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PRINCIPAL AND INDUCED FIBRATIONS

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In this paper, the following is proved.

Theorem. Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration in which E and B have the homotopy type of CW complexes. Suppose that F is $(n-1)$ connected and B is $(m-1)$ connected, where $m, n \geqq 2$. Let $l = \min(m, n), k = \min(2m-1, 2n)$. Suppose that there exists a map $E \times F \to E$ of type $(1, i)$. If $\pi_q(B) = 0$ for all $q \geqq n + l$, then the fibration is Ganea principal. If further $\pi_q(F) = 0$ for all $q \geqq n + k$, then the fibration is induced by some map $f: B \to Y$ for some space Y. The dual is also true.

1. All spaces in this paper are provided with a base point, and all maps and homotopies are assumed to preserve base points. In [2], Ganea proved the following.

THEOREM 1. *teL $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration in which E and B have the homotopy type of CW complexes. Suppose that B is $(m-1)$ connected and F is $(n-1)$ connected, where $m, n \geqq 2$. Let $l = \min(m, n)$. Suppose that i maps ΩF into the centre of ΩE. If $\pi_q(B) = 0$ for all $q \geqq n + l$ and $\pi_q(F) = 0$ for all $q \geqq n + 2l - 1$, then the fibration is principal and induced by some map $f: B \to Y$.*

In [2], Ganea calls a fibration $F \xrightarrow{i} E \xrightarrow{p} B$ principal if there exists a map $\phi: E \times F \to E$ and an H-structure $m: F \times F \to F$ such that $\phi(i \times 1) = im$ and $p\phi = P$ where $P: E \times F \to B$ is defined by $P(x, y) = p(x)$. It is said to be induced by a map $f: B \to Y$ for some space Y such that $F \cong \Omega Y$ if it is equivalent to the pull back $\Omega Y \to W \xrightarrow{\pi} B$ by f of the path space fibration $\Omega Y \to PY \to Y$, that is, if there exists a homotopy equivalence $g: E \to W$ such that $\pi g = p$. In the rest of the paper, we shall refer to a fibration which is principal in the sense of Ganea as being Ganea-principal.

Various other people have considered principal fibrations slightly differently. In particular, Meyer [4], Porter [6], [7] and Nowlan [5] have considered these questions from various other points of view and have obtained interesting results. In §3, we shall briefly indicate the connection between their work and our results.

In [2], Ganea says that a map $f: A \to X$ maps ΩA into the centre of ΩX if $(\Omega f)_\#: [Z, \Omega A] \to [Z, \Omega X]$ has image contained in the centre of $[Z, \Omega X]$ for all spaces Z. It is proved there that this is equivalent to the following. Let $XbA \xrightarrow{L} X \vee A$ be the fibre of the usual inclusion $X \vee A \rightarrow X \times A$. Then f maps ΩA into the centre of ΩX if and only if $\nabla(1 \vee f)L \cong *: XbA \rightarrow X$ where $\nabla: X \vee X \rightarrow X$ is the folding map. Examples are given in [2] to show that the dimensions imposed on the homotopy of B and F are best possible.

The question of whether or not a given fibration is induced is equivalent to the question of whether or not a map is homotopic to the inclusion of the fibre of some fibration. Thus a fibration $F \rightarrow E \rightarrow B$ is induced means that we can fit it into a sequence $F \rightarrow E \rightarrow B \rightarrow Y$ where any two consecutive maps form a fibre triple. Obviously, a necessary condition is that $F \cong \Omega Y$. Another necessary condition is that $F \rightarrow E$ must be homotopic to the "boundary" map in the Puppe sequence of $E \rightarrow B \rightarrow Y$. Since this may be taken to be $\rho / \Omega Y$ where $\rho: E \times \Omega Y \rightarrow E$ is the operation of the loop space of the base space on the fibre E, it follows that ρ is a map of type $(1, \partial)$ where ∂ is the "boundary". We make the following definition.

DEFINITION. Let $f: A \rightarrow X$ be a map. We say that f is cyclic if $\nabla(1 \vee f): X \vee A \rightarrow X$ extends to $X \times A$, that is, if there exists a map $\phi: X \times A \rightarrow X$ of type $(1, f)$.

The property of being cyclic is a property of the homotopy class of f. We observe that if $F \xrightarrow{i} E \xrightarrow{p} B$ is induced by some map $f: B \rightarrow Y$, then $F \cong \Omega Y$, and i may be taken to be the boundary ∂ in the Puppe sequence of $E \xrightarrow{p} B \xrightarrow{f} Y$. Hence i is cyclic.

We note that if $f: A \rightarrow X$ is cyclic, then f maps ΩA into the centre of ΩX. This follows from the fibration $XbA \rightarrow X \vee A \rightarrow X \times A$. If $\nabla(1 \vee f): X \vee A \rightarrow X$ extends to $X \times A$, then clearly $\nabla(1 \vee f)L \cong *$. We intend to replace the condition "i maps ΩF into the centre of ΩE" in Theorem 1 by the stronger condition "i is cyclic." This is intended to enable us to deduce a stronger conclusion. However, we observe that, under the conditions of Theorem 1, the two statements are equivalent. This follows from the following.

THEOREM 2. *Let $f: A \rightarrow X$ be a map and suppose that A is $(m-1)$ coonected and X is $(n-1)$ connected. Let $l = \min(m, n)$. Suppose that f maps ΩA into the centre of ΩX and that $\pi_j(X) = 0$ for all $j \geqq m + n + l - 1$. Then f is cyclic.*

Proof. Consider the fibration $XbA \xrightarrow{L} X \vee A \rightarrow X \times A$. By hypothesis, $\nabla(1 \vee f)L \cong *$. We may factor the inclusion $X \vee A \rightarrow X \times A$ as

$$\begin{tikzcd}
X \vee A \arrow[rr] \arrow[d, "j"'] \arrow[dr, phantom, "\copyright"] & & X \times A \\
X \vee A \bigcup_L C(XbA) \arrow[urr, "k"'] & &
\end{tikzcd}$$

where k extends the inclusion $X \vee A \subset X \times A$ by mapping $C(XbA)$ to the base point. Now from the cofibration $XbA \xrightarrow{L} X \vee A \xrightarrow{j} X \vee A \bigcup_L C(XbA)$, since $\nabla(1 \vee f)L \cong *$, we have a map $g\colon X \vee A \bigcup_L C(XbA) \to X$ such that $gj \cong \nabla(1 \vee f)$. Now consider the following situation

$$\begin{tikzcd}
XbA \arrow[r, "L"] & X \vee A \arrow[r] \arrow[d, "j"] \arrow[dr, phantom, "\copyright"] & X \times A \arrow[r] \arrow[dr, phantom, "\ominus"] & X \wedge A \\
& X \vee A \bigcup_L C(XbA) \arrow[ur, "k"'] \arrow[dr] & & \\
& & \Sigma(XbA) \arrow[uur, dashed, "u"'] &
\end{tikzcd}$$

where $X \times A \to X \wedge A$ is the cofibre of $X \vee A \to X \times A$ and $XbA \xrightarrow{L} X \vee A \xrightarrow{j} X \vee A \bigcup_L C(XbA) \to \Sigma(XbA) \to$ is the Puppe sequence of the cofibration, and u is determined in the obvious way. Since $XbA \cong \Sigma(\Omega X \wedge \Omega A)$ (see [1]) it is easily calculated that XbA is $n + m - 2$ connected. Hence $X \vee A \to X \times A$ is $n + m - 1$ connected. Also $X \times A$ is $(l-1)$ connected. Applying the Serre theorem, which is dual to the Blakers-Massey theorem (see [3]), we see that u is $n + m + l - 1$ connected. Hence by the 5-lemma, it follows that k is $n + m + l - 1$ connected. Since $\pi_j(X) = 0$ for all $j \geqq n + m + l - 1$, by obstruction theory, we can find a map $\phi\colon X \times A \to X$ such that $\phi k \cong g$. Hence $\phi kj \cong gj \cong \nabla(1 \vee f)$, where $kj\colon X \vee A \subset X \times A$ is the inclusion. Hence f is cyclic.

REMARK. Thus in Theorem 1, we may replace the statement "i maps ΩF into the centre of ΩE" by "i is cyclic."

We need the following two facts due to Ganea [2]. Suppose $F \xrightarrow{i} E \xrightarrow{p} B$ is a fibration and suppose that $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a triple, that is, $gf = *$. Suppose that have maps $\phi\colon X \to F$, $\varepsilon\colon Y \to E$ such that $\varepsilon f \cong i\phi$. Let $h_t\colon X \to E$ be a homotopy such that $h_0 = i\phi$, $h_1 = \varepsilon f$. Then ϕ, ε and h_t define a map $\lambda\colon Y \bigcup_f CX \to E \bigcup_i CF$ by $\lambda(y) = \varepsilon(y)$ and

$$\lambda(sx) = \begin{cases} 2s\phi(x) & 0 \leqq s \leqq \dfrac{1}{2} \\ h_{2s-1}(x) & \dfrac{1}{2} \leqq s \leqq 1 \,. \end{cases}$$

Let $k: Y \cup CX \to Z$ extend g by mapping CX to the base point, and let $r: E \cup CF \to B$ extend p by mapping CF to the base point. Then the triangles in the following diagram commute.

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[ddd, "\phi"'] \arrow[dddr, phantom, "\ominus"] & Y \arrow[rr, "g"] \arrow[rr, phantom, "\copyright"', shift right=3ex] \arrow[dr] \arrow[ddd, "\varepsilon"] \arrow[ddr, phantom, "\copyright"] & & Z \arrow[ddd, dashed, "\beta"] \arrow[ddl, phantom, "\ominus"] \\
& & Y \cup CX \arrow[ur, "k"'] \arrow[d, "\lambda"] & \\
& & E \cup CF \arrow[dr, "r"] & \\
F \arrow[r, "i"'] & E \arrow[rr, "p"'] \arrow[rr, phantom, "\copyright", shift left=3ex] \arrow[ur] & & B .
\end{tikzcd}
$$

We have the following result.

LEMMA 1 (Ganea [2]). *Suppose that in the above situation there exist a map $\beta: Z \to B$ such that $\beta k \cong r\lambda$. Then we can find maps $\phi_1 \cong \phi$, $\varepsilon_1 \cong \varepsilon$ making the squares in the following diagram commutative.*

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\phi_1"] & Y \arrow[r, "g"] \arrow[d, "\varepsilon_1"] & Z \arrow[d, "\beta"] \\
F \arrow[r, "i"'] & E \arrow[r, "p"'] & B .
\end{tikzcd}
$$

Proof. See Lemma 1.1 of [2].

We also need the following.

THEOREM 3 (Ganea [2]). *Suppose that $F \xrightarrow{i} E \xrightarrow{p} B$ is a fibration with E and B having the homotopy type of CW complexes. Suppose that B is $(m-1)$ connected and F is $(n-1)$ connected, $m, n \geqq 1$. Suppose that $\pi_q(F) = 0$ for all $q \geqq n + 2m - 1$. If the fibration is Ganea-principal and if there there exists a space Y and a homotopy equivalence $F \to \Omega Y$ which is also an H-map, then the fibration is induced by some map $f: B \to Y$.*

We now state our main result.

THEOREM 4. *Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration in which E and B have the homotopy type of CW complexes. Suppose that F is*

$(n-1)$ connected and B is $(m-1)$ connected, $m, n \geqq 2$. Let $l = \min(m, n)$, $k = \min(2m-1, 2n)$. Suppose that i is cyclic. If $\pi_q(B) = 0$ for all $q \geqq n + l$, then the fibration is Ganea-principal. Further, if $\pi_q(F) = 0$ for all $q \geqq n + k$, then the fibration is induced by some map $f: B \to Y$ for some space Y.

Proof. We assume that $\pi_q(B) = 0$ for all $q \geqq n + l$. Since i is cyclic, we can find a map $\phi: E \times F \to E$ of type $(1, i)$. We factor

$$\begin{tikzcd} E \vee F \arrow[rr] \arrow[dr, "j"'] & {\scriptstyle ©} & E \times F \\ & E \vee F \bigcup\limits_{L} C(EbF) \arrow[ur, "k"'] & \end{tikzcd}$$

that is, $kj: E \vee F \to E \times F$ is the usual inclusion. Let $P: E \times F \to B$ be given by $P(x, y) = p(x)$. Then $Pkj = p\phi kj$. Now, in the Puppe sequence of the cofibration

$$EbF \xrightarrow{L} E \vee F \xrightarrow{j} E \vee F \cup C(EbF) \longrightarrow \Sigma(EbF) \longrightarrow \cdots$$

there is an operation

$$[E \vee F \cup C(EbF), B] \times [\Sigma(EbF), B] \longrightarrow [E \vee F \cup C(EbF), B]$$

and $Pkj = p\phi kj$ if and only if we can find a map $\beta: \Sigma(EbF) \to B$ such that $Pk \cong p\phi k \top \beta$ rel. $E \vee F$, where we denote the operation by $\top$. It is easily calculated that $\Sigma(EbF)$ is $n + l - 1$ connected. Since $\pi_q(B) = 0$ for all $q \geqq n + l$, there is no obstruction to null-homotopy of β, that is, $\beta \cong *$. Let $\varepsilon: \Sigma(EbF) \to E$ be the constant map. Then $\beta \cong p\varepsilon$, that is, $Pk \cong p\phi k \top p\varepsilon$ rel. $E \vee F$. Let $g_0 = \phi k \top \varepsilon: E \vee F \cup C(EbF) \to E$. Then $Pk \cong pg_0$ rel. $E \vee F$. Note also that $g_0 j = (\phi k \top \varepsilon) j = \phi kj$. Since $Pk \cong pg_0$ rel. $E \vee F$, and since p is a fibration, we can find a homotopy $g_t: E \vee F \cup C(EbF) \to E$ such that $pg_1 = Pk$, and $pg_t d = *$ where $d = j(i \vee 1): F \vee F \to E \vee F \to E \vee F \cup C(EbF)$. Hence we can find a homotopy $\nabla_t: F \vee F \to F$ such that $i\nabla_t = g_t d$. Hence $i\nabla_0 = g_0 d = g_0 j(i \vee 1) = \phi kj(i \vee 1) = \nabla(1 \vee i)(i \vee 1) = i\nabla$, that is, $\nabla_0 = \nabla$ and $\nabla_1 \cong \nabla$. We can form the following diagram

$$\begin{tikzcd} F \vee F \arrow[r, "d"] \arrow[d, "\nabla_1"'] \arrow[dr, phantom, "©"] & E \vee F \cup C(EbF) \arrow[r] \arrow[d, "g_1"] \arrow[dr, phantom, "©"] & E \vee F \cup C(EbF) \cup C(F \vee F) \arrow[r, "S"] \arrow[d, "G"] \arrow[dr, phantom, "©"] & B \arrow[d, equal] \\ F \arrow[r, "i"'] & E \arrow[r] & E \cup CF \arrow[r, "r"'] & B \end{tikzcd}$$

where G is induced by ∇_1 and g_1, r extends $p: E \to B$ by mapping CF to the base point, and S extends $Pk: E \vee F \cup C(EbF) \to B$ by mapping $C(F \vee F)$ to the base point. We observe that $g_1 \cong g_0 \cong \phi k$.

Let $H_t\colon E \vee F \cup C(EbF) \to E$ be a homotopy such that $H_0 = g_1$, $H_1 = \phi k$. Let $h_t = H_t d\colon F \vee F \to E \vee F \cup C(EbF) \to E$. Then $h_0 = H_0 d = g_1 d = i\nabla_1$, $h_1 = H_1 d = \phi k d$. Thus we have the following diagram

$$\begin{tikzcd}
F \vee F \arrow[r, "kd"] \arrow[dddd, "\nabla_1"'] \arrow[ddddr, phantom, "\ominus"] & E \times F \arrow[rr, "P"] \arrow[dr] \arrow[dddd, "\phi"] \arrow[ddr, phantom, "\copyright"] & \arrow[d, phantom, "\copyright"] & B \arrow[dddd, equal] \arrow[dl, phantom, ""] \\
 & & E \times F \cup C(F \vee F) \arrow[ur, "R"'] \arrow[dd, "\eta"] & \\
 & & & \\
 & & E \cup CF \arrow[dr] \arrow[d, phantom, "\copyright"] & \\
F \arrow[r, "i"'] & E \arrow[ur] \arrow[rr, "p"'] & \ & B .
\end{tikzcd}$$

Here r extends p by mapping CF to the base point, and R extends P by mapping $C(F \vee F)$ to the base point. Also the maps ∇_1, ϕ and the homotopy h_t induce η. We claim that $r\eta \cong R$. In fact, the map $k\colon E \vee F \cup C(EbF) \to E \times F$ gives the following diagram

$$\begin{tikzcd}
F \vee F \arrow[r, "d"] \arrow[d, "1"'] \arrow[dr, phantom, "\copyright"] & E \vee F \cup C(EbF) \arrow[r] \arrow[d] \arrow[dr, phantom, "\copyright"] & E \vee F \cup C(EbF) \cup C(F \vee F) \arrow[d, "\psi"] \\
F \vee F \arrow[r, "kd"'] & E \times F \arrow[r] & E \times F \cup C(F \vee F)
\end{tikzcd}$$

where ψ is induced by k and 1. Then we check that $\eta\psi \cong G$, $R\psi = S$. Hence $r\eta\psi \cong rG = S = R\psi$. Since k is $(n + 2l - 1)$ connected, by the 5-lemma, it follows that ψ is also $n + 2l - 1$ connected. Since $\pi_q(B) = 0$ for all $q \geqq n + l$, there is no obstruction to a homotopy between $r\eta$ and R. Hence $r\eta \cong R$. We now apply Lemma 1 and conclude that we have maps $\nabla_1'\colon F \vee F \to F$, $\phi'\colon E \times F \to E$ with $\nabla_1' \cong \nabla_1 \cong \nabla$ and $\phi' \cong \phi$, and

$$\begin{tikzcd}
F \vee F \arrow[r, "kd"] \arrow[d, "\nabla_1'"] \arrow[dr, phantom, "\copyright"] & E \times F \arrow[r, "P"] \arrow[d, "\phi'"] \arrow[dr, phantom, "\copyright"] & B \arrow[d, equal] \\
F \arrow[r, "i"'] & E \arrow[r, "p"'] & B .
\end{tikzcd}$$

Hence $p\phi'(i \times 1) = P(i \times 1) = *$. This means that we can find a map $m\colon F \times F \to F$ with $im = \phi'(i \times 1)$. Let $t\colon F \vee F \to F \times F$ be the inclusion. Then $imt = \phi'(i \times 1)t = \phi' kd$ since $(i \times 1)t = kd$, that is, $imt = \phi' kd = i\nabla_1'$. Hence $mt = \nabla_1' \cong \nabla$. Thus m is an H-structure. Since $im = \phi'(i \times 1)$ we have the diagram

$$\begin{tikzcd} F \times F \arrow[r, "i \times 1"] \arrow[d, "m"] \arrow[dr, phantom, "©"] & E \times F \arrow[r, "P"] \arrow[d, "\phi'"] \arrow[dr, phantom, "©"] & B \arrow[d, equal] \\ F \arrow[r, "i"'] & E \arrow[r, "p"'] & B \end{tikzcd}$$

that is, the fibration is Ganea-principal.

Now suppose in addition that $\pi_q(F) = 0$ for all $q \geqq n + k$. Then $\pi_q(F) = 0$ for $q \geqq 3n$. Since F is an H-space and is $(n-1)$ connected, it follows by Theorem C of [3] that there is a homotopy equivalence $\theta: F \to \Omega Y$ which is also an H-map, for some space Y. In fact, Y may be constructed as follows. Applying the Hopf construction to the multiplication $m: F \times F \to F$, we get a map $\Sigma(F \wedge F) \to \Sigma F$, and hence $\Sigma(F \wedge F) \to \Sigma F \xrightarrow{v} \Sigma F \cup C\Sigma(F \wedge F)$. Here $\Sigma F \cup C\Sigma(F \wedge F) = FP$ is the F-projective plane. Let $(FP)_{3n}$ be the $3n$-Postnikov section of FP and let $\pi: FP \to (FP)_{3n}$ be the projection. We take $Y = (FP)_{3n}$ and θ to be the map $F \xrightarrow{\bar{v}} \Omega(FP) \xrightarrow{\Omega\pi} \Omega(FP)_{3n}$, $\bar{v}$ being the adjoint of v. It is easily seen that $\bar{v}$ is an H-map and hence θ is an H-map. The connectivity of $\bar{v}$ may be calculated by the Blakers-Massey theorem, and hence, it may be seen that θ is a homotopy equivalence. The proof of the theorem is now completed by applying Theorem 3.

REMARK 1. We observe that we have separated the conditions on B and F, and from each, we have deduced a conclusion. Ganea's Theorem 1, above, uses both the conditions on B and F to deduce the conclusion that the fibration is Ganea-principal. Our proof shows that the conclusion that the fibration is Ganea-principal uses only the condition on the homotopy of B.

REMARK 2. If $m \leqq n$, our theorem and that of Ganea are the same. However, if $m < n$, our theorem improves that of Ganea by allowing F to have an extra homotopy group. Thus our condition allows the fibration to be Ganea-principal even if $\pi_{3n-1}(F) \neq 0$, while Ganea's theorem requires that $\pi_{3n-1}(F) = 0$.

REMARK 3. The dimension condition on the homotopy of B is best possible. This is shown by the example given in [2]. Let Q be the rationals and let $n \geqq 4$ be even. Consider the fibration $K(Q, n) b K(Q, n) \to K(Q, n) \vee K(Q, n) \to K(Q, n) \times K(Q, n)$. Now $K(Q, n) b K(Q, n) \cong \Sigma(K(Q, n-1) \wedge K(Q, n-1))$. Since n is even, $(n-1)$ is odd, and hence $K(Q, n-1) \cong K'(Q, n-1)$. Hence

$$K(Q, n) b K(Q, n) \cong \Sigma K'(Q, 2n-2) = K'(Q, 2n-1) \cong K(Q, 2n-1)\,.$$

Thus we have a fibration

$$K(Q, 2n-1) \longrightarrow K(Q, n) \vee K(Q, n) \longrightarrow K(Q, n) \times K(Q, n) .$$

Since the fibre is a single Eilenberg-MacLane complex, by a classical result of Serre, this fibration is induced. It can only be induced by a map $K(Q, n) \times K(Q, n) \rightarrow K(Q, 2n)$. Thus we have a fibration

$$K(Q, n) \vee K(Q, n) \longrightarrow K(Q, n) \times K(Q, n) \longrightarrow K(Q, 2n) .$$

Observe that here $\pi_{n+l}(K(Q, 2n)) \neq 0$, $n + l$ being $2n$ here. All the other conditions of the theorem are satisfied. This fibration is not Ganea-principal since $K(Q, n) \vee K(Q, n)$ is not an H-space.

REMARK 4. We do not know if the dimension condition on the homotopy of F in Theorem 4 is best possible or not. However, we can say that if it is not best possible, then the best possible is the condition $\pi_q(F) = 0$ for all $q \geqq n + k + 1$. This is because we have the following example. Let $F = K(Z_3, 3; Z_9, 10; \lambda u(\beta u)^2)$ be the 2-stage Postnikov system, where $u \in H^3(Z_3, 3; Z_3)$ is the fundamental class, β is Bockstein operator, and λ is induced by the coefficient homomorphism $Z_3 \subset Z_9$. Then F is an H-space but not a loop-space (see page 599 of [3]). Thus $F \rightarrow F \rightarrow *$ is not induced. The map $F \rightarrow F$ here is cyclic since an identity map is cyclic if and only if the space is an H-space. Thus all the conditions of Theorem 4 are satisfied except that $\pi_{n+k+1}(F) \neq 0$. In fact, we only have $\pi_q(F) = 0$ for all $q \geqq n + k + 2$. Thus the best possible condition on F in Theorem 4 is either $\pi_q(F) = 0$ for all $q \geqq n + k$ or $\pi_q(F) = 0$ for all $q \geqq n + k + 1$.

Theorem 4 admits the following application. We assume here that all our spaces have the homotopy type of CW complexes. We recall that if we have a fibration $F \xrightarrow{i} E \xrightarrow{p} B$ where E and B are H-spaces, then if p is an H-map, it follows that F can be given an H-structure so that i is an H-map. Stasheff in [8] gives a converse under some restrictions.

THEOREM 5 (Stasheff). *Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration in which E and B are H-spaces. If F has a multiplication, then p is an H-map with respect to some multiplication on E, provided that E is $(n-1)$ connected, $\pi_q(E) = 0$ for $q \geqq n + m$, where $m \geqq n + 1$, and B is $(m-1)$ connected and $\pi_q(B) = 0$ for $q \geqq n + m$.*

We observe that if E is an H-space, then i is automatically cyclic. In fact, if $m: E \times E \rightarrow E$ is the H-structure in E, then the map $E \times F \rightarrow E$ of type $(1, i)$ required can be taken to be $m(1 \times i)$. Using this fact, it follows from Theorems 4 and 5 that we have the following.

THEOREM 6. *Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration in which F is $(n-1)$ connected, B is $(m-1)$ connected, and $\pi_q(B) = 0$ for $q \geqq 2n$, where $m \geqq n+1$. Suppose that E and B are H-spaces. Then p is an H-map for some multiplication on E.*

2. These theorems can be dualized. We state the dual of Theorem 4. Recall that a cofibration $A \xrightarrow{d} X \xrightarrow{f} C$ is Ganea-principal if we can find a co-H-structure $m\colon C \to C \vee C$ and a map $\phi\colon X \to X \vee C$ such that the following diagram commutes

$$\begin{tikzcd}
C \vee C & X \vee C \arrow[l, "f \vee 1"'] & A \arrow[l, "D"'] \\
C \arrow[u, "m"] \arrow[ur, phantom, "©"] & X \arrow[l, "f"] \arrow[u, "\phi"] \arrow[ur, phantom, "©"] & A \arrow[l, "d"] \arrow[u, equal]
\end{tikzcd}$$

where $D(a) = (d(a), *)$. We are following the terminology of [2]. The cofibration is induced if there is a space Y and a map $g\colon Y \to A$ such that the cofibration is equivalent to the cofibration strictly induced by g from $Y \to CY \to \Sigma Y$, that is, to the triple $A \to A \cup_g CY \to \Sigma Y$, This means that there is a homotopy-equivalence $X \to A \cup_g CY$ such that

$$\begin{tikzcd}
A \arrow[r] \arrow[dr, "d"'] \arrow[dr, phantom, shift left=4, "©"] & A \cup_g CY \\
& X . \arrow[u]
\end{tikzcd}$$

For a 1-connected CW complex K, we write $\dim K \leqq n$ to indicate that $H_n(K)$ is free and $H_q(K) = 0$ for $q > n$. We say that a map $f\colon X \to A$ is cocyclic if the map $(1 \times f)\Delta\colon X \to X \times A$ is compressible into $X \vee A$. This is homotopy property of f. The dual of Theorem 4 is the following.

THEOREM 7. *Let $A \xrightarrow{d} X \xrightarrow{f} C$ be a cofibration in which (X, A) is a CW pair. Suppose that A is $(m-1)$ connected, and that C is n-connected, $m \geqq 2$, $n \geqq 1$. Suppose that f is cocyclic. If $\dim A \leqq n + \min(m-1, n)$, then the cofibration is Ganea-principal. If further, $\dim C \leqq n + \min(2m-1, 2n)$, then the cofibration is induced.*

Proof. Dualize the proof of Theorem 4.

3. We now briefly consider the connection between our results and the work of Meyer [4], Nowlan [5] and Porter [6], [7]. We refer the reader to these papers for detailed definitions, but we shall

indicate how their notions can be expressed in our terminology. Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration. Then it is an H-fibration in the sense of Meyer [4] if it is Ganea-principal, with i being cyclic and ϕ being a map of type $(1, i)$. A principal fibration in the sense of Meyer [4] is an induced fibration in the sense of Ganea [2]. All the results in [4] concerning principal fibrations hold for induced fibrations, and all the results concerning H-fibrations hold for Ganea-principal fibrations.

An H-fibration in the sense of Porter [6] is also Ganea-principal with i being cyclic and ϕ being of type $(1, i)$. We might emphasise that our definition of a Ganea-principal fibration does not require i to be cyclic and does not require ϕ to be of type $(1, i)$. However, a simple examination shows that most of the results in Meyer [4] on H-fibrations and most of those in Porter [6] on H-fibrations do not require these extra conditions. They merely require that the fibrations be Ganea-principal. In particular, we mention that Theorem 1 of Porter [6] holds for Ganea-principal fibrations. There is no need to assume that i is cyclic or that ϕ is of type $(1, i)$. The proof carries over word for word. Thus we have the following which we shall attribute to Porter.

THEOREM 8 (Porter). *Let $F \xrightarrow{i} E \xrightarrow{p} B$ be Ganea-principal, with B being path-connected. Let $g_1, g_2\colon X \to E$ be maps with $pg_1 \simeq pg_2$, X being a CW complex. Then there exists a map $u\colon X \to F$ with $\phi(g_2 \times u)\Delta \simeq g_1$, where $\Delta\colon X \to X \times X$ is the diagonal map.*

In [7] Porter defines his principal fibrations to be Ganea-principal with (F, m) being an associative H-space, and the map $\phi\colon E \times F \to E$ of type $(1, i)$ is required to be an associative action of E. Let $F \xrightarrow{i} E \xrightarrow{p} B$ be such a principal fibration and let $f\colon B' \to B$ be a map. Let $E_f = \{(b, e) \text{ in } B' \times E \mid f(b) = p(e)\}$. Then $F \to E_f \to B'$ is also a principal fibration and

$$\begin{tikzcd} F \arrow[r] \arrow[d, equal] & E_f \arrow[r] \arrow[d] & B' \arrow[d, "f"] \\ F \arrow[r] & E \arrow[r] & B \end{tikzcd}$$

is a homomorphism of principal fibrations. We ask the reader to refer to Porter [7] for the definitions of the various terms we shall be using.

Let $F \xrightarrow{i} E \xrightarrow{p} B$ and $F' \xrightarrow{i'} E' \xrightarrow{p'} B'$ be fibrations. Then we have the principal fibrations $\Omega B \to E_p \to E$, $\Omega B' \to E_{p'} \to E'$. Theorem 12 of Porter [7] says that there exists a map $f\colon B \to B'$ such that $F \to E \to B$ is equivalent to the fibration induced by f from $F' \to E' \to B'$

if and only if there exists a strong homotopy homomorphism of principal fibrations

$$\begin{array}{ccccc} \Omega B & \longrightarrow & E_p & \longrightarrow & E \\ \downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ \Omega B' & \longrightarrow & E_{p'} & \longrightarrow & E' \end{array}$$

with g being a homotopy equivalence. Our results concern the case where $F' \to E' \to B'$ is a path space fibration $\Omega B' \to PB' \xrightarrow{\pi} B'$. Thus under the conditions stated there on the homotopy of B and F, our Theorem 4 says that there exists a strong homotopy homomorphism

$$\begin{array}{ccccc} \Omega B & \longrightarrow & E_p & \longrightarrow & E \\ \downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ \Omega B' & \longrightarrow & E_\pi & \longrightarrow & PB' \end{array}$$

with g being a homotopy equivalence.

Nowlan's H-fibrations [5] are our Ganea-principal fibrations. Nowlan considers fibrations in which an associative H-space (F, m) operates, but not necessarily associatively. Such fibrations are called A_1-principal fibre spaces. For example, all the various H-fibrations are such fibre spaces, and Ganea-principal fibrations are also such fibre spaces if (F, m) is associative. If instead of requiring that the action be associative, we only require that it be homotopy associative, that is, that the following diagram homotopy commutes

$$\begin{array}{ccc} E \times F \times F & \xrightarrow{\phi \times 1} & E \times F \\ \downarrow{\scriptstyle 1 \times m} & & \downarrow{\scriptstyle \phi} \\ E \times F & \xrightarrow[\phi]{} & E \end{array}$$

then we get an A_2-principal fibre space. Thus a principal fibration in the sense of Porter [7] is an A_2-principal fibre space. An A_n-principal fibre space is one where the action of F on E satisfies higher homotopy conditions. Nowlan [5] obtains a classification theorem. The notion of an A_∞-principal fibre space is also obtained. Nowlan proves the following.

THEOREM (Nowlan). *$p: E \to B$ is fibre homotopy equivalent to an induced fibre space if and only if E admits an A_∞-action of ΩB.*

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