PACIFIC JOURNAL OF MATHEMATICS
Vol 66, No 2, 1976

WIENER INTEGRALS OVER THE SETS BOUNDED BY SECTIONALLY CONTINUOUS BARRIERS

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Let $C_W \equiv C[0, T]$ denote the Wiener space on $[0, T]$. The Wiener integrals of various functionals $F[x]$ over the space C_W are well-known. In this paper we establish formulas for the Wiener integrals of $F[x]$ over the subsets of C_W bounded by sectionally continuous functions.

1. Introduction. Let $C_W \equiv C[0, T]$ be the Wiener space on $[0, T]$, i.e., the space of all real-valued continuous functions on $[0, T]$ vanishing at the origin. The standard Wiener process $\{X(t) \equiv X(t, \cdot): 0 \leqq t \leqq T\}$ and C_W are related by $X(t, x) = x(t)$ for each x in C_W. Evaluation formulas for the Wiener integral

$$\int_{C_W} F[x] d_W x \equiv E\{F[x]\}$$

of various functionals $F[x]$ are of course well-known (for example see **[7]** for some of these formulas). Now, consider sets of the type

$$\Gamma_f \equiv \left\{ \sup_{0 \leqq t \leqq T} X(t) - f(t) < 0 \right\}$$
$$= \left\{ x \in C_W : \sup_{0 \leqq t \leqq T} x(t) - f(t) < 0 \right\}$$

where $f(t)$ is sectionally continuous on $[0, T]$ and $f(0) \geqq 0$.

It is well-known that for $b \geqq 0$

$$P[\Gamma_b] = 2\Phi(bT^{-1/2}) - 1$$

and

$$P[\Gamma_{at+b}] = \Phi[(aT + b)T^{-1/2}] - e^{-2ab}\Phi[(aT - b)T^{-1/2}]$$

where Φ is the standard normal distribution function. In **[3]**, **[5]**, and **[6]** more general functions $f(t)$ are considered and formulas given for the probabilities of the sets Γ_f.

The main purpose of this paper is to derive formulas for Wiener integrals over the sets Γ_f. In §2 we state and prove the main results, while in §3 we discuss some applications and examples.

2. Integration formulas. Our first theorem is preliminary; it plays a key role in the proof of Theorem 2.

THEOREM 1. *Let* $0 = t_0 < t_1 < \cdots < t_n = T$ *be a partition of* $[0, T]$. *For* $j = 1, 2, \cdots, n$ *let* $f_j(t)$ *be continuous on* $[t_{j-1}, t_j]$. *Then the conditional probability*

$$P\left[\sup_{t_{j-1}\leqq t\leqq t_j} X(t) - f_j(t) < 0,\ 1 \leqq j \leqq n \,|\, X(t_j) = u_j,\ 1 \leqq j \leqq n\right]$$
$$= \prod_{j=1}^{n} P\left[\sup_{0\leqq t\leqq \Delta t_j} X(t) - \{f_j(t + t_{j-1}) - u_{j-1}\} < 0 \,|\, X(\Delta t_j) = \Delta u_j\right]$$
$$= \prod_{j=1}^{n} P\left(\sup_{0\leqq t<\infty} X(t) - \left\{\frac{t + t\Delta t_j}{\Delta t_j}\left[f_j\left(\frac{\Delta t_j}{1 + t\Delta t_j} + t_{j-1}\right) - u_{j-1}\right] - \frac{\Delta u_j}{\Delta t_j}\right\} < 0\right)$$

where $\Delta t_j = t_j - t_{j-1}$ *and* $\Delta u_j = u_j - u_{j-1}$ *with* $u_0 = 0$.

Proof. First we note that

$$P\left[\sup_{t_{j-1}\leqq t\leqq t_j} X(t) - f_j(t) < 0,\ 1 \leqq j \leqq n \,|\, X(t_j) = u_j,\ 1 \leqq j \leqq n\right]$$
$$= P\left\{\sup_{t_{j-1}\leqq t\leqq t_j} X(t) - X(t_{j-1}) - [f_j(t) - u_{j-1}] < 0,\right.$$
$$\left. 1 \leqq j \leqq n \,|\, X(t_j) - X(t_{j-1}) = \Delta u_j,\ 1 \leqq j \leqq n\right\}.$$

Now since the Wiener process has independent increments, the above expression equals

$$\prod_{j=1}^{n} P\left\{\sup_{t_{j-1}\leqq t\leqq t_j} X(t) - X(t_{j-1}) - [f_j(t) - u_{j-1}] < 0 \,|\, X(t_j) - X(t_{j-1}) = \Delta u_j\right\}.$$

Hence the first equality in the theorem follows from the fact that stationarity implies that $X(t) - X(t_{j-1})$ is the same process as $X(t - t_{j-1})$. To prove the second equality in the theorem, we note that $X(t)$ and $tX(1/t)$ are identical Wiener processes for $t > 0$ by checking the covariance function. Thus

$$P\left\{\sup_{0\leqq t\leqq \Delta t_j} X(t) - [f_j(t + t_{j-1}) - u_{j-1}] < 0 \,|\, X(\Delta t_j) = \Delta u_j\right\}$$
$$= P\left\{\sup_{0< t\leqq \Delta t_j} X\left(\frac{1}{t}\right) - \frac{1}{t}[f_j(t + t_{j-1}) - u_{j-1}] < 0 \,|\, X\left(\frac{1}{\Delta t_j}\right) = \frac{\Delta u_j}{\Delta t_j}\right\}$$
$$= P\left\{\sup_{0< t\leqq \Delta t_j} X\left(\frac{1}{t}\right) - X\left(\frac{1}{\Delta t_j}\right) - \frac{1}{t}[f_j(t + t_{j-1}) - u_{j-1}] + \frac{\Delta u_j}{\Delta t_j} < 0\right\}.$$

The result now follows by the transformation

$$t^{-1}-(\Delta t_j)^{-1}\to t.$$

THEOREM 2. *Let* $0=t_0<t_1<\cdots<t_n=T$ *be a partition of* $[0,T]$. *Let* $g(u_1,\cdots,u_n)$ *be a Lebesgue measurable function on* R^n *and for* $x\in C_W$ *let* $G[x]=g(x(t_1),\cdots,x(t_n))$. *For* $j=1,2,\cdots,n$ *let* $f_j(t)$ *be a continuous function on* $[t_{j-1},t_j]$. *Then the Wiener integral of* $G[x]$ *over the set*

$$\Gamma_f\equiv\left\{x\in C_W:\ \sup_{t_{j-1}\leqq t\leqq t_j} x(t)-f_j(t)<0,\ j=1,2,\cdots,n\right\}$$

is given by

$$\int_{\Gamma_f} G[x]d_Wx=\int_{-\infty}^{\lambda_1}\cdot(n)\cdot\int_{-\infty}^{\lambda_n} g(u_1,\cdots,u_n)H(u_1,\cdots,u_n)du_n\cdots du_1$$

in the sense the existence of either side implies that of their equality, where

$$\lambda_j=\begin{cases}\min\{f_j(t_j),f_{j+1}(t_j)\}, & j=1,2,\cdots,n-1\\ \\ f_n(T), & j=n\end{cases}$$

$$f(t)=\begin{cases} f_j(t), & t_{j-1}<t<t_j,\quad j=1,2,\cdots,n\\ f_1(0), & t=0\\ \lambda_j\quad, & t=t_j,\qquad j=1,2,\cdots,n\end{cases},$$

and

$$H(u_1,\cdots,u_n)=\prod_{j=1}^{n}(2\pi\Delta t_j)^{-1/2}\exp\{-(\Delta u_j)^2/(2\Delta t_j)\}$$
$$P\left[\sup_{0\leqq t\leqq\Delta t_j} X(t)-\{f_j(t+t_{j-1})-u_{j-1}\}<0\,|\,X(\Delta t_j)=\Delta u_j\right].$$

Proof. First consider the case where $G[x]$ is the characteristic function of a Wiener interval I. That is to say I has the form

$$I=\{x\in C_W\,|\,[x(t_1),\cdots,x(t_n)]\in E\}$$

for some Lebesgue measurable set E in R^n. Then $G[x]=\chi_I(x)=\chi_E[x(t_1),\cdots,x(t_n)]$ and so in this case

$$\int_{\Gamma_f} G[x]d_W x = \int_{\Gamma_f} \chi_I(x)d_W x = P[\Gamma_f \cap I]$$

$$= \int_{-\infty}^{\lambda_1} \cdot (n) \cdot \int_{-\infty}^{\lambda_n} P\Big\{ \sup_{t_{j-1} \leqq t \leqq t_j} X(t) - f_j(t) < 0,\ 1 \leqq j \leqq n,$$

$$[x(t_1), \cdots, x(t_n)] \in E \,|\, x(t_j) = u_j,\ 1 \leqq j \leqq n\Big\} K(\vec{t}, \vec{u}) du_n \cdots du_1$$

where

$$K(\vec{t}, \vec{u}) \equiv \prod_{j=1}^{n} (2\pi \Delta t_j)^{-1/2} \exp\{-(\Delta u_j)^2/(2\Delta t_j)\}.$$

Next we observe that

$$P\Big\{ \sup_{t_{j-1} \leqq t \leqq t_j} X(t) - f_j(t) < 0,\ 1 \leqq j \leqq n, [x(t_1), \cdots, x(t_n)] \in E \,|$$

$$x(t_j) = u_j,\ 1 \leqq j \leqq n\Big\} = \chi_E(u_1, \cdots, u_n) P\Big[\sup_{t_{j-1} \leqq t \leqq t_j} X(t) - f_j(t) < 0,$$

$$1 \leqq j \leqq n \,|\, x(t_j) = u_j,\ 1 \leqq j \leqq n\Big].$$

Next, applying Theorem 1 to the last conditional probability above gives the desired result for this case. The general case follows by the usual arguments in integration theory.

THEOREM 3. *Let $f(t)$ be sectionally continuous on $[0, T]$ with $f(0) \geqq 0$. Let $g(u_1, \cdots, u_n)$ be a Lebesgue measurable function on R^n, and let $\alpha(t) \in BV[0, T]$. Then*

$$\int_{\Gamma_f} g[x(t_1), \cdots, x(t_n)] e^{\int_0^T \alpha(t)dx(t)} d_W x$$

$$= e^{1/2 \int_0^T \alpha^2(t)dt} \int_{\Gamma_{f(t) - \int_0^t \alpha(s)ds}} g\left[x(t_1) + \int_0^{t_1} \alpha(s)ds, \cdots, x(t_n) + \int_0^{t_n} \alpha(s)ds\right] d_W x$$

in the sense the existence of either side implies that of the other and their equality.

COROLLARY. *If f and α satisfy the conditions in Theorem 3, then*

$$\int_{\Gamma_f} e^{\int_0^T \alpha(t)dx(t)} d_W x = e^{1/2 \int_0^T \alpha^2(t)dt} P\Big\{ \sup_{0 \leqq t \leqq T} X(t) - \Big[f(t) - \int_0^t \alpha(s)ds\Big] < 0\Big\}.$$

Proof (of Theorem 3). Using the Cameron–Martin translation theorem (see **[1]** or **[7]**) with the translation

$$x(t) \to x(t) + \int_0^t \alpha(u)du$$

we obtain

$$\int_{\Gamma_f} g[x(t_1), \cdots, x(t_n)] e^{\int_0^T \alpha(t)dx(t)} d_W x$$

$$= e^{-1/2 \int_0^T \alpha^2(t)dt} \int_{\Gamma_{f(t)-\int_0^t \alpha(s)ds}} g\left[x(t_1) + \int_0^{t_1} \alpha(u)du, \cdots, x(t_n) + \int_0^{t_n} \alpha(u)du\right]$$

$$\cdot e^{\int_0^T \alpha(t)d[x(t)+\int_0^t \alpha(u)du]} e^{-\int_0^T \alpha(t)dx(t)} d_W x.$$

The result now follows by simplifying the last expression.

THEOREM 4. *Assume that $g(z) = \Sigma_0^\infty a_n z^n$ is an entire function such that for some M, N and $\gamma \in (0,2)$, $|g(z)| \leqq M \exp(N|z|^\gamma)$ for all complex numbers z. For some $r \in (2,\infty]$ and $b > 0$ assume that $\theta(t,u) \in L_{1r}([0,T] \times (-\infty, b])$, i.e., $\int_{-\infty}^b |\theta(t,u)| du \in L_r[0,T]$. Then for any $\psi(u) \in L_1(-\infty, b] \cup L_\infty(-\infty, b]$,*

$$\int_{\Gamma_b} g\left[\int_0^T \theta(t, x(t))dt\right] \psi(x(T)) d_W x = \sum_0^\infty a_n J_n(T) n! \tag{1}$$

where for $n = 0, 1, 2, \cdots$

$$J_n(T) \equiv \int_0^{t_{n+1}} \cdot (n) \cdot \int_0^{t_2} \int_{-\infty}^b \cdot (n+1) \cdot \int_{-\infty}^b \psi(u_{n+1}) \prod_{j=1}^n \theta(t_j, u_j)$$

$$\begin{aligned} &\prod_{j=1}^{n+1} (2\pi \Delta t_j)^{-1/2} \exp\{-(\Delta u_j)^2/(2\Delta u_j)\} \\ &\cdot [1 - \exp\{-2(b - u_{j-1})(b - u_j)/\Delta t_j\}] \\ &\cdot du_{n+1} \cdots du_1 dt_1 \cdots dt_n \end{aligned} \tag{2}$$

and where $u_0 \equiv 0$, and $0 = t_0 < t_1 < \cdots < t_{n+1} = T$.

Proof. Proceeding formally we obtain

$$\int_{\Gamma_b} g\left[\int_0^T \theta(t, x(t))dt\right] \psi(x(T)) d_W x$$

$$= \sum_0^\infty a_n \int_{\Gamma_b} \left[\int_0^T \theta(t, x(t))dt\right]^n \psi(x(T)) d_W x.$$

But for $n = 1, 2, \cdots$

$$\begin{aligned}
&\int_{\Gamma_b} \left[\int_0^T \theta(t, x(t))dt\right]^n \psi(x(T))d_W x \\
&= \int_{\Gamma_b} \left[\int_0^T \cdot(n)\cdot \int_0^T \prod_{j=1}^n \theta(t_j, x(t_j))dt_1 \cdots dt_n\right] \psi(x(T))d_W x \\
&= n! \int_0^{t_{n+1}} \cdot(n)\cdot \int_0^{t_2} \left[\int_{\Gamma_b} \psi(x(t_{n+1})) \prod_{j=1}^n \theta(t_j, x(t_j))d_W x\right] dt_1 \cdots dt_n \\
&= n! \int_0^{t_{n+1}} \cdot(n)\cdot \int_0^{t_2} \int_{-\infty}^b \cdot(n+1)\cdot \int_{-\infty}^b \psi(u_{n+1}) \prod_{j=1}^n \theta(t_j, u_j) \prod_{j=1}^{n+1} \{(2\pi\Delta t_j)^{-1/2} \\
&\quad \cdot \exp[-(\Delta u_j)^2/(2\Delta t_j)][1 - \exp(-2(b - u_{j-1})(b - u_j)/\Delta t_j)]\} \\
&\quad \cdot du_{n+1} \cdots du_1 dt_1 \cdots dt_n
\end{aligned}$$

where the last equality above is obtained using Theorem 2, Theorem 1, and the fact that

$$\begin{aligned}
P\Big(\sup_{0 \leqq t < \infty} X(t) - \{(b_j - u_{j-1})t + (b_j - u_j)/\Delta t_j\} < 0\Big) \\
= 1 - \exp[-2(b - u_{j-1})(b - u_j)/\Delta t_j].
\end{aligned}$$

Thus proceeding formally we have obtained equation (1). The Theorem follows readily once the absolute convergence of the series

$$\sum_0^\infty n!\, a_n J_n(T)$$

is established.

Recall that $r \in (2, \infty]$. We will establish the absolute convergence when $2 < r < \infty$ and $\psi \in L_1(-\infty, b]$; then other cases are similar, but easier. Let p satisfy $1/r + 1/p = 1$. Then $1 < p < 2$ and by Hölder's inequality we obtain

$$\begin{aligned}
|J_n(T)| &\leqq \|\psi\|_1 \int_0^{t_{n+1}} \cdot(n)\cdot \int_0^{t_2} \prod_{j=1}^n \|\theta(t_j, \cdot)\|_1 \prod_{j=1}^{n+1} (2\pi\Delta t_j)^{-1/2} dt_1 \cdots dt_n \\
&\leqq \|\psi\|_1 \left\{\int_0^{t_{n+1}} \cdot(n)\cdot \int_0^{t_2} [t_1(t_2 - t_1)\cdots(t_{n+1} - t_n)]^{-p/2} dt_1 \cdots dt_n\right\}^{1/p} \\
&\quad \cdot \left\{\int_0^{t_{n+1}} \cdot(n)\cdot \int_0^{t_2} \prod_{j=1}^n \|\theta(t_j, \cdot)\|_1^r dt_1 \cdots dt_n\right\}^{1/r} (2\pi)^{-(n+1)/2}.
\end{aligned}$$

But

$$\left\{\int_0^{t_{n+1}} \cdot (n) \cdot \int_0^{t_2} \prod_{j=1}^n \|\theta(t_j, \cdot)\|_1^r dt_1 \cdots dt_n\right\}^{1/r}$$
$$= \left\{\frac{1}{n!}\int_0^{T} \cdot (n) \cdot \int_0^{T} \prod_{j=1}^n \|\theta(t_j, \cdot)\|_1^r dt_1 \cdots dt_n\right\}^{1/r}$$
$$= \left(\frac{1}{n!}\right)^{1/r} \left(\int_0^T \|\theta(t, \cdot)\|_1^r dt\right)^{n/r}$$
$$= \left(\frac{1}{n!}\right)^{1/r} \|\theta\|_{1r}^n.$$

In addition

$$\int_0^{t_{n+1}} \cdot (n) \cdot \int_0^{t_2} [t_1(t_2 - t_1) \cdots (t_{n+1} - t_n)]^{-p/2} dt_1 \cdots dt_n$$
$$= \frac{T^{n(2-p)/2}\{\Gamma(1-p/2)\}^{n+1}}{T^{p/2}\Gamma[(n+1)(1-p/2)]}$$

where $\Gamma(z)$ denotes the Gamma function.

Thus the series $\Sigma_0^\infty |a_n J_n(T)| n!$ is dominated by the series

$$\sum_0^\infty \frac{n! |a_n| (n!)^{-1/r} \|\theta\|_{1r}^n T^{n(2-p)/2p}\{\Gamma(1-p/2)\}^{(n+1)/p}}{T^{1/2}\{\Gamma[(n+1)(1-p/2)]\}^{1/p}}$$
(3)
$$= \sum_0^\infty \frac{|a_n| \|\theta\|_{1r}^n T^{n(2-p)/2p}\{\Gamma(1-p/2)\}^{(n+1)/p}}{T^{1/2}} \left\{\frac{n!}{\Gamma[(n+1)(1-p/2)]}\right\}^{1/p}.$$

But since $g(z)$ is an entire function of order at most γ we know that

$$\limsup_{n\to\infty} \left(\frac{n ln n}{-ln|a_n|}\right) \leqq \gamma < \frac{\gamma+2}{2}$$

and so for n sufficiently large we obtain that

$$|a_n| < n^{-2n/(\gamma+2)}.$$

But $\Gamma(z) = z^{z-1/2}e^{-z}(2\pi)^{1/2}(1+0(1))$ and hence for positive z sufficiently large

$$\frac{1}{\Gamma(z)} < \frac{2e^z z^{1/2}}{(2\pi)^{1/2} z^z}.$$

Also by Stirling's formula

$$n! \leqq (n/e)^n (2\pi n)^{1/2} \exp\left(\frac{1}{12n}\right).$$

Thus for n sufficiently large we obtain

$$\begin{aligned} &|a_n| \left\{\frac{n!}{\Gamma[(n+1)(1-p/2)]}\right\}^{1/p} \\ (4) \qquad &\leqq 2^{1/p} \exp\left(\frac{12n+1}{12np}\right) n^{1/2} e^{-(n+1)/2} \left(\frac{2}{2-p}\right) \\ &\times [(n+1)(2-p)-1]/2p_n^{-n(2-\gamma)/2(\gamma+2)}. \end{aligned}$$

Now using inequality (4) the convergence of the series (3) follows by the root test.

COROLLARY 1. *Let $\theta(t, u)$ be as in Theorem 4. Then*

$$\int_{\Gamma_b} \exp\left[\int_0^T \theta(s, x(s))ds\right] d_W x = \sum_0^\infty J_n(T)$$

where $J_n(T)$ is given by (2) *with $\psi \equiv 1$.*

COROLLARY 2. *Let $\alpha(t)$ be of bounded variations on* $[0, T]$. *Then for any $b > 0$,*

$$\int_{\Gamma_b} e^{\int_0^T \alpha(t)dx(t)} d_W x = \sum_0^\infty (-1)^n K_n(T)$$

where

$$\begin{aligned} K_n(T) \equiv &\int_0^{t_{n+1}} \cdot (n) \cdot \int_0^{t_2} \int_{-\infty}^b \cdot (n+1) \cdot \int_{-\infty}^b e^{\alpha(T)u_{n+1}} \prod_{j=1}^n u_j \prod_{j=1}^{n+1} (2\pi \Delta t_j)^{-1/2} \\ (5) \qquad &\cdot \exp[-(\Delta u_j)^2/(2\Delta t_j)]\{1 - \exp[-2(b-u_{j-1})(b-u_j)/\Delta t_j]\} \\ &\cdot du_{n+1} \cdots du_1 d\alpha(t_1) \cdots d\alpha(t_n). \end{aligned}$$

Proof (of Corollary 2). The Corollary follows quite readily once the absolute convergence of the series $\Sigma_0^\infty (-1)^n K_n(T)$ is established. Now proceeding formally we see that

$$\begin{aligned} \int_{\Gamma_b} e^{\int_0^T \alpha(t)dx(t)} d_W x &= \int_{\Gamma_b} e^{\alpha(T)x(T) - \int_0^T x(t)d\alpha(t)} d_W x \\ &= \int_{\Gamma_b} \sum_{n=0}^\infty \frac{(-1)^n}{n!} e^{\alpha(T)x(T)} \left[\int_0^T x(t)d\alpha(t)\right]^n d_W x \end{aligned}$$

and so

$$|K_n(T)| = \left| \frac{1}{n!} \int_{\Gamma_b} e^{\alpha(T)x(T)} \left[\int_0^T x(t) d\alpha(t) \right]^n d_W x \right|$$
$$\leqq \frac{1}{n!} \int_{C_W} e^{\alpha(T)x(T)} \left| \int_0^T x(t) d\alpha(t) \right|^n d_W x.$$

Thus

$$\sum_0^\infty |K_n(T)| \leqq \sum_0^\infty \frac{1}{n!} \int_{C_W} e^{\alpha(T)x(T)} \left| \int_0^T x(t) d\alpha(t) \right|^n d_W x$$
$$= \int_{C_W} e^{\alpha(T)x(T)} e^{|\int_0^T x(t)d\alpha(t)} d_W x$$
$$\leqq \int_{C_W} e^{2|\alpha(T)x(T)|} e^{|\int_0^T \alpha(t)dx(t)|} d_W x$$
$$\leqq \left[\int_{C_W} e^{4|\alpha(T)x(T)|} d_W x \right]^{1/2} \left[\int_{C_W} e^{2|\int_0^T \alpha(t)dx(t)|} d_W x \right]^{1/2}$$
$$< \infty \quad \text{since} \quad \alpha(t) \in L^2[0, T].$$

3. Applications and examples.

A. Application 1. For our first application we obtain a formula for the probability that a Wiener path always stays below the broken line segments $f_j(t) = a_j t + b_j$, $t_{j-1} \leqq t \leqq t_j$, $1 \leqq j \leqq n$, where $b_1 > 0$. Using Theorems 1 and 2 we obtain

$$P\left[\sup_{t_{j-1} \leqq t \leqq t_j} X(t) - (a_j t + b_j) < 0, 1 \leqq j \leqq n \right]$$
$$= \int_{-\infty}^{\lambda_1} \cdot (n) \cdot \int_{-\infty}^{\lambda_n} \prod_{j=1}^n P\Big(\sup_{0 \leqq t < \infty} X(t) - \Big\{ \frac{1 + t\Delta t_j}{\Delta t_j} \Big[a_j \Big(\frac{\Delta t_j}{1 + t\Delta t_j} + t_{j-1} \Big)$$
$$+ b_j - u_{j-1} \Big] - \frac{\Delta u_j}{\Delta t_j} \Big\} < 0 \Big) K(\vec{t}, \vec{u}) du_n \cdots du_1$$

where

$$\lambda_j = \begin{cases} \min\{a_j t_j + b_j, a_{j+1} t_j + b_{j+1}\}, & 1 \leqq j < n \\ a_n T + b_n, & j = n \end{cases}$$

and

$$K(\vec{t}, \vec{u}) = \prod_{j=1}^{n} (2\pi\Delta t_j)^{-1/2} \exp\{-(\Delta u_j)^2/(2\Delta t_j)\}.$$

But the probability in the integrand simplifies into

$$P\Big(\sup_{0 \leqq t < \infty} X(t) - \{(a_j t_{j-1} + b_j - u_{j-1})t + (a_j t_j + b_j - u_j)/\Delta t_j\} < 0\Big)$$

which, using Doob [**2**, p. 397], equals the expression

$$[1 - \exp\{-2(a_j t_{j-1} + b_j - u_{j-1})(a_j t_j + b_j - u_j)/\Delta t_j\}].$$

Thus, we finally obtain the formula

$$P\Big[\sup_{t_{j-1} \leqq t \leqq t_j} X(t) - (a_j t + b_j) < 0,\ 1 \leqq j \leqq n\Big]$$

$$= \int_{-\infty}^{\lambda_1} \cdot (n) \cdot \int_{-\infty}^{\lambda_n} \prod_{j=1}^{n} [1 - \exp\{-2(a_j t_{j-1} + b_j - u_{j-1})(a_j t_j + b_j - u_j)/\Delta t_j\}]$$

$$\cdot [2\pi\Delta t_j]^{-1/2} \exp\{-(\Delta u_j)^2/(2\Delta t_j)\} du_n \cdots du_1.$$

B. *Example* 1. For $j = 1, 2, \cdots, n$ let the a_j and b_j be as above. Let

$$E \equiv \Big\{x \in C_W \;\Big|\; \sup_{t_{j-1} \leqq t \leqq t_j} X(t) - (a_j t + b_j) < 0,\ 1 \leqq j \leqq n\Big\}.$$

Let $g(u_1, \cdots, u_n)$ be Lebesgue measurable on R^n. Then

$$\int_E g(x(t_1), \cdots, x(t_n)) d_W x$$

$$= \int_{-\infty}^{\lambda_1} \cdot (n) \cdot \int_{-\infty}^{\lambda_n} g(u_1, \cdots, u_n) \prod_{j=1}^{n} [2\pi\Delta t_j]^{-1/2} \exp\{-(\Delta u_j)^2/(2\Delta t_j)\}$$

$$\cdot \prod_{j=1}^{n} [1 - \exp\{-2(a_j t_{j-1} + b_j - u_{j-1})(a_j t_j + b_j - u_j)/\Delta t_j\}] du_n \cdots du_1$$

in the sense the existence of either side implies that of the other and their equality.

C. *Application* 2. Assume $\alpha(t)$ is of bounded variation on $[0, T]$ and let $f(t) \equiv \int_0^t \alpha(s) ds$. For $b > 0$ we want to find the probability of the set

$$\Gamma_{f(\cdot)+b} = \left\{ \sup_{0 \leqq t \leqq T} X(t) - f(t) < b \right\}.$$

First, using the Corollary to Theorem 3, we have that

$$P\,[\Gamma_{f(\cdot)+b}] = e^{-1/2 \int_0^T \alpha^2(t)dt} \int_{\Gamma_b} e^{-\int_0^T \alpha(t)dx(t)} d_W x.$$

Next, using Corollary 2 of Theorem 4, we obtain

$$P[\Gamma_{f(\cdot)+b}] = e^{-1/2 \int_0^T \alpha^2(t)dt} \sum_{n=0}^{\infty} K_n(T)$$

where $K_n(T)$ is given by equation (5). This expression is an entirely different series expansion of the probability than the ones given by Park and Paranjape in **[5]**.

D. Application 3. The Corollary to Theorem 3 is also useful to evaluate integrals of the type

$$\int_{\Gamma_f} e^{\int_0^T \alpha(t)dx(t)} d_W x$$

numerically for given $\alpha(t)$, $f(t)$, and T. By the Corollary, the above integral is equal to

$$e^{1/2 \int_0^T \alpha^2(t)dt} P\left\{ \sup_{0 \leqq t \leqq T} X(t) - \left[f(t) - \int_0^t \alpha(s)ds \right] < 0 \right\},$$

and the last probability can be evaluated numerically using the Park–Schuurmann method **[6]**.

The following table was computed by an IBM/168 with the unit interval divided into 2^9 equal subintervals.

Estimates of $\int_{\Gamma_f} e^{\int_0^1 \alpha(t)dx(t)} d_W x$

$\alpha(t)$	$\sin t$	e^t	t	$\sqrt{t}$
$f(t)$	$t+1$	t^2+t+1	$\cos t$	$ln(t+1)+1$
The integral	.976414	3.729278	.467819	.939285

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Received April 2, 1976.

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