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THE CENTRALISER OF $E \otimes_\lambda F$

A. W. WICKSTEAD

If E is a real Banach space then $\mathscr{B}(E)$ is the space of all bounded linear operators on E, and $\mathscr{Z}(E)$ the subspace of M-bounded operators, i.e. the centraliser of E. Two Banach spaces E and F are considered as well as the tensor product $E \otimes_\lambda F$. There is a natural mapping of the algebraic tensor product $\mathscr{Z}(E) \odot \mathscr{Z}(F)$ into $\mathscr{Z}(E \otimes_\lambda F)$. It is shown that $\mathscr{Z}(E \otimes_\lambda F)$ is precisely the strong operator closure, in $\mathscr{B}(E \otimes_\lambda F)$, of its image.

1. Definitions and statement of results. A linear operator T on a real Banach space E is *M-bounded* if there is $\lambda > 0$ such that if $e \in E$ and D is a closed ball in E containing λe and $-\lambda e$, then $Te \in D$. The *centraliser* of E, $\mathscr{Z}(E)$, is the commutative Banach algebra of all M-bounded linear operators on E. Let K denote the unit ball of E^*, the Banach dual of E, equipped with the weak* topology. We denote the set of extreme points of a convex set C by $\mathscr{E}(C)$. In [2], Theorem 4.8 it is shown that a bounded linear operator T on E is M-bounded if and only if each point of $\mathscr{E}(K)$ is an eigenvalue for T^*, the adjoint of T. Thus there is a real valued function $\tilde{T}$ on $\mathscr{E}(K)$ such that $T^*p = \tilde{T}(p)p (p \in \mathscr{E}(K))$.

An *L-ideal* in a real Banach space is a subspace I with a complementary direct summand J such that $\|i\| + \|j\| = \|i + j\| (i \in I, j \in J)$. The sets $I \cap \mathscr{E}(K)$ for I a weak*-closed L-ideal in E^* form the closed sets of the structure topology on $\mathscr{E}(K)$. The map $T \mapsto \tilde{T}$ is an isometric algebra isomorphism of $\mathscr{Z}(E)$ onto the bounded structurally continuous real valued functions on $\mathscr{E}(K)$ with the supremum norm and pointwise multiplication ([2], Theorem 4.9).

We shall consider two Banach spaces E and F, K will retain its meaning and M will denote the corresponding subset of F^*. We use $E \odot F$ to denote the algebraic tensor product of E and F. We shall consider the norm

$$\left\| \sum_{i=1}^n e_i \otimes f_i \right\|_\lambda = \sup \left\{ \left| \sum_{i=1}^n k(e_i) m(f_i) \right| : k \in K,\ m \in M \right\} .$$

$E \odot_\lambda F$ will denote $E \odot F$ with this norm, and $E \otimes_\lambda F$ its completion.

We may identify $E \otimes_\lambda F$ concretely in a number of ways. The formula $(k, m) \mapsto \sum_{i=1}^n k(e_i) m(f_i)$ defines a real valued function on $K \times M$. Such functions are continuous and affine in each variable. $\|\sum_{i=1}^n e_i \otimes f_i\|_\lambda$ is the same as the supermum norm for such a function, so we may identify $E \otimes_\lambda F$ with a subspace H, the closure of these functions, in $C(K \times M)$, the continuous real valued functions on $K \times M$. We shall have need to call upon:

LEMMA. *Every extreme point of the unit ball of H^* is of the form $h \mapsto h(p, q)(p \in \mathscr{E}(K), q \in \mathscr{E}(M))$.*

Let $R: C(K \times M)^* \to H^*$ be the restriction map, and let B be the unit ball of $C(K \times M)^*$. If f is an extreme point of the unit ball of H^*, then $R^{-1}f \cap B$ is a weak* closed face of B which is nonempty by the Hahn-Banach theorem. By the Krein-Milman theorem, $R^{-1}f \cap B$ has an extreme point, which must be extreme in the unit ball of $C(K \times M)^*$, so is of the form $h \mapsto \pm h(p, q)$ for $p \in K, q \in M$. By replacing p *by* $-p$, if necessary, we may ensure a positive sign. If p (say) is not extreme, then $p = 1/2(p_1 + p_2)$, $p_1, p_2 \in K$, $p_1 \neq p_2$. $h(p, q) = 1/2h(p_1, q) + 1/2h(p_2, q)(h \in H)$ as these functions are affine in each variable. As the functions of H separate the points of $K \times M$, this contradicts the extremality.

COROLLARY.

$$\left\| \sum_{i=1}^{n} e_i \otimes f_i \right\|_\lambda = \sup \left\{ \left| \sum_{i=1}^{n} p(e_i) q(f_i) \right| : p \in \mathscr{E}(K), q \in \mathscr{E}(M) \right\} .$$

We consider the centraliser of $E \otimes_\lambda F$. We have quite easily:

PROPOSITION. *If $S_i \in \mathscr{Z}(E)$, $T_i \in \mathscr{Z}(F)(1 \leqq i \leqq n)$ there is $U \in \mathscr{Z}(E \otimes_\lambda F)$ such that if $e_j \in E, f_j \in F(1 \leqq j \leqq m)$ then $U(\sum_{j=1}^{m} e_j \otimes f_j) = \sum_{j=1}^{m} \sum_{i=1}^{n} (S_i e_j) \otimes (T_i e_j)$.*

To show that U exists (as a bounded linear operator) we need only show that the linear operator defined on $E \odot_\lambda F$ by this formula is bounded. This is so because,

$$\begin{aligned}
&\left\| \sum_{i,j} (S_i e_j) \otimes (T_i f_j) \right\|_\lambda \\
&\quad = \sup \left\{ \left| \sum_{i,j} p(S_i e_j) q(T_i e_j) \right| : p \in \mathscr{E}(K), q \in \mathscr{E}(M) \right\} \\
&\quad = \sup \left\{ \left| \sum_{i,j} \widetilde{S}_i(p) \widetilde{T}_i(p) p(e_j) q(f_j) \right| : p \in \mathscr{E}(K), q \in \mathscr{E}(M) \right\} \\
&\quad \leqq \sup \left\{ \sum_i |\widetilde{S}_i(p)| \, |\widetilde{T}_i(p)| \left| \sum_j p(e_j) q(f_j) \right| : p \in \mathscr{E}(K), q \in \mathscr{E}(M) \right\} \\
&\quad \leqq \sum_i \|S_i\| \, \|T_i\| \sup \left\{ \left| \sum_j p(e_j) q(f_j) \right| : p \in \mathscr{E}(K), q \in \mathscr{E}(M) \right\} \\
&\quad = \sum_i \|S_i\| \, \|T_i\| \, \|\sum_j e_j \otimes f_j\|_\lambda .
\end{aligned}$$

It remains to show that each extreme point of the unit ball of $(E \bigotimes_\lambda F)^*$ is an eigenvalue for U^*. If we denote by $p \otimes q$ the functional $\sum_j e_j \otimes f_j \mapsto \sum_j p(e_j)q(f_j)$ then we have

$$\begin{aligned} U^*(p \otimes q)\Big(\sum_j e_j \otimes f_j\Big) &= (p \otimes q)U\Big(\sum_j e_j \otimes f_j\Big) \\ &= (p \otimes q) \sum_{i,j} (S_i e_j) \otimes (T_i f_j) \\ &= \sum_{i,j} p(S_i e_j) q(T_i f_j) \\ &= \sum_{i,j} \tilde{S}_i(p) \tilde{T}_i(p) p(e_j) q(f_j) \\ &= \Big[\sum_i \tilde{S}_i(p) \tilde{T}_i(p)\Big]\Big[(p \otimes q)\Big(\sum_j e_j \otimes f_j\Big)\Big] . \end{aligned}$$

It is immediate that $U^*(p \otimes q) = [\sum_i \tilde{S}_i(p)\tilde{T}_i(p)](p \otimes q)$.

We thus have an embedding of $\mathscr{Z}(E) \odot \mathscr{Z}(F)$ in $\mathscr{Z}(E \bigotimes_\lambda F)$ in an obvious way. The remainder of this paper is devoted to a proof of the following result.

THEOREM. *$\mathscr{Z}(E \bigotimes_\lambda F)$ is the closure, for the strong operator topology, of the canonical copy of $\mathscr{Z}(E) \odot \mathscr{Z}(F)$ in $\mathscr{B}(E \bigotimes_\lambda F)$.*

2. The proof. For this proof we shall identify the element $\sum_{i=1}^n e_i \otimes f_i \in E \odot F$ with the function $k \mapsto \sum_{i=1}^n k(e_i) f_i$ from K into F. This is continuous affine function vanishing at 0. The set of all F-valued continuous affine functions of K which vanish at 0 we shall denote by $A_0(K, F)$, and norm it by $\|a\| = \sup\{\|a(k)\| : k \in K\}$, which corresponds to the norm on $E \odot_\lambda F$. We may thus identify $E \bigotimes_\lambda F$ whith the closure, H, in $A_0(K, F)$ of the functions with finite dimensional range.

If $\sum_{i=1}^n S_i \otimes T_i \in \mathscr{Z}(E) \odot \mathscr{Z}(F)$ then $\pi : p \mapsto \sum_{i=1}^n \tilde{S}_i(p) T_i$ is a function from $\mathscr{E}(K)$ into $\mathscr{Z}(F)$ which is bounded and continuous for the structure topology on $\mathscr{E}(K)$ and the strong operator topology on $\mathscr{Z}(F)$. If U is the image of $\sum_{i=1}^n S_i \otimes T_i$ in $\mathscr{Z}(H)$ (using the proposition and the identification of H with $E \otimes_\lambda F$) then we have

$$(Uh)(p) = \pi(p)h(p) \quad (h \in H,\, p \in \mathscr{E}(K)) .$$

This is because, if $\varepsilon > 0$, we may find $\sum_{j=1}^m e_j \otimes f_j \in E \odot F$ with $\|h - \sum_{j=1}^m e_j \otimes f_j\|_\lambda < \varepsilon$ and then

$$\begin{aligned} \|(Uh)(p) - \pi(p)h(p)\| \leqq \Big\| (Uh)(p) - U\Big(\sum_{j=1}^m e_j \otimes f_j\Big)(p)\Big\| \\ + \Big\| U\Big(\sum_{j=1}^m e_j \otimes f_j\Big)(p) - \pi(p)h(p) \Big\| . \end{aligned}$$

But

$$\begin{aligned} U\Big(\sum_{j=1}^{m} e_j \otimes f_j\Big)(p) &= \sum_{i,j} (S_i e_j) \otimes (T_i e_j)(p) \\ &= \sum_{i,j} p(S_i e_j)(T_i e_j) \\ &= \sum_{i,j} \tilde{S}_i(p) p(e_j)(T_i e_j) \\ &= \Big(\sum_i \tilde{S}_i(p) T_i\Big)\Big(\sum_j p(e_j) f_j\Big) \\ &= \pi(p)\Big(\Big(\sum_{j=1}^{m} e_j \otimes f_j\Big)(p)\Big)\,. \end{aligned}$$

Thus $\|(Uh)(p) - \pi(p)h(p)\| \leqq \|U\|\varepsilon + \|\pi(p)\|\, \|\sum_{j=1}^{m} e_j \otimes f_j - h)(p)\| \leqq (\|U\| + \|\pi(p)\|)\varepsilon$, which can be made as small as desired, so that $(Uh)(p) = \pi(p)h(p)$.

Let $V(K)$ denote the set of extreme points, p, of K for which there is $x \in E$ with $p(x) = \|x\|$, then $V(K)$ is weak* dense in $\mathscr{E}(K)$. To show this it will suffice to prove that $K = \overline{\mathrm{co}}(V(K))$, the weak* closed convex hull of $V(K)$, for then $\mathscr{E}(K) \subset \overline{V(K)}$ by Milman's theorem. If $\overline{\mathrm{co}}(V(K)) \neq K$ we may, by Hahn-Banach separation, find $x \in E$ with $k(x) \leqq \alpha < k_0(x)$ for some real α, all $k \in \overline{\mathrm{co}}(V(K))$ and some $k_0 \in K$. Then $\{k \in K: k(x) = \|x\|\}$ is a nonempty weak* closed face of K. This possesses an extreme point, which cannot lie in $\overline{\mathrm{co}}(V(K))$, yet which is in $V(K)$ by its construction, a contradiction.

If $p \in V(K)$, $q \in V(M)$ then $p \otimes q$ is extreme in the unit ball of $(E \bigotimes_\lambda F)^*$. Fix $e \in E$, $f \in F$ with $\|e\| = e(p) = 1$, $\|f\| = f(p) = 1$. Define injections $P: E \to E \bigotimes_\lambda F$, $Q: F \to E \bigotimes_\lambda F$ by $P(x) = x \otimes f$, $Q(y) = e \otimes y$. P, Q are isometric injections so the image of the unit ball of $(E \bigotimes_\lambda F)^*$ under P^* (respectively Q^*) is K (respectively M). P^*, Q^* are continuous and affine, so $P^{*-1}(p)$ and $Q^{*-1}(q)$ intersect the unit ball of $(E \bigotimes_\lambda F)^*$ in weak* closed faces, as must $P^{*-1}(p) \cap Q^{*-1}(q)$. This intersection is nonempty, for $P^*(p \otimes q) = p$, $Q^*(p \otimes q) = q$. This is because for $x \in E$, $(P^*(p \otimes q))(x) = (p \otimes q)(Px) = (p \otimes q)(x \otimes f) = p(x)q(f) = p(x)$, with a similar proof for Q^*. This face must have an extreme point which is extreme in the unit ball of $(E \bigotimes_\lambda F)^*$, so is $p' \otimes q'$ for $p' \in \mathscr{E}(K)$, $q' \in \mathscr{E}(M)$. But now $p = P^*(p \otimes q) = P^*(p' \otimes q') = p'$ and also $q = q'$, so that $p \otimes q$ is itself extreme.

It follows that if $U \in \mathscr{Z}(H)$ then all points $p \otimes q$ for $p \in \mathscr{E}(K)$, $q \in \mathscr{E}(M)$ are eigenvectors for U^*. For let $p_\gamma \to p$, $q_\delta \to q$ be nets with $p_\gamma \in V(K)$, $q_\delta \in V(M)$. The continuity of the map $(k, m) \mapsto k \otimes m$ from $K \times M$ into $(E \bigotimes_\lambda F)^*$ implies that $p_\gamma \otimes q_\delta \to p \otimes q$. But $U^*(p_\gamma \otimes q_\delta) = \tilde{U}(p_\gamma \otimes q_\delta)(p_\gamma \otimes q_\delta)$. The reals $\tilde{U}(p_\gamma \otimes q_\delta)$ are bounded (by $\|U\|$) so we may suppose (by choosing a subnet if necessary) that $\tilde{U}(p_\gamma \otimes q_\delta) \to \lambda$. Now $U^*(p \otimes q) = \lim U^*(p_\gamma \otimes q_\delta) = \lim \tilde{U}(p_\gamma \otimes q_\delta) \lim (p_\gamma \otimes q_\delta) = \lambda(p \otimes q)$.

Suppose $U \in \mathscr{Z}(H)$, $p \in \mathscr{E}(K)$ and $h, h' \in H$ with $h(p) = h'(p)$. If

$q \in \mathscr{E}(M)$ then

$$\begin{aligned} q((Uh)(p)) &= (p \otimes q)(Uh) = \tilde{U}(p \otimes q)((p \otimes q)(h)) \\ &= \tilde{U}(p \otimes q)(q(h(p))) \\ &= \tilde{U}(p \otimes q)(q(h'(p))) = q((Uh')(p)) . \end{aligned}$$

Thus $(Uh)(p) = (Uh')(p)$. We may thus define a linear operator $\pi(p)$ on F by $\pi(p)y = (Uh)(p)$ whenever $h(p) = y$. $\pi(p)$ is clearly linear, is well defined, and has domain the whole of F since we may take $h = e \otimes y$ where $e(p) = 1$.

$\pi(p)$ has norm at most $\|U\|$, for we may find $e_n \in E$ with $e_n(p) = 1$, $\|e_n\| \leqq (n+1)/n$, and then

$$\begin{aligned} \|\pi(p)y\| = \|U(e_n \otimes y)(p)\| &\leqq \|U(e_n \otimes y)\| \\ &\leqq \|U\| \, \|e_n \otimes y\| = \|U\| \, \|y\|(n+1)/n . \end{aligned}$$

Thus $\|\pi(p)y\| \leqq \|U\| \, \|y\|$. In fact $\pi(p) \in \mathscr{E}(F)$ because if $y \in F$, $q \in \mathscr{E}(M)$ and $e \in E$ with $p(e) = 1$ then

$$\begin{aligned} q(\pi(p)y) &= q(U(e \otimes y)(p)) = (p \otimes q)(U(e \otimes y)) \\ &= \tilde{U}(p \otimes q)(p \otimes q)(e \otimes y) = \tilde{U}(p \otimes q)q(y) . \end{aligned}$$

We thus have a function $\pi: \mathscr{E}(K) \to \mathscr{Z}(F)$ with $(Uh)(p) = \pi(p)h(p)(p \in \mathscr{E}(K))$. Also π is norm bounded, and we let $\|\pi\|$ denote $\sup\{\|\pi(p)\|: p \in \mathscr{E}(K)\}$.

π is continuous for the structure topology on $\mathscr{E}(K)$ and the weak operator topology on $\mathscr{Z}(F)$. Suppose $y \in F$, $g \in F^*$ and $x \in E$ then $k \mapsto g(U(x \otimes y)(k))$ is a continuous affine function on K vanishing at 0, so may be identified with an element of E. If $p \in \mathscr{E}(K)$ then

$$\begin{aligned} g(U(x \otimes y)(p)) &= g(\pi(p)(x \otimes y)(p)) \\ &= g(\pi(p)x(p)y) = x(p)(g(\pi(p)y)) . \end{aligned}$$

Thus $x \mapsto g(U(x \otimes y))$ is an element of $\mathscr{Z}(E)$, so the function $p \mapsto g(\pi(p)y)$ is structurally continuous.

By [2], Proposition 3.10 π has an extension, $\bar{\pi}$, to $\overline{\mathscr{E}(K)}\backslash\{0\}$ which is continuous for the weak* topology on $\overline{\mathscr{E}(K)}\backslash\{0\}$ and the weak operator topology on $\mathscr{Z}(F)$ (the result there is stated for real valued functions but the proof remains valid in this context). We note for later reference that $\pi\mathscr{E}(K)) = \bar{\pi}(\overline{\mathscr{E}(K)}\backslash\{0\})$. We propose now to show $\bar{\pi}$ is still continuous when $\mathscr{Z}(F)$ is given its strong operator topology.

Provisionally we define $\tilde{\pi}(k)$, for $k \in \overline{\mathscr{E}(K)}\backslash\{0\}$, to be that linear operator on F such that

$$\tilde{\pi}(k)y = U(x \otimes y)(k)/k(x)$$

with $x \in E$, $k(x) > 0$. This definition coincides with that of π if $k \in \mathscr{E}(K)$, and is well defined because if $k_\gamma \in \mathscr{E}(K)$ and $k_\gamma \to k$ for the weak* topology then

$$\tilde{\pi}(k)y = U(x \otimes y)(k)/k(x) = \lim U(x \otimes y)(k_\gamma)/k_\gamma(x)$$
$$= \lim \pi(k_\gamma)y \; .$$

Clearly $\tilde{\pi}(k)$ acts linearly on F, and it is bounded because

$$\begin{aligned} ||(\tilde{\pi}(k)y)|| &= ||U(x \otimes y)(k)||/|k(x)| \\ &= \lim ||U(x \otimes y)(k_\gamma)||/|k_\gamma(x)| \\ &= \lim ||\pi(k_\gamma)y|| \leqq ||\pi||\,||y|| \; . \end{aligned}$$

Also $||\tilde{\pi}|| = \sup\{||\pi(k)||: k \in \overline{\mathscr{E}(K)}\backslash\{0\}\} = ||\pi||$. $\tilde{\pi}$ is locally a quotient of a function that is clearly strong operator continuous and a non-vanishing scalar function, so is strong operator continuous. In fact $\tilde{\pi}$ is the same as $\bar{\pi}$ as both are extensions of π to $\overline{\mathscr{E}(K)}\backslash\{0\}$ which are continuous for the weak* topology on $\overline{\mathscr{E}(K)}\backslash\{0\}$ and the weak operator topology on $\mathscr{Z}(F)$.

We do not know if π itself is continuous when $\mathscr{Z}(F)$ is given the strong operator topology. All that we shall require is that if $D \subset \mathscr{E}(K)$ and 0 does not lie in the weak* closure of D, then $\pi|_D$ is continuous for the structure topology on D and the strong operator topology on $\mathscr{Z}(F)$. For suppose d_γ, $d \in D$ and $d_\gamma \to d$ for the structure topology, then $\pi(d_{\gamma'}) \to \pi(d)$ for the weak operator topology whenever $(d_{\gamma'})$ is a subnet of (d_γ). Let $(d_{\gamma''})$ be a weak* convergent subnet of $(d_{\gamma'})$ with limit $d' \neq 0$, which exists as K is weak* compact. Then $\pi(d_{\gamma''}) \to \pi(d)$ for the weak operator topology whilst $\pi(d_{\gamma''}) = \bar{\pi}(d_{\gamma''}) \to \bar{\pi}(d')$ for the strong operator topology, and hence also for the weak operator topology. Thus $\pi(d) = \bar{\pi}(d')$ and $\pi(d_{\gamma''}) \to \pi(d)$ for the strong operator topology. I.e. every subnet of $(\pi(d_\gamma))$ has a subnet converging to $\pi(d)$, so in fact $\pi(d_\gamma) \to \pi(d)$ for the strong operator topology.

We now seek, given $h_i \in H (i = 1, 2, \cdots, n)$ and $\varepsilon > 0$, to find $\pi'\colon \mathscr{E}(K) \to \mathscr{Z}(F)$ which is of finite dimensional range and continuous for the structure topology, such that

$$||\pi'(p)h_i(p) - \pi(p)h_i(p)|| \leqq \varepsilon \quad (p \in \mathscr{E}(K), 1 \leqq i \leqq n) \; .$$

π' is the image of an element of $\mathscr{Z}(E) \odot \mathscr{Z}(F)$ so defines an element U' of the copy of $\mathscr{Z}(E) \odot \mathscr{Z}(F)$ in $\mathscr{E}(E \otimes_\lambda F)$. We then have

$$||(U'h_i)(p) - (Uh_i)(p)|| \leqq \varepsilon \quad (p \in \mathscr{E}(K), 1 \leqq i \leqq n) \; .$$

The function $k \mapsto ||(U'h_i)(k) - (Uh_i)(k)||$ on K is continuous and convex, so by [1], Lemma II.7.1, $||(U'h_i) - (Uh_i)|| \leqq \varepsilon (1 \leqq i \leqq n)$. This will show that U is in the strong operator closure of the copy of $\mathscr{Z}(E) \odot$

$\mathscr{Z}(F)$ in $\mathscr{Z}(E \otimes_\lambda F)$.

We first prove that [3], Proposition 4.8 remains valid in this context. I.e. if $x \in E$ then $P = \{p \in \mathscr{E}(K): |p(x)| \geqq \alpha\}$ is structurally compact provided $\alpha > 0$. If $(C_s)_{s \in S}$ is a family of nonempty structurally closed subsets of P with the finite intersection property, let $C_s = P \cap F_s$ with each F_s a weak* closed L-ideal in E^*. Set $Q = \{k \in K: |k(x)| \geqq \alpha\}$ then each $F_s \cap Q$ is nonempty and this family has the finite intersection property. As Q is weak* compact and these sets are weak* closed, $\bigcap(F_s \cap Q) = (\bigcap F_s) \cap Q \neq \varnothing$. $\bigcap F_s$ is a weak* closed L-ideal and for some $k \in K \cap (\bigcap F_s) |k(x)| \geqq \alpha$. But x attains its supremum at an extreme point, p, of $K \cap (\bigcap F_s)$ which is an extreme point of K by [2], Proposition 1.15. As $K \cap (\bigcap F_s)$ is symmetric, $p(x) \geqq \alpha$ so that $p \in E(K) \cap (\bigcap F_s) = \bigcap (p \cap F_s) = \bigcap C_s$. We note also that such a set P does not contain 0 in its weak* closure, so $\pi|_P$ is continuous for the strong operator topology.

Given $h_i \in H$, $\delta > 0$, we may find a weak* closed subset Q_i of $\overline{\mathscr{E}(K)}$, not containing 0 and with $Q_i \cap \mathscr{E}(K)$ structurally compact, such that $\|h_i(k)\| < \delta$ if $k \in \mathscr{E}(K) \backslash Q_i$. For we can find $\sum_{j=1}^m e_j \otimes f_j \in E \odot F$ with $\|\sum_{j=1}^m k(e_j)f_j - h_i(k)\| < \delta/2 (k \in K)$. Now let $P_j = \{k \in \mathscr{E}(K): |k(e_j)| \, \|f_j\| \geqq \delta/2m\}$, which is weak* closed, does not contain 0, and is such that $P_j \cap \mathscr{E}(K)$ is structurally compact. Define $Q_i = \bigcup_{i=1}^m P_j$, then Q_i will have all the desired properties except possibly that on the norm. If $k \in \overline{\mathscr{E}(K)} \backslash Q_i$ then

$$\begin{aligned} \|h_i(k)\| &\leqq \left\|\sum_{j=1}^m k(e_j)f_j\right\| + \left\|\sum_{j=1}^m k(e_j)f_j - h_i(k)\right\| \\ &< \sum_{j=1}^m |k(e_j)| \, \|f_j\| + \delta/2 \\ &\leqq m(\delta/2m) + \delta/2 = \delta\,. \end{aligned}$$

We may thus find a weak* open neighbourhood of 0 in $\overline{\mathscr{E}(K)}$, O_0, with structurally compact complement in $\mathscr{E}(K)$, such that $O_0 \subset \{k \in \overline{\mathscr{E}(K)}: \|h_i(k)\| < \varepsilon/(2\|\pi\| + 1)(1 \leqq i \leqq n)\}$. Indeed if we take $\delta = \varepsilon/(2\|\pi\| + 1)$ and choose Q_i as above we take O_0 to be $\overline{\mathscr{E}(K)} \backslash \bigcup_{i=1}^n Q_i$, which has the desired properties. If $k \in \overline{\mathscr{E}(K)}$ we let $U_k = \{T \in \mathscr{Z}(F): \|T(h_i(k))\| < \varepsilon/3 (1 \leqq i \leqq n)\}$, an open symmetric neighbourhood of the origin in $\mathscr{Z}(F)$ for the strong operator topology. Thus $\bar{\pi}^{-1}(\bar{\pi}(k) + U_k)$ is an open subset of $\overline{\mathscr{E}(K)} \backslash \{0\}$ (by the continuity of $\bar{\pi}$ for the strong operator topology) and hence of $\overline{\mathscr{E}(K)}$. The set $\overline{\mathscr{E}(K)} \cap \bigcap_{i=1}^n h_i^{-1}(h_i(k) + B)$ (where B is the open ball in F of centre the origin and radius $\varepsilon/(3(\|\pi\| + 1)))$ is also weak* open, hence so is

$$O_k = (\bar{\pi}^{-1}(\bar{\pi}(k) + U_k)) \cap \bigcap_{i=1}^n h_i^{-1}(h_i(k) + B)$$

for each $k \in \overline{\mathscr{E}(K)}\backslash\{0\}$, and we have $k \in O_k$. Now let $\{0, k_1, k_2, \cdots, k_r\}$ be a finite set of distinct points of $\overline{\mathscr{E}(K)}$ with $\overline{\mathscr{E}(K)} = O_0 \cup \bigcup_{j=1}^{r} O_{k_j}$.

Let $W = \bigcap_{j=1}^{r} U_{k_j}$, an open convex symmetric neighbourhood of the origin in $\mathscr{L}(F)$ for the strong operator topology. Because $\mathscr{E}(K)\backslash O_0$ is structurally compact and π is continuous on this for the strong operator topology on $\mathscr{L}(F)$, $\pi(\mathscr{E}(K)\backslash O_0)$ is strong operator compact. Thus there exist $\{T_1, T_2, \cdots, T_s\} \subset \mathscr{L}(F)$ such that $\bigcup_{i=1}^{s}(T_i + W/2) \supset \pi(\mathscr{E}(K)\backslash O_0)$. Define G to be the linear span of $\{T_i: 1 \leqq i \leqq s\}$ in $\mathscr{L}(F)$, and let Φ be defined on $\pi(\mathscr{E}(K)\backslash O_0)$ with values in 2^G by

$$\Phi(S) = \{g \in G: \|g\| < \|\pi\| + 1, g - S \in W/2\}^- .$$

For some i, $T_i - S \in W/2$ and $T_i \in \pi(\mathscr{E}(K)\backslash O_0)$ so $\|T_i\| \leqq \|\pi\|$, so that $\Phi(S)$ is certainly nonempty. It is clear that $\Phi(S)$ is closed and convex.

We show that Φ is lower semi-continuous, for the unique vector topology on G, and the weak and strong operator topologies on $\pi(\mathscr{E}(K)\backslash O_0)$ which coincide by the compactness of $\pi(\mathscr{E}(K)\backslash O_0)$ for the latter topology. If $D \subset G$ is open we must show that $\{S \in \pi(\mathscr{E}(K)\backslash O_0): \Phi(S) \cap D \neq \emptyset\}$ is open. Suppose $S_0 \in \pi(\mathscr{E}(K)\backslash O_0)$ with $\Phi(S_0) \cap D \neq \emptyset$. By the definition of Φ, we can find $x_0 \in D$ with $\|x_0\| < \|\pi\| + 1$, $x_0 - S_0 \in W/2$. As W is open, there is a symmetric strong operator neighbourhood of the origin in $\mathscr{L}(F)$, V, such that $x_0 - S_0 + V \subset W/2$. Now if $S \in (S_0 + V) \cap \pi(\mathscr{E}(K)\backslash O_0)$ we claim $\Phi(S) \cap D \neq \emptyset$, for $x_0 - S = (x_0 - S_0) + (S_0 - S) \in (x_0 - S_0) + V \subset W/2$. It is now clear that $x_0 \in \Phi(S) \cap D$, completing the proof that Φ is lower semi-continuous.

As G is finite dimensional we can apply a selection theorem (e.g. [4], Theorem 3.2′) to assert the existence of a continuous selection for Φ, ϕ. We note that $\phi(\pi(\mathscr{E}(K)\backslash O_0))$ is contained in the closed ball in G of centre the origin and radius $\|\pi\| + 1$. We extend ϕ to ψ defined on the whole of $\pi(\mathscr{E}(K))$ with values in the same ball and with ψ continuous for the weak operator topology on $\pi(\mathscr{E}(K))$. Let $\beta(\pi(\mathscr{E}(K)))$ be the Stone-Čech compactification of $\pi(\mathscr{E}(K))$ (for the weak operator topology), and ρ the natural injection of $\pi(\mathscr{E}(K))$ into $\beta(\pi(\mathscr{E}(K)))$. Since the weak operator topology is uniformisable ρ is a homeomorphism, so that $\phi \circ \rho^{-1}$ is a continuous function from the closed set $\rho(\pi(\mathscr{E}(K)\backslash O_0))$ into G. Let σ be a continuous extension of $\phi \circ \rho^{-1}$ to the whole of $\beta(\pi(\mathscr{E}(K)))$ with values in the required ball in G, which exists by Tietze's extension theorem. Now $\psi = \sigma \circ \rho$ is the desired function. Define $\pi' = \psi \circ \pi$, a function from $\mathscr{E}(K)$ into G that is bounded and continuous for the structure topology on $\mathscr{E}(K)$, since π is continuous for the structure topology on $\mathscr{E}(K)$ and the weak operator topology on $\mathscr{L}(F)$ whilst ψ is continuous for the weak operator topology on $\pi(\mathscr{E}(K))$. We claim π' has the required property.

If $p \in \mathscr{E}(K) \backslash O_0$ then $p \in O_{k_j}$ for some j. Then $\|h_i(p) - h_i(k_j)\| < \varepsilon/3(\|\pi\| + 1)$ and we also have $\pi'(p) - \pi(p) \in \overline{W/2} \subset W$. Thus for $1 \leqq i \leqq n$,

$$
\begin{aligned}
&\|\pi(p)h_i(p) - \pi'(p)h_i(p)\| \\
&\qquad \leqq \|\pi(p)h_i(p) - \pi(p)h_i(k_j)\| + \|\pi(p)h_i(k_j) - \pi'(p)h_i(k_j)\| \\
&\qquad\quad + \|\pi'(p)h_i(k_j) - \pi'(p)h_i(p)\| \\
&\qquad \leqq \|\pi(p)\|\,\|h_i(p) - h_i(k_j)\| + (\varepsilon/3) + \|\pi'(p)\|\,\|h_i(k_j) - h_i(p)\| \\
&\qquad\qquad (\text{since } \pi(p) - \pi'(p) \in W \subset U_{k_j}) \\
&\qquad \leqq \|\pi\|(\varepsilon/3(\|\pi\| + 1)) + (\varepsilon/3) + (\|\pi\| + 1)(\varepsilon/3(\|\pi\| + 1)) \\
&\qquad < \varepsilon\,.
\end{aligned}
$$

On the other hand if $p \in O_0 \cap \mathscr{E}(K)$ then

$$
\begin{aligned}
&\|\pi(p)h_i(p) - \pi'(p)h_i(p)\| \\
&\qquad \leqq (\|\pi'(p)\| + \|\pi(p)\|)\|h_i(p)\| \\
&\qquad \leqq (2\|\pi\| + 1)(\varepsilon/(2\|\pi\| + 1)) = \varepsilon\,.
\end{aligned}
$$

Thus π' has the desired properties.

So far we have shown that $\mathscr{Z}(E \otimes_\lambda F)$ is contained in the strong operator closure in $\mathscr{B}(E \otimes_\lambda F)$ of the copy of $\mathscr{Z}(E) \odot \mathscr{Z}(F)$ there. It remains only to show that for any Banach space, X, $\mathscr{Z}(X)$ is strong operator closed in $\mathscr{B}(X)$. Indeed if $T_\lambda \to T$ for the strong operator topology with $T_\gamma \in \mathscr{Z}(X)$, p is an extreme point of the unit ball of X^* and $x \in X$, then

$$(T^*p)(x) = \lim (T_\gamma^* p)(x) = \lim \tilde{T}_\gamma(p)p(x)\,.$$

Thus $\lim \tilde{T}_\gamma(p)$ exists and $T^*p = (\lim \tilde{T}_\gamma(p))p$, so $T \in \mathscr{Z}(X)$.

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THE QUEEN'S UNIVERSITY OF BELFAST, NORTHERN IRELAND