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RATIONAL APPROXIMATION TO x^n

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This note is concerned with the approximations of x^n on $[0, 1]$ by polynomials and rational functions having only non-negative coefficients and of degree at most $k (1 \leqq k \leqq n-1)$. It is shown that the best approximating polynomial of degree k on $[0, 1]$ to x^n is of the form

$$p_k(x) = dx^k ,$$

where $d > 0$ and satisfies the assumption that

$$n(1-d) = (n-k)\left(\frac{k}{n}\right)^{k/(n-k)} d^{n/(n-k)} ,$$

with an error $\varepsilon_k = 1 - d$, for each fixed $k = 1, 2, 3, \cdots, n-1$. It is also shown that dx^k is a best approximating rational function of degree k to x^n on $[0, 1]$.

More than one hundred years ago Chebyshev showed that x^n can be uniformly approximated on $[-1, 1]$ by polynomials of degree at most $(n-1)$ with an error of exactly 2^{-n+1}.

Just recently D. J. Newman [1] has shown that x^n can be uniformly approximated on $[-1, 1]$ by rational functions of degree at most $(n-1)$ with an error roughly $\sqrt{n}(3\sqrt{3})^{-n}$.

If one looks carefully at the above results, then the following questions arise naturally.

Q.1: How close can one approximate x^n uniformly on $[0, 1]$ by polynomials of degree $(n-1)$ having only non-negative coefficients?

Q.2: Is the error obtained by rational functions of degree $(n-1)$ having only nonnegative coefficients in approximating x^n on $[0, 1]$ less than the error obtained by polynomials of degree $(n-1)$ having only nonnegative coefficients?

We answer these questions in this note.

Let

$$(1) \qquad \varepsilon_k = \inf_{p \in \pi_k^*} ||x^n - p(x)||_{L\infty[0,1]}$$

where $\pi_k^* (1 \leqq k < n)$ denotes the class of all algebraic polynomials of degree at most k having only nonnegative coefficients.

$$(1') \qquad \theta_k = \inf_{p, q \in \pi_k^*} \left|\left| x^n - \frac{p(x)}{q(x)} \right|\right|_{L\infty[0,1]} .$$

THEOREM 1. *If* $p_k(x) = dx^k$, $1 \leqq k < n$, *where* $d > 0$ *and satisfies the assumption that*

$$n(1-d) = (n-k)\left(\frac{k}{n}\right)^{k/(n-k)} d^{n/(n-k)} \tag{2}$$

then $p_k(x)$ *is a best approximating polynomial to* x^n *in the sense of* (1). *In fact, we get*

$$n\varepsilon_k = (n-k)\left(\frac{k}{n}\right)^{k/(n-k)}(1-\varepsilon_k)^{n/(n-k)} . \tag{3}$$

Proof. Let

$$p_k(x) = d\,x^k \tag{4}$$

then it is easy to see by finding a point where $|x^n - p_k(x)|$ attains its maximum on [0, 1], that

$$\varepsilon_k \leqq \|x^n - p_k(x)\|_{L\infty[0,1]} = \max\left\{(1-d),\ \left(\frac{n-k}{n}\right)\left(\frac{k}{n}\right)^{k/(n-k)} d^{n/(n-k)}\right\} . \tag{5}$$

From (2), it is clear that

$$\varepsilon_k \leqq \|x^n - p_k(x)\|_{L\infty[0,1]} = (1-d) . \tag{6}$$

So that, again by (2), we obtain

$$n\,\varepsilon_k \leqq (1-\varepsilon_k)^{n/(n-k)}(n-k)\left(\frac{k}{n}\right)^{k/(n-k)} . \tag{7}$$

Now we get the lower bound to $n\,\varepsilon_k$.

From (1) and the nonnegativity of the coefficients we get

$$\begin{aligned}\varepsilon_k \geqq p(x) - x^n &\geqq [p(1)]x^k - x^n = [p(1)-1]x^k + x^k - x^n \\ &\geqq x^k(-\varepsilon_k + 1 - x^{n-k})\end{aligned}$$

i.e.,

$$\varepsilon_k \geqq \frac{x^k(1-x^{n-k})}{1+x^k} . \tag{8}$$

$\dfrac{(1-x^{n-k})x^k}{1+x^k}$ attains its maximum for values of x satisfying

$$x^{n-k} = \frac{k}{n}\left(\frac{1+x^n}{1+x^k}\right) .$$

Hence for this value of x, we obtain

$$(9) \quad \varepsilon_k \geqq x^k\left(\frac{n-k}{k}\right)x^{n-k} = \frac{x^n(n-k)}{k} = \frac{k - n\,x^{n-k}}{k} = 1 - \frac{n}{k}\,x^{n-k}\,.$$

From (9) we get

$$x^{n-k} \geqq (1-\varepsilon_k)\,\frac{k}{n}$$

i.e.,

$$(10) \qquad x \geqq \left[(1-\varepsilon_k)\frac{k}{n}\right]^{1/(n-k)}\,.$$

From (9) and (10) we obtain

$$(11) \qquad \varepsilon_k \geqq (1-\varepsilon_k)^{n/(n-k)}\left(\frac{k}{n}\right)^{n/(n-k)}\left(\frac{n-k}{k}\right)\,.$$

From (7) and (11) we get

$$n\,\varepsilon_k = (1-\varepsilon_k)^{n/(n-k)}(n-k)\left(\frac{k}{n}\right)^{k/(n-k)}\,.$$

Hence, $p_k(x) = d\,x^k$ is a best approximating polynomial in the sense of (1).

THEOREM 2.

$$(12) \qquad \varepsilon_k = \theta_k \textit{ for all } k(1 \leqq k < n)\,.$$

Proof. By definition, for a $p(x)$ and $q(x)$, we have

$$(13) \qquad \left\|x^n - \frac{p(x)}{q(x)}\right\|_{L\infty[0,1]} = \theta_k\,.$$

From (13) we get as earlier

$$(14) \qquad \theta_k \geqq \frac{p(x)}{q(x)} - x^n \geqq \frac{p(1)x^k}{q(1)} - x^n$$
$$= \left(\frac{p(1)}{q(1)} - 1\right)x^k + x^k - x^n \geqq x^k(1 - x^{n-k} - \theta_k)\,.$$

i.e.,

$$(15) \qquad \theta_k \geqq \frac{x^k(1-x^{n-k})}{1+x^k}$$

(8) and (15) being the same in terms of x, n and k, we get

$$(16) \qquad n\,\theta_k \geqq (n-k)\left(\frac{k}{n}\right)^{k/(n-k)}(1-\theta_k)^{n/(n-k)}\,.$$

From Theorem 1 and (16), we obtain

$$(17) \qquad (1-\varepsilon_k)^{n/(n-k)}\left(\frac{k}{n}\right)^{k/(n-k)} \geqq \varepsilon_k\left(\frac{n}{n-k}\right) \geqq \left(\frac{n}{n-k}\right)\theta_k$$
$$\geqq (1-\theta_k)^{n/(n-k)}\left(\frac{k}{n}\right)^{k/(n-k)} \geqq (1-\varepsilon_k)^{n/(n-k)}\left(\frac{k}{n}\right)^{k/(n-k)} .$$

(12) follows easily from (17). Hence the result is proved.

Remarks on Theorems 1 and 2. According to ([2], Theorem 6) p_k of our Theorem 1 is unique. Hence p_k is the best approximating polynomial in the sense of (1). (ii) As a result of Theorems 1 and 2 a best approximation to x^n in the sense of (1′) is also

$$p_k(x) - dx^k ,$$

where $d > 0$, satisfies (2). (iii) Let us suppose $\varepsilon_k < 1 - d$, then from (2) and (3), we get $\varepsilon_k > 1 - d$. Similarly, assume $\varepsilon_k > 1 - d$, then we get from (2) and (3), $\varepsilon_k < 1 - d$. Hence we have from (2) and (3),

$$\varepsilon_k = 1 - d, \text{ for each fixed } k = 1, 2, \cdots, n-1 .$$

(iv) For the case $k = n - 1$, we get

$$\theta_{n-1} = \varepsilon_{n-1} \sim \frac{c}{n} ,$$

where c satisfies the equation $ce^{c+1} = 1$.

References

1. D. J. Newman, *Rational approximation to x^n*, J. Approximation Theory, to appear.
2. J. A. Roulier and G. D. Taylor, *Uniform approximation having bounded coefficients*, Abhand. aus dem Math. Sem. der Univ. Hamburg band, **36** (1971), 126–135.

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