa^{-1}\tau a^{-1}a$ for all $a$ in $S$. But this is clear by letting $e = (aa^{-1})(a^{-1}a)$. Therefore, $\xi \subseteq \tau$.

Now if $S$ is in fact a semilattice of groups then $\eta = \mathscr{H}$ and $\tau$ is clearly the identity on $S$. Hence by Lemma 1.2, $\tau \subseteq \xi$. Consequently $\tau = \xi$.

On an orthodox semigroup, the minimum inverse-congruence $\mathcal{Y}$ has been described by Hall **[4]** and Schein **[13]** as follows:

$$a\mathcal{Y}b \leftrightarrow V(a) = V(b),$$

where $V(x)$ is the set of all inverses of $x$. For any $a$, $a\mathcal{Y}$ will denote the $\mathcal{Y}$-class containing $a$.

THEOREM 2.2. *Let $S$ be an orthodox semigroup. Then $\xi$ can be defined on $S$ as follows*:

$$a\xi b \leftrightarrow a\eta b \quad \text{and} \quad eae = ebe \quad \text{for some} \quad e^2 = e\eta a.$$

*Proof.* Since $\mathcal{Y}$ is the minimum inverse-congruence then $\mathcal{Y} \subseteq \eta$. Thus for $a, b \in S$, $a\mathcal{Y}\eta b\mathcal{Y}$ implies $a\eta b$.

Now, $S/\mathcal{Y}$ is the maximum inverse homomorphic image of $S$, and therefore, letting $\xi'$ be the minimum $SG$-congruence on $S/\mathcal{Y}$, we have, via Theorem 2.1,

$$(*) \qquad a\xi b \leftrightarrow a\mathcal{Y}\xi' b\mathcal{Y} \leftrightarrow a\mathcal{Y}\eta b\mathcal{Y} \quad \text{and} \quad x\mathcal{Y}a\mathcal{Y} = x\mathcal{Y}b\mathcal{Y},$$

for some $(x\mathcal{Y})^2 = x\mathcal{Y}$ with $x\mathcal{Y}\eta a\mathcal{Y}$.

Since $(x\mathcal{Y})^2 = x\mathcal{Y}$, there exists an idempotent $f$ such that $f\mathcal{Y}x$, and thus $f\eta a$. Therefore, using the fact that $\mathcal{Y}$ is a congruence, $(*)$ is equivalent to

$$a\mathcal{Y}\eta b\mathcal{Y} \quad \text{and} \quad (fa)\mathcal{Y} = (fb)\mathcal{Y} \quad \text{for some} \quad f^2 = f\eta a.$$

By definition of $\mathcal{Y}$, this means

$$a\xi b \leftrightarrow a\eta b \quad \text{and} \quad V(fa) = V(fb) \quad \text{for some} \quad f^2 = f\eta a.$$

The rest of the proof that $\xi$ can be defined as in the statement of the theorem is very similar to that of Lemma 3.2 of **[7]**, using the additional fact that $\eta$ is a semilattice-congruence.

COROLLARY 2.3. (See **[14]** or [**11**; Theorem 3.2].) *Let $S$ be a regular semigroup. Then $\beta \cap \xi = \iota_S$ if and only if $S$ is an orthodox band of groups.*

*Proof.* Let $\beta \cap \xi = \iota_S$. Now we know that $\mathcal{H} \subseteq \beta \subseteq \eta$. We will show that $\beta$ is idempotent-separating. Let $e$ and $f$ be idempotents with $e\beta f$. Then $e\eta f$ and $(ef)e(ef) = (ef)f(ef)$ with $ef\eta e$. That is, by Theorem 2.2, $e\xi f$. Since $\beta \cap \xi = \iota_S$, we have $e = f$. Therefore, $\beta \subseteq \mu \subseteq \mathcal{H}$.

But $\mathcal{H} \subseteq \beta$ so $\beta = \mathcal{H}$ and $S$ is a band of groups. The converse follows easily from the fact that $\beta = \mathcal{H}$ and $\mathcal{H} \cap \xi = \iota$.

**3.** $\kappa \cap \xi = \iota$. In this section we characterize those semigroups $S$ which are a subdirect product of a unitary semigroup and a semilattice of groups, that is, those semigroups $S$ for which $\kappa \cap \xi$ is the identity. Clearly, since a unitary semigroup and a semilattice of groups are both orthodox, then $S$ is again an orthodox semigroup.

Recall that a semigroup is $\eta$-simple if it has exactly one $\eta$-class.

LEMMA 3.1. *Let $S$ be an $\eta$-simple orthodox semigroup. Then $S$ is unitary if and only if $\kappa \cap \xi = \iota_S$.*

*Proof.* If $S$ is unitary then $\kappa = \iota_S$ so $\kappa \cap \xi = \iota_S$. Conversely, let $\kappa \cap \xi = \iota_S$. Then, using Theorem 2.2 and the fact that $S$ is $\eta$-simple, we have

$$a \xi b \leftrightarrow a \eta b \quad \text{and} \quad eae = ebe \quad \text{for some} \quad e^2 = e \eta a$$
$$\leftrightarrow eae = ebe \quad \text{for some} \quad e^2 = e \leftrightarrow a \sigma b.$$

That is, $\xi = \sigma$ and $\kappa \cap \sigma = \iota_S$. But $\sigma$ is a unitary congruence so $\kappa \cap \sigma = \kappa$. Therefore $\kappa = \iota_S$ and $S$ is unitary.

LEMMA 3.2. *Let $S$ be a semigroup with $\kappa \cap \xi = \iota_S$. Then $S$ is a semilattice of $\eta$-simple unitary semigroups.*

*Proof.* Since $\eta$ is a semilattice-congruence, we know that $S$ is a semilattice $Y$ of $\eta$-simple semigroups $S_\alpha$, $\alpha \in Y$. Now, on $S_\alpha$, $\kappa \mid S_\alpha$ is a unitary-congruence and $\xi \mid S_\alpha$ is a $SG$-congruence. Hence, on $S_\alpha$, $(\kappa \mid S_\alpha) \cap (\xi \mid S_\alpha) = \iota$, and thus the intersection of the minimum unitary-congruence on $S_\alpha$ and the minimum $SG$-congruence on $S_\alpha$ is also the identity. By Lemma 3.1, $S_\alpha$ is unitary.

LEMMA 3.3 [**5**; Theorem 3.9]. *If $S$ is a unitary semigroup then $\mathcal{H} \cap \sigma = \iota_S$.*

LEMMA 3.4. *Let $S$ be a regular semigroup. Then $\mathcal{H} \cap \sigma \subseteq \kappa$.*

*Proof.* Let $\mathcal{K}$ be the class of all unitary semigroups. Letting $\tau$ be the congruence generated by $\mathcal{H} \cap \sigma$, then $\tau = \iota_S$ for any $S \in \mathcal{K}$, by Lemma 3.3. Therefore, by Lemma 1.2, $\tau \subseteq \kappa$. That is, $\mathcal{H} \cap \sigma \subseteq \kappa$.

THEOREM 3.5. *Let $S$ be a regular semigroup. The following statements are equivalent.*

(i) $\kappa \cap \xi = \iota_S$, *where* $\kappa$ *is the minimum unitary-congruence and* $\xi$ *is the minimum SG-congruence.*

(ii) *S is a semilattice of unitary semigroups and* $\kappa = \mathcal{H} \cap \sigma = \mu \cap \sigma$.

(iii) *S is a semilattice of unitary semigroups and* $\mathcal{H} \cap \sigma$ *is a unitary congruence on S.*

(iv) *S is a subdirect product of a unitary semigroup and a semilattice of groups.*

*Proof.* (i) implies (ii). Let $\kappa \cap \xi = \iota$. By Lemma 3.2, $S$ is a semilattice of unitary semigroups. Since $\eta$ and $\sigma$ are both unitary-congruences, $\kappa$ is contained in both $\eta$ and $\sigma$. We shall show that $\kappa$ is idempotent-separating. For, let $e\kappa f$, with $e, f \in E_S$. Then $e\eta f$ and $(ef)e(ef) = (ef)f(ef)$ with $ef\eta e$. That is, by Theorem 2.2, $e\xi f$. Since $\kappa \cap \xi = \iota$, then $e = f$. Hence $\kappa$ is idempotent-separating and $\kappa \subseteq \mu$. Consequently, using Lemma 3.4, we have $\mathcal{H} \cap \sigma \subseteq \kappa \subseteq \mu \cap \sigma$. But $\mu \subseteq \mathcal{H}$, so equality holds.

(ii) implies (iii). Clear.

(iii) implies (iv). Let $S$ be a semilattice of $\eta$-simple unitary semigroups $S_\alpha$, $\alpha \in Y$, with $\mathcal{H} \cap \sigma$ unitary. Then by Lemma 3.4, $\kappa = \mathcal{H} \cap \sigma$. Therefore

$$\kappa \cap \xi = (\mathcal{H} \cap \sigma) \cap \xi = \mathcal{H} \cap (\sigma \cap \xi) = \mathcal{H} \cap \xi.$$

Let $a\mathcal{H} \cap \xi b$. Then $a, b \in S_\alpha$ for some $\alpha$. Thus, since $S_\alpha$ is $\eta$-simple, in $S_\alpha$, $a\mathcal{H} \cap \sigma b$. But $S_\alpha$ is unitary, so by Lemma 3.3, on $S_\alpha$, $\mathcal{H} \cap \sigma = \iota$. Hence $a = b$. Consequently $\kappa \cap \xi = \iota_S$. Therefore $S$ is a subdirect product of $S/\kappa$ and $S/\xi$ (see II.1.4 of **[10]**).

(iv) implies (i). Let $S$ be a subdirect product of a unitary semigroup $U$ and a semilattice of groups $T$. Then the congruences induced on $S$ by the two projection maps are, respectively, a unitary-congruence $\lambda$, and a $SG$-congruence, $\rho$, and $\lambda \cap \rho = \iota_S$. Thus $\kappa \cap \xi \subseteq \lambda \cap \rho = \iota_S$.

It is not possible to eliminate either one of the two conditions:

(1) $S$ is a semilattice of unitary semigroups,

(2) $\mathcal{H} \cap \sigma$ is unitary.

For, any unitary semigroup (which is not a group) with a zero adjoined, satisfies (1), but for such a semigroup, $\kappa = \beta$ and $\beta \cap \xi \neq \iota$ by Corollary 2.3. On the other hand, let $S = B(G, \alpha)$ be any bisimple $\omega$-semigroup for which $\alpha$ is not one-to-one. Then $S$ is not unitary but $\kappa = \mathcal{H} \cap \sigma$ and $\kappa \cap \xi = \kappa \cap \sigma = \mathcal{H} \cap \sigma \neq \iota$.

COROLLARY 3.6. *Let S be a fundamental regular semigroup. Then* $\kappa \cap \xi = \iota$ *if and only if S is unitary.*

*Proof.* Since $S$ is fundamental, then $\mu = \iota$. Therefore, $\mu \cap \sigma = \iota$.

Every regular semigroup which is a semilattice $Y$ of semigroups $S_\alpha$, $\alpha \in Y$, can be constructed via certain homomorphisms $\phi_{\alpha,\beta}$ from $S_\alpha$ into $\Omega(S_\beta)$, the translational hull of $S_\beta$, for all $\alpha > \beta$; in this case, we shall denote $S$ by $(Y, S_\alpha, \phi_{\alpha,\beta})$. For a full description of this structure the reader is referred to [**10**; III.7.5]. In light of Theorem 3.5, it is of interest to know how the condition $\kappa = \mathcal{H} \cap \sigma$ can be expressed in terms of the structure homomorphisms $\phi_{\alpha,\beta}$. To do this we need to explore the translational hull $\Omega(T)$ of a unitary semigroup $T$. For the elementary properties of the translational hull, see Chapter V of [**10**]. Recall [**10**; III.7.5] that in $S = (Y, S_\alpha, \phi_{\alpha,\beta})$, if $s \in S_\alpha$, $t \in S_\beta$, with $\alpha > \beta$ then

$$st = \phi^s_{\alpha,\beta} t = \lambda^s t \quad \text{and} \quad ts = t\phi^s_{\alpha,\beta} = t\rho^s,$$

where $\phi^s_{\alpha,\beta} = (\lambda^s, \rho^s) \in \Omega(S_\beta)$.

LEMMA 3.7. *Let $S$ be unitary and $(\lambda, \rho) \in \Omega(S)$. If there exists an idempotent $e$ such that $\lambda e \in E_S$ or $e\rho \in E_S$ then $\lambda(E_S) \subseteq E_S$, $(E_S)\rho \subseteq E_S$.*

*Proof.* Let $e$ and $\lambda e$ be in $E_S$. Then $(e\rho)e = e(\lambda e) \in E_S$, and since $S$ is unitary, by Proposition 1.1, $e\rho \in E_S$.

Let $f$ be in $E_S$. Then $e(\lambda f) = (e\rho)f \in E_S$, so again $\lambda f$ is in $E_S$. Thus $\lambda(E_S) \subseteq E_S$. Since $\lambda f \in E_S$ implies $f\rho \in E_S$, then also $(E_S)\rho \subseteq E_S$.

LEMMA 3.8. *Let $S$ be a unitary semigroup. Define*

$$K(S) = \{(\lambda, \rho) \in \Omega(S) \mid \lambda(E_S) \subseteq E_S,\ (E_S)\rho \subseteq E_S\}.$$

*Then $K(S)$ is a subsemigroup of $\Omega(S)$ which contains $E_{\Omega(S)}$.*

*Proof.* That $K(S)$ is a semigroup is clear. Let $(\lambda, \rho)$ be in $E_{\Omega(S)}$. Then $\lambda^2 = \lambda$, $\rho^2 = \rho$. Let $a \in S$ and $\lambda a = b$. Let $b'$ be an inverse of $b$; then we have $\lambda(ab') = (\lambda a)b' = bb' \in E_S$. Hence there exists $x$ in $S$ such that $\lambda x = f \in E_S$. Moreover, $f = \lambda x = \lambda^2 x = \lambda(\lambda x) = \lambda f$. Therefore $\lambda f \in E_S$, and by Lemma 3.7, $\lambda(E_S) \subseteq E_S$. Similarly $(E_S)\rho \subseteq E_S$.

If $S$ is an inverse semigroup then $E_{\Omega(S)} = K(S)$, [**1**; Lemma 2.1]. However, in general, strict containment is possible. For, if $S$ is a rectangular group, we may assume $S = L \times G \times R$ where $L$ $(R)$ is a left (right) zero semigroup and $G$ is a group. Then $\Omega(S) = T(L) \times G \times T'(R)$, where $T(L)$ $(T'(R))$ is the semigroup of all transformations of $L$ $(R)$ written on the left (right) [**10**; V.3.12]. Under this isomorphism,

$$E_{\Omega(S)} = \{(f, 1, f') \mid f, f' \text{ are retractions}\},$$

where a retraction is any mapping which is the identiy on its range. On the other hand, $K(S) = T(L) \times 1 \times T'(R)$ which is not equal to $E_{\Omega(S)}$.

From **[5]** we recall that a congruence $\tau$ is unitary if $x^2 \tau x$ and $(sx)^2 \tau sx$ implies $s^2 \tau s$. For regular semigroups this is equivalent to:

$$\text{for } e, f \in E_S, \ se\tau f \text{ implies } s^2 \tau s.$$

We now explore the properties of $\phi_{\alpha,\beta}$ which make $\kappa \cap \xi$ the identity. For a semigroup $S_\alpha$, we denote $E_{S_\alpha}$ by $E_\alpha$, and $K(S_\alpha)$ by $\dot{K}_\alpha$. For $S = (Y, S_\alpha, \phi_{\alpha,\beta})$ a semilattice of unitary semigroups $S_\alpha$, let $\Gamma_{\alpha,\beta} = K_\beta \phi^{-1}_{\alpha,\beta}$ for all $\alpha > \beta$ and $\Gamma = \bigcup_{\alpha > \beta} \Gamma_{\alpha,\beta}$.

THEOREM 3.9. *Let $S$ be a regular semigroup. Then $\kappa \cap \xi = \iota_S$ if and only if $S = (Y, S_\alpha, \phi_{\alpha,\beta})$ is a semilattice of unitary semigroups satisfying the properties*:

(i) *$\Gamma$ is a band of groups,*

(ii) *for $s$ in $\Gamma \cap H_e$, $e \in E_\alpha$, if $f < e$ with $f \in E_\beta$ then $\phi^s_{\alpha,\beta}(f) \mathscr{H} f$, where $\phi^s_{\alpha,\beta}(f)$ means both $\phi^s_{\alpha,\beta} f$ and $f \phi^s_{\alpha,\beta}$.*

*Proof.* Let $\kappa \cap \xi = \iota_S$. Then $\mathscr{H} \cap \sigma = \mu \cap \sigma$ is unitary. Let $\alpha > \beta$ and $s$ be in $S_\alpha$ with $\phi^s_{\alpha,\beta} \in K_\beta$. Then $\phi^s_{\alpha,\beta} = (\lambda, \rho)$ and $\lambda(E_\beta) \subseteq E_\beta$. Let $g$ be in $E_\beta$. Then $\lambda g = f = f^2 \in E_\beta$, so by definition of multiplication in $S$, $sg = \phi^s_{\alpha,\beta} g = \lambda g = f$. Hence $se\mu \cap \sigma f$, and $\mu \cap \sigma$ is unitary, so $s\mu \cap \sigma s^2$. Thus $s$ is contained in a group. Now since $\mu \cap \sigma$ is a congruence, the $\mu \cap \sigma$-classes which contain idempotents form a band of groups, $T$, and since $\phi_{\alpha,\beta}$ is a homomorphism, then $K_\beta \phi^{-1}_{\alpha,\beta} = \Gamma_{\alpha,\beta}$ is a band of groups contained in $T$. Thus $\Gamma$ is a band of groups.

Now let $s$ be in $\Gamma_{\alpha,\beta}$. Then $s$ is in a group so there exists an idempotent $h$ such that $ss' = s's = h$ for some $s' \in V(s)$. Since $s\mu s^2$, then $s\mu h$ and for all idempotents $f$, $sfs' = hfh$, $s'fs = hfh$, **[6]**.

Let $f < h$, $f \in S_\gamma$. Then $sfs' = f$ and $fs'sf = fhf = f$. Thus if $\phi^s_{\alpha,\gamma} = (\lambda^s, \rho^s)$ and $\phi^{s'}_{\alpha,\gamma} = (\lambda^{s'}, \rho^{s'})$, $f = fs'sf = (f\rho^{s'})(\lambda^s f)$. Therefore,

$$f\rho^{s'} = f(f\rho^{s'}) = (f\rho^{s'})(\lambda^s f)(f\rho^{s'}), \qquad \lambda^s f = (\lambda^s f) f = (\lambda^s f)(f\rho^{s'})(\lambda^s f);$$

that is, $f\rho^{s'}$ is an inverse of $\lambda^s f$. Now $sfs' = f = fs'sf$ can be expressed by

$$(\lambda^s f)(f\rho^{s'}) = f = (f\rho^{s'})(\lambda^s f).$$

Thus $\lambda^s f \mathscr{H} f$. By considering $s'fs = fss'f = f$ we find $f\rho^s \mathscr{H} f$. Thus $\phi^s_{\alpha,\gamma}(f) \mathscr{H} f$.

Conversely, to show $\kappa \cap \xi = \iota_S$, using Theorem 3.5, we need only show that $\mu \cap \sigma$ is a unitary congruence. Let $se\mu \cap \sigma f$ with $e, f \in E_S$.

Letting $s$ be in $S_\alpha$, $e$ in $E_\beta$, then $f \in E_{\alpha\beta}$. Since $se\sigma f$, there exists $g \in E_\gamma$, $\gamma \leqq \alpha\beta$ such that $g(se)g = gfg \in E_\gamma$. This means that $(g\phi^s_{\alpha,\gamma})(\phi^e_{\beta,\gamma}g) \in E_\gamma$. Since $e$ is idempotent so is $\phi^e_{\beta,\gamma}$ and by Lemma 3.8, $\phi^e_{\beta,\gamma}g$ is in $E_\gamma$. Thus since $S_\gamma$ is unitary, $g\phi^s_{\alpha,\gamma}$ is idempotent by Proposition 1.1; by Lemma 3.7, $\phi^s_{\alpha,\gamma}$ is in $K_\gamma$. Consequently by property (i), $s$ is contained in a band $E_\alpha$ of groups $G_b$, $b \in E_\alpha$. In particular, there exists $s' \in V(s)$ such that $ss' = s's = h$ for some $h \in E_\alpha$. We need to show that $s\mu h$, and to do this it is sufficient to show that $sfs' = f$, $s'fs = f$ for all $f \leqq h$. Now if $f$ is in $E_\alpha$, $f \leqq h$, then $s$ and $s'$ are in the group $G_h$ and

$$sfs' \in G_hG_fG_h \subseteq G_{hfh} = G_f,$$

so that $sfs' = f$. Similarly $s'fs = f$. Now let $f$ be in $E_\delta$, $\delta < \alpha$, with $f < h$. Let $\phi^s_{\alpha,\delta} = (\lambda^s, \rho^s)$, $\phi^{s'}_{\alpha,\delta} = (\lambda^{s'}, \rho^{s'})$. By property (ii), $\lambda^s f \mathcal{H} f$, $f\rho^s \mathcal{H} f$. That is, $f(\lambda^s f) = \lambda^s f$, $(f\rho^s)f = f\rho^s$. Now since $f < h = ss'$, then $f = hf = \lambda^{ss'}f = \lambda^s\lambda^{s'}f$, and thus

$$\begin{aligned} sfs' &= (\lambda^s f)(f\rho^{s'}) = f(\lambda^s f)(f\rho^{s'}) = (f\rho^s)f(f\rho^{s'}) = (f\rho^s)(f\rho^{s'}) \\ &= [(f\rho^s)f]\rho^{s'} = (f\rho^s)\rho^{s'} = f\rho^{ss'} = f. \end{aligned}$$

Similarly $s'fs = f$. Therefore $s\mu h$.

Since $\sigma$ is always unitary, $se\sigma f$ implies $s\sigma h$. Consequently, $s\mu \cap \sigma h$, and $\mu \cap \sigma$ is a unitary congruence. By Theorem 3.5, $\kappa \cap \xi = \iota_S$.

A regular semigroup $S = (Y, S_\alpha, \phi_{\alpha,\beta})$ is a strong semilattice of the semigroups $S_\alpha$, if $\phi_{\alpha,\beta}$ maps $S_\alpha$ into $S_\beta$ for all $\alpha > \beta$. The conditions in Theorem 3.9 can be simplified considerably for strong semilattices of unitary semigroups.

COROLLARY 3.10. *Let $S = (Y, S_\alpha, \phi_{\alpha,\beta})$ be a strong semilattice of unitary semigroups $S_\alpha$. Then $\kappa \cap \xi = \iota_S$ if and only if $E_\beta\phi^{-1}_{\alpha,\beta}$ is a band of groups for all $\alpha > \beta$.*

*Proof.* It can be easily seen that $K_\beta \cap \Pi(S_\beta) \simeq E_\beta$ where $\Pi(S_\beta)$ is the semigroup of inner bitranslations of $S_\beta$. Thus if $\phi_{\alpha,\beta}$ maps $S_\alpha$ into $S_\beta \simeq \Pi(S_\beta)$, then $K_\beta\phi^{-1}_{\alpha,\beta} = E_\beta\phi^{-1}_{\alpha,\beta}$. Property (ii) automatically holds since homomorphisms preserve $\mathcal{H}$-classes.

The author would like to express her appreciation to Professor Mario Petrich for the suggestions he made on the preliminary copy of this paper.

## REFERENCES


1. Janet E. Ault, *The translational hull of an inverse semigroup*, Glasgow Math. J., **14** (1973), 56–64.
2. A. H. Clifford, *Radicals in semigroups*, Semigroup Forum, **1** (1970), 103–128.

3. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Amer. Math. Soc., Math. Surveys No. 7, Vol. I, Providence, R.I., 1961.

4. T. E. Hall, *On regular semigroups whose idempotents form a subsemigroup*, Bull. Austral. Math. Soc., **1** (1969), 195–208.

5. J. M. Howie and G. Lallement, *Certain fundamental congruences on a regular semigroup*, Proc. Glasgow Math. Assoc., **7** (1966), 145–159.

6. John Meakin, *Congruences on orthodox semigroups*, J. Austral. Math. Soc., **12** (1971), 323–341.

7. ———, *Congruences on orthodox semigroups II*, J. Austral. Math. Soc., **13** (1972), 259–266.

8. D. B. McAlister, *Groups, semilattices and inverse semigroups*, Trans. Amer. Math. Soc., **192** (1974), 227–244.

9. ———, *Groups, semilattices and inverse semigroups, II*, Trans. Amer. Math. Soc., **196** (1974), 351–370.

10. Mario Petrich, *Introduction to Semigroups*, Charles E. Merrill Publishing Co., Columbus, Ohio, 1973.

11. ———, *Regular semigroups which are subdirect products of a band and a semilattice of groups*, Glasgow Math. J., **14** (1973), 27–49.

12. N. R. Reilly and H. E. Scheiblich, *Congruences on regular semigroups*, Pacific J. Math., **23** (1967), 349–360.

13. B. M. Schein, *On the theory of generalized groups and generalized heaps*, Theory of semigroups and appl. I, 286–324, Izdat. Saratov. Univ., Saratov, 1965 (Russian).

14. ———, *A note on radicals in regular semigroups*, Semigroup Forum, **3** (1971), 84–85.

Received September 29, 1975.

MADISON COLLEGE