PRINCIPAL AND INDUCED FIBRATIONS

C. S. Hoo

In this paper, the following is proved.

THEOREM. Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration in which E and B have the homotopy type of CW complexes. Suppose that F is (n-1) connected and B is (m-1) connected, where $m, n \ge 2$. Let $l = \min(m, n), k = \min(2m - 1, 2n)$. Suppose that there exists a map $E \times F \to E$ of type (1, i). If $\pi_q(B) = 0$ for all $q \ge n + l$, then the fibration is Ganea principal. If further $\pi_q(F) = 0$ for all $q \ge n + k$, then the fibration is induced by some map $f: B \to Y$ for some space Y. The dual is also true.

1. All spaces in this paper are provided with a base point, and all maps and homotopies are assumed to preserve base points. In [2], Ganea proved the following.

THEOREM 1. teL $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration in which E and Bhave the homotopy type of CW complexes. Suppose that B is (m-1)connected and F is (n-1) connected, where $m, n \ge 2$. Let l =min(m, n). Suppose that i maps ΩF into the centre of ΩE . If $\pi_q(B) = 0$ for all $q \ge n + l$ and $\pi_q(F) = 0$ for all $q \ge n + 2l - 1$, then the fibration is principal and induced by some map $f: B \to Y$.

In [2], Ganea calls a fibration $F \xrightarrow{i} E \xrightarrow{p} B$ principal if there exists a map $\phi: E \times F \to E$ and an *H*-structure $m: F \times F \to F$ such that $\phi(i \times 1) = im$ and $p\phi = P$ where $P: E \times F \to B$ is defined by P(x, y) =p(x). It is said to be induced by a map $f: B \to Y$ for some space *Y* such that $F \cong \Omega Y$ if it is equivalent to the pull back $\Omega Y \to W \xrightarrow{\pi} B$ by *f* of the path space fibration $\Omega Y \to PY \to Y$, that is, if there exists a homotopy equivalence $g: E \to W$ such that $\pi g = p$. In the rest of the paper, we shall refer to a fibration which is principal in the sense of Ganea as being Ganea-principal.

Various other people have considered principal fibrations slightly differently. In particular, Meyer [4], Porter [6], [7] and Nowlan [5] have considered these questions from various other points of view and have obtained interesting results. In $\S3$, we shall briefly indicate the connection between their work and our results.

In [2], Ganea says that a map $f: A \to X$ maps ΩA into the centre of ΩX if $(\Omega f)_{\sharp}: [Z, \Omega A] \to [Z, \Omega X]$ has image contained in the centre of $[Z, \Omega X]$ for all spaces Z. It is proved there that this is equivalent to the following. Let $XbA \xrightarrow{L} X \lor A$ be the fibre of the usual inclusion $X \lor A \to X \times A$. Then f maps ΩA into the centre of ΩX if and only if $\mathcal{V}(1 \lor f)L \cong *: XbA \to X$ where $\mathcal{V}: X \lor X \to X$ is the folding map. Examples are given in [2] to show that the dimensions imposed on the homotopy of B and F are best possible.

The question of whether or not a given fibration is induced is equivalent to the question of whether or not a map is homotopic to the inclusion of the fibre of some fibration. Thus a fibration $F \rightarrow E \rightarrow B$ is induced means that we can fit it into a sequence $F \rightarrow E \rightarrow B \rightarrow Y$ where any two consecutive maps form a fibre triple. Obviously, a necessary condition is that $F \cong \Omega Y$. Another necessary condition is that $F \rightarrow E$ must be homotopic to the "boundary" map in the Puppe sequence of $E \rightarrow B \rightarrow Y$. Since this may be taken to be $\rho/\Omega Y$ where $\rho: E \times \Omega Y \rightarrow E$ is the operation of the loop space of the base space on the fibre E, it follows that ρ is a map of type $(1, \partial)$ where ∂ is the "boundary". We make the following definition.

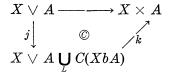
DEFINITION. Let $f: A \to X$ be a map. We say that f is cyclic if $V(1 \lor f): X \lor A \to X$ extends to $X \times A$, that is, if there exists a map $\phi: X \times A \to X$ of type (1, f).

The property of being cyclic is a property of the homotopy class of f. We observe that if $F \xrightarrow{i} E \xrightarrow{p} B$ is induced by some map $f: B \rightarrow Y$, then $F \cong \Omega Y$, and i may be taken to be the boundary ∂ in the Puppe sequence of $E \xrightarrow{p} B \xrightarrow{f} Y$. Hence i is cyclic.

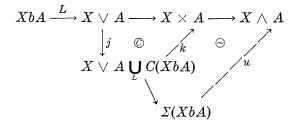
We note that if $f: A \to X$ is cyclic, then f maps $\mathcal{Q}A$ into the centre of $\mathcal{Q}X$. This follows from the fibration $XbA \to X \lor A \to X \times A$. If $\mathcal{V}(1 \lor f): X \lor A \to X$ extends to $X \times A$, then clearly $\mathcal{V}(1 \lor f)L \cong^*$. We intend to replace the condition "*i* maps $\mathcal{Q}F$ into the centre of $\mathcal{Q}E$ " in Theorem 1 by the stronger condition "*i* is cyclic." This is intended to enable us to deduce a stronger conclusion. However, we observe that, under the conditions of Theorem 1, the two statements are equivalent. This follows from the following.

THEOREM 2. Let $f: A \to X$ be a map and suppose that A is (m-1) connected and X is (n-1) connected. Let $l = \min(m, n)$. Suppose that f maps ΩA into the centre of ΩX and that $\pi_j(X) = 0$ for all $j \ge m + n + l - 1$. Then f is cyclic.

Proof. Consider the fibration $XbA \xrightarrow{L} X \lor A \longrightarrow X \times A$. By hypothesis, $\mathcal{V}(1 \lor f)L \cong *$. We may factor the inclusion $X \lor A \longrightarrow X \times A$ as



where k extends the inclusion $X \vee A \subset X \times A$ by mapping C(XbA) to the base point. Now from the cofibration $XbA \xrightarrow{L} X \vee A \xrightarrow{j} X \vee A \bigcup_{L} C(XbA)$, since $V(1 \vee f)L \cong *$, we have a map $g: X \vee A \bigcup_{L} C(XbA) \to X$ such that $gj \cong V(1 \vee f)$. Now consider the following situation



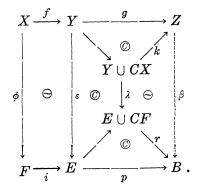
where $X \times A \to X \wedge A$ is the cofibre of $X \vee A \to X \times A$ and $XbA \xrightarrow{L} X \vee A \xrightarrow{j} X \vee A \bigcup_{L} C(XbA) \to \Sigma(XbA) \to is$ the Puppe sequence of the cofibration, and u is determined in the obvious way. Since $XbA \cong \Sigma(\Omega X \wedge \Omega A)$ (see [1]) it is easily calculated that XbA is n + m - 2 connected. Hence $X \vee A \to X \times A$ is n + m - 1 connected. Also $X \times A$ is (l-1) connected. Applying the Serre theorem, which is dual to the Blakers-Massey theorem (see [3]), we see that u is n + m + l - 1 connected. Hence by the 5-lemma, it follows that k is n + m + l - 1 connected. Since $\pi_j(X) = 0$ for all $j \ge n + m + l - 1$, by obstruction theory, we can find a map $\phi: X \times A \to X$ such that $\phi k \cong g$. Hence $\phi kj \cong gj \cong F(1 \vee f)$, where $kj: X \vee A \subset X \times A$ is the inclusion. Hence f is cyclic.

REMARK. Thus in Theorem 1, we may replace the statement "*i* maps ΩF into the centre of ΩE " by "*i* is cyclic."

We need the following two facts due to Ganea [2]. Suppose $F \xrightarrow{i} E \xrightarrow{p} B$ is a fibration and suppose that $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a triple, that is, gf = *. Suppose that have maps $\phi: X \to F, \varepsilon: Y \to E$ such that $\varepsilon f \cong i\phi$. Let $h_i: X \to E$ be a homotopy such that $h_0 = i\phi$, $h_1 = \varepsilon f$. Then ϕ, ε and h_i define a map $\lambda: Y \bigcup_f CX \to E \bigcup_i CF$ by $\lambda(y) = \varepsilon(y)$ and

$$\lambda(sx) = egin{cases} 2s\phi(x) & 0 \leq s \leq rac{1}{2} \ h_{2s-1}(x) & rac{1}{2} \leq s \leq 1 \ . \end{cases}$$

Let $k: Y \cup CX \rightarrow Z$ extend g by mapping CX to the base point, and let $r: E \cup CF \rightarrow B$ extend p by mapping CF to the base point. Then the triangles in the following diagram commute.



We have the following result.

LEMMA 1 (Ganea [2]). Suppose that in the above situation there exist a map $\beta: Z \to B$ such that $\beta k \cong r\lambda$. Then we can find maps $\phi_1 \cong \phi, \varepsilon_1 \cong \varepsilon$ making the squares in the following diagram commutative.

$$\begin{array}{cccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ & & & & \downarrow^{\varphi_1} & & \downarrow^{\varepsilon_1} & & \downarrow^{\beta} \\ F & \xrightarrow{i} & E & \xrightarrow{p} & B \end{array}$$

Proof. See Lemma 1.1 of [2].

We also need the following.

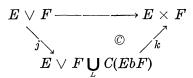
THEOREM 3 (Ganea [2]). Suppose that $F \xrightarrow{i} E \xrightarrow{p} B$ is a fibration with E and B having the homotopy type of CW complexes. Suppose that B is (m-1) connected and F is (n-1) connected, $m, n \ge 1$. Suppose that $\pi_q(F) = 0$ for all $q \ge n + 2m - 1$. If the fibration is Ganea-principal and if there there exists a space Y and a homotopy equivalence $F \rightarrow \Omega Y$ which is also an H-map, then the fibration is induced by some map $f: B \rightarrow Y$.

We now state our main result.

THEOREM 4. Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration in which E and B have the homotopy type of CW complexes. Suppose that F is

(n-1) connected and B is (m-1) connected, $m, n \ge 2$. Let $l = \min(m, n), k = \min(2m-1, 2n)$. Suppose that i is cyclic. If $\pi_q(B) = 0$ for all $q \ge n + l$, then the fibration is Ganea-principal. Further, if $\pi_q(F) = 0$ for all $q \ge n + k$, then the fibration is induced by some map $f: B \to Y$ for some space Y.

Proof. We assume that $\pi_q(B) = 0$ for all $q \ge n + l$. Since *i* is cyclic, we can find a map $\phi: E \times F \to E$ of type (1, *i*). We factor



that is, $kj: E \lor F \rightarrow E \times F$ is the usual inclusion. Let $P: E \times F \rightarrow B$ be given by P(x, y) = p(x). Then $Pkj = p\phi kj$. Now, in the Puppe sequence of the cofibration

$$EbF \xrightarrow{L} E \lor F \xrightarrow{j} E \lor F \cup C(EbF) \longrightarrow \Sigma(EbF) \longrightarrow \cdots$$

there is an operation

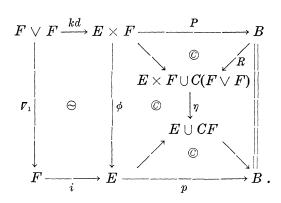
$$[E \lor F \cup C(EbF), B] \times [\Sigma(EbF), B] \longrightarrow [E \lor F \cup C(EbF), B]$$

and $Pkj = p\phi kj$ if and only if we can find a map $\beta: \Sigma(EbF) \to B$ such that $Pk \cong p\phi k \top \beta$ rel. $E \lor F$, where we denote the operation by \top . It is easily calculated that $\Sigma(EbF)$ is n + l - 1 connected. Since $\pi_q(B) = 0$ for all $q \ge n + l$, there is no obstruction to nullhomotopy of β , that is, $\beta \cong *$. Let $\varepsilon: \Sigma(EbF) \to E$ be the constant map. Then $\beta \cong p\varepsilon$, that is, $Pk \cong p\phi k \top p\varepsilon$ rel. $E \lor F$. Let $g_0 = \phi k \top \varepsilon: E \lor F \cup C(EbF) \to E$. Then $Pk \cong pg_0$ rel. $E \lor F$. Note also that $g_0j = (\phi k \top \varepsilon)j = \phi kj$. Since $Pk \cong pg_0$ rel. $E \lor F$, and since pis a fibration, we can find a homotopy $g_i: E \lor F \cup C(EbF) \to E$ such that $pg_1 = Pk$, and $pg_id = *$ where $d = j(i \lor 1): F \lor F \to E \lor F \to$ $E \lor F \cup C(EbF)$. Hence we can find a homotopy $\nabla_i: F \lor F \to F$ such that $i\nabla_t = g_td$. Hence $i\nabla_0 = g_0d = g_0j(i \lor 1) = \phi kj(i \lor 1) = V(1 \lor i)(i \lor 1) =$ $i\nabla$, that is, $\nabla_0 = \nabla$ and $\nabla_1 \cong \nabla$. We can form the following diagram

$$\begin{array}{cccc} F \lor F \stackrel{d}{\longrightarrow} E \lor F \cup C(EbF) \stackrel{\longrightarrow}{\longrightarrow} E \lor F \cup C(EbF) \cup C(F \lor F) \stackrel{S}{\longrightarrow} B \\ \downarrow & & & & \\ F_{1} \downarrow & & & \\ \downarrow & & & \\ F \stackrel{d}{\longrightarrow} E \stackrel{G}{\longrightarrow} E \stackrel{G}{\longrightarrow} E \cup CF \stackrel{G}{\longrightarrow} B \end{array}$$

where G is induced by V_1 and g_1 , r extends $p: E \to B$ by mapping CF to the base point, and S extends $Pk: E \lor F \cup C(EbF) \to B$ by mapping $C(F \lor F)$ to the base point. We observe that $g_1 \cong g_0 \cong \phi k$.

Let $H_t: E \vee F \cup C(EbF) \to E$ be a homotopy such that $H_0 = g_1, H_1 = \phi k$. Let $h_t = H_t d: F \vee F \to E \vee F \cup C(EbF) \to E$. Then $h_0 = H_0 d = g_1 d = i \mathcal{V}_1, h_1 = H_1 d = \phi k d$. Thus we have the following diagram



Here r extends p by mapping CF to the base point, and R extends P by mapping $C(F \lor F)$ to the base point. Also the maps \mathcal{V}_i, ϕ and the homotopy h_i induce η . We claim that $r\eta \cong R$. In fact, the map $k: E \lor F \cup C(EbF) \to E \times F$ gives the following diagram

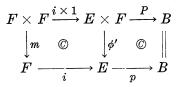
$$\begin{array}{cccc} F \lor F \stackrel{a}{\longrightarrow} E \lor F \cup C(EbF) \longrightarrow E \lor F \cup C(EbF) \cup C(F \lor F) \\ 1 & & & & & \\ f \lor F \stackrel{a}{\longrightarrow} E \times F \stackrel{c}{\longrightarrow} E \times F \cup C(F \lor F) \end{array}$$

where ψ is induced by k and 1. Then we check that $\eta \psi \cong G$, $R\psi = S$. Hence $r\eta \psi \cong rG = S = R\psi$. Since k is (n + 2l - 1) connected, by the 5-lemma, it follows that ψ is also n + 2l - 1 connected. Since $\pi_q(B) = 0$ for all $q \ge n + l$, there is no obstruction to a homotopy between $r\eta$ and R. Hence $r\eta \cong R$. We now apply Lemma 1 and conclude that we have maps $V'_1 \cong F \lor F \mapsto F$, $\phi' \colon E \times F \to E$ with $V'_1 \cong V_1 \cong V_1 \cong V$ and $\phi' \cong \phi$, and

$$F \lor F \xrightarrow{kd} E \times F \xrightarrow{P} B$$

 $\downarrow F'_1 @ \downarrow \phi' @ \parallel$
 $F \xrightarrow{i} E \xrightarrow{p} B.$

Hence $p\phi'(i \times 1) = P(i \times 1) = *$. This means that we can find a map $m: F \times F \to F$ with $im = \phi'(i \times 1)$. Let $t: F \vee F \to F \times F$ be the inclusion. Then $imt = \phi'(i \times 1)t = \phi'kd$ since $(i \times 1)t = kd$, that is, $imt = \phi'kd = i\Gamma'_1$. Hence $mt = \Gamma'_1 \cong \Gamma$. Thus m is an H-structure. Since $im = \phi'(i \times 1)$ we have the diagram



that is, the fibration is Ganea-principal.

Now suppose in addition that $\pi_q(F) = 0$ for all $q \ge n + k$. Then $\pi_q(F) = 0$ for $q \ge 3n$. Since F is an H-space and is (n - 1) connected, it follows by Theorem C of [3] that there is a homotopy equivalence $\theta: F \to \Omega Y$ which is also an H-map, for some space Y. In fact, Y may be constructed as follows. Applying the Hopf construction to the multiplication $m: F \times F \to F$, we get a map $\Sigma(F \wedge F) \to \Sigma F$, and hence $\Sigma(F \wedge F) \to \Sigma F \xrightarrow{v} \Sigma F \cup C\Sigma(F \wedge F)$. Here $\Sigma F \cup C\Sigma(F \wedge F) = FP$ is the F-projective plane. Let $(FP)_{3n}$ be the 3n-Postnikov section of FP and let $\pi: FP \to (FP)_{3n}$ be the projection. We take $Y = (FP)_{3n}$ and θ to be the map $F \xrightarrow{v} \Omega(FP) \xrightarrow{\Omega \pi} \Omega(FP)_{3n}, \overline{v}$ being the adjoint of v. It is easily seen that \overline{v} is an H-map and hence θ is an H-map. The connectivity of \overline{v} may be calculated by the Blakers-Massey theorem, and hence, it may be seen that θ is a homotopy equivalence. The proof of the theorem is now completed by applying Theorem 3.

REMARK 1. We observe that we have separated the conditions on B and F, and from each, we have deduced a conclusion. Ganea's Theorem 1, above, uses both the conditions on B and F to deduce the conclusion that the fibration is Ganea-principal. Our proof shows that the conclusion that the fibration is Ganea-principal uses only the condition on the homotopy of B.

REMARK 2. If $m \leq n$, our theorem and that of Ganea are the same. However, if m < n, our theorem improves that of Ganea by allowing F to have an extra homotopy group. Thus our condition allows the fibration to be Ganea-principal even if $\pi_{3n-1}(F) \neq 0$, while Ganea's theorem requires that $\pi_{3n-1}(F) = 0$.

REMARK 3. The dimension condition on the homotopy of B is best possible. This is shown by the example given in [2]. Let Q be the rationals and let $n \ge 4$ be even. Consider the fibration $K(Q, n)bK(Q, n) \rightarrow K(Q, n) \lor K(Q, n) \rightarrow K(Q, n) \times K(Q, n)$. Now $K(Q, n)bK(Q, n) \cong \Sigma(K(Q, n-1) \land K(Q, n-1))$. Since n is even, (n-1) is odd, and hence $K(Q, n-1) \cong K'(Q, n-1)$. Hence

 $K(Q, n)bK(Q, n) \cong \Sigma K'(Q, 2n - 2) = K'(Q, 2n - 1) \cong K(Q, 2n - 1)$. Thus we have a fibration

C. S. HOO

 $K(Q, 2n-1) \longrightarrow K(Q, n) \lor K(Q, n) \longrightarrow K(Q, n) \times K(Q, n)$.

Since the fibre is a single Eilenberg-MacLane complex, by a classical result of Serre, this fibration is induced. It can only be induced by a map $K(Q, n) \times K(Q, n) \rightarrow K(Q, 2n)$. Thus we have a fibration

 $K(Q, n) \lor K(Q, n) \longrightarrow K(Q, n) \times K(Q, n) \longrightarrow K(Q, 2n)$.

Observe that here $\pi_{n+l}(K(Q, 2n)) \neq 0$, n+l being 2n here. All the other conditions of the theorem are satisfied. This fibration is not Ganea-principal since $K(Q, n) \vee K(Q, n)$ is not an *H*-space.

REMARK 4. We do not know if the dimension condition on the homotopy of F in Theorem 4 is best possible or not. However, we can say that if it is not best possible, then the best possible is the condition $\pi_q(F) = 0$ for all $q \ge n + k + 1$. This is because we have the following example. Let $F = K(Z_3, 3; Z_9, 10; \lambda u(\beta u)^2)$ be the 2stage Postnikov system, where $u \in H^3(Z_3, 3; Z_3)$ is the fundamental class, β is Bockstein operator, and λ is induced by the coefficient homomorphism $Z_3 \subset Z_9$. Then F is an H-space but not a loop-space (see page 599 of [3]). Thus $F \to F \to *$ is not induced. The map $F \to F$ here is cyclic since an identity map is cyclic if and only if the space is an H-space. Thus all the conditions of Theorem 4 are satisfied except that $\pi_{n+k+1}(F) \neq 0$. In fact, we only have $\pi_q(F) = 0$ for all $q \ge n + k + 2$. Thus the best possible condition on F in Theorem 4 is either $\pi_q(F) = 0$ for all $q \ge n + k$ or $\pi_q(F) = 0$ for all $q \ge n + k + 1$.

Theorem 4 admits the following application. We assume here that all our spaces have the homotopy type of CW complexes. We recall that if we have a fibration $F \xrightarrow{i} E \xrightarrow{p} B$ where E and B are *H*-spaces, then if p is an *H*-map, it follows that F can be given an *H*-structure so that i is an *H*-map. Stasheff in [8] gives a converse under some restrictions.

THEOREM 5 (Stasheff). Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration in which E and B are H-spaces. If F has a multiplication, then p is an H-map with respect to some multiplication on E, provided that E is (n-1) connected, $\pi_q(E) = 0$ for $q \ge n + m$, where $m \ge n + 1$, and B is (m-1) connected and $\pi_q(B) = 0$ for $q \ge n + m$.

We observe that if E is an *H*-space, then i is automatically cyclic. In fact, if $m: E \times E \to E$ is the *H*-structure in E, then the map $E \times F \to E$ of type (1, i) required can be taken to be $m(1 \times i)$. Using this fact, it follows from Theorems 4 and 5 that we have the following.

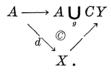
396

THEOREM 6. Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration in which F is (n-1) connected, B is (m-1) connected, and $\pi_q(B) = 0$ for $q \ge 2n$, where $m \ge n+1$. Suppose that E and B are H-spaces. Then p is an H-map for some multiplication on E.

2. These theorems can be dualized. We state the dual of Theorem 4. Recall that a cofibration $A \xrightarrow{d} X \xrightarrow{f} C$ is Ganea-principal if we can find a co-H-structure $m: C \to C \lor C$ and a map $\phi: X \to X \lor C$ such that the following diagram commutes

$$\begin{array}{c} C \lor C \xleftarrow{f \lor 1} X \lor C \xleftarrow{D} A \\ \uparrow m & \textcircled{O} & \uparrow \phi & \textcircled{O} \\ C \xleftarrow{f} X \xleftarrow{d} A \end{array}$$

where D(a) = (d(a), *). We are following the terminology of [2]. The cofibration is induced if there is a space Y and a map $g: Y \to A$ such that the cofibration is equivalent to the cofibration strictly induced by g from $Y \to CY \to \Sigma Y$, that is, to the triple $A \to A \bigcup_g CY \to \Sigma Y$, This means that there is a homotopy-equivalence $X \to A \bigcup_g CY$ such that



For a 1-connected CW complex K, we write dim $K \leq n$ to indicate that $H_n(K)$ is free and $H_q(K) = 0$ for q > n. We say that a map $f: X \to A$ is cocyclic if the map $(1 \times f) \varDelta: X \to X \times A$ is compressible into $X \vee A$. This is homotopy property of f. The dual of Theorem 4 is the following.

THEOREM 7. Let $A \xrightarrow{d} X \xrightarrow{f} C$ be a cofibration in which (X, A)is a CW pair. Suppose that A is (m-1) connected, and that C is n-connected, $m \ge 2, n \ge 1$. Suppose that f is cocyclic. If dim $A \le n + \min(m-1, n)$, then the cofibration is Ganea-principal. If further, dim $C \le n + \min(2m-1, 2n)$, then the cofibration is induced.

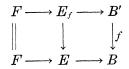
Proof. Dualize the proof of Theorem 4.

3. We now briefly consider the connection between our results and the work of Meyer [4], Nowlan [5] and Porter [6], [7]. We refer the reader to these papers for detailed definitions, but we shall indicate how their notions can be expressed in our terminology. Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration. Then it is an *H*-fibration in the sense of Meyer [4] if it is Ganea-principal, with *i* being cyclic and ϕ being a map of type (1, *i*). A principal fibration in the sense of Meyer [4] is an induced fibration in the sense of Ganea [2]. All the results in [4] concerning principal fibrations hold for induced fibrations, and all the results concerning *H*-fibrations hold for Ganea-principal fibrations.

An *H*-fibration in the sense of Porter [6] is also Ganea-principal with *i* being cyclic and ϕ being of type (1, *i*). We might emphasise that our definition of a Ganea-principal fibration does not require *i* to be cyclic and does not require ϕ to be of type (1, *i*). However, a simple examination shows that most of the results in Meyer [4] on *H*-fibrations and most of those in Porter [6] on *H*-fibrations do not require these extra conditions. They merely require that the fibrations be Ganea-principal. In particular, we mention that Theorem 1 of Porter [6] holds for Ganea-principal fibrations. There is no need to assume that *i* is cyclic or that ϕ is of type (1, *i*). The proof carries over word for word. Thus we have the following which we shall attribute to Porter.

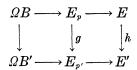
THEOREM 8 (Porter). Let $F \xrightarrow{i} E \xrightarrow{p} B$ be Ganea-principal, with B being path-connected. Let $g_1, g_2: X \to E$ be maps with $pg_1 \cong pg_2, X$ being a CW complex. Then there exists a map $u: X \to F$ with $\phi(g_2 \times u) \Delta \cong g_1$, where $\Delta: X \to X \times X$ is the diagonal map.

In [7] Porter defines his principal fibrations to be Ganea-principal with (F, m) being an associative *H*-space, and the map $\phi: E \times F \to E$ of type (1, i) is required to be an associative action of *E*. Let $F \xrightarrow{i} E \xrightarrow{p} B$ be such a principal fibration and let $f: B' \to B$ be a map. Let $E_f = \{(b, e) \text{ in } B' \times E | f(b) = p(e)\}$. Then $F \to E_f \to B'$ is also a principal fibration and



is a homomorphism of principal fibrations. We ask the reader to refer to Porter [7] for the definitions of the various terms we shall be using.

Let $F \xrightarrow{i} E \xrightarrow{p} B$ and $F' \xrightarrow{i'} E' \xrightarrow{p'} B'$ be fibrations. Then we have the principal fibrations $\Omega B \to E_p \to E$, $\Omega B' \to E_{p'} \to E'$. Theorem 12 of Porter [7] says that there exists a map $f: B \to B'$ such that $F \to E \to B$ is equivalent to the fibration induced by f from $F' \to E' \to B'$ if and only if there exists a strong homotopy homomorphism of principal fibrations



with g being a homotopy equivalence. Our results concern the case where $F' \to E' \to B'$ is a path space fibration $\Omega B' \to PB' \xrightarrow{\pi} B'$. Thus under the conditions stated there on the homotopy of B and F, our Theorem 4 says that there exists a strong homotopy homomorphism

$$\begin{array}{ccc} \Omega B & \longrightarrow & E_p \longrightarrow & E \\ & & & & \downarrow g & & \downarrow h \\ \Omega B' \longrightarrow & E_{\pi} \longrightarrow & PB' \end{array}$$

with g being a homotopy equivalence.

Nowlan's *H*-fibrations [5] are our Ganea-principal fibrations. Nowlan considers fibrations in which an associative *H*-space (*F*, *m*) operates, but not necessarily associatively. Such fibrations are called A_1 -principal fibre spaces. For example, all the various *H*-fibrations are such fibre spaces, and Ganea-principal fibrations are also such fibre spaces if (*F*, *m*) is associative. If instead of requiring that the action be associative, we only require that it be homotopy associative, that is, that the following diagram homotopy commutes

$$egin{array}{c} E imes F imes F imes F & \longrightarrow F \ & \downarrow 1 imes m & \downarrow \phi \ & E imes F & \longrightarrow E \end{array}$$

then we get an A_2 -principal fibre space. Thus a principal fibration in the sense of Porter [7] is an A_2 -principal fibre space. An A_n principal fibre space is one where the action of F on E satisfies higher homotopy conditions. Nowlan [5] obtains a classification theorem. The notion of an A_{∞} -principal fibre space is also obtained. Nowlan proves the following.

THEOREM (Nowlan). $p: E \to B$ is fibre homotopy equivalent to an induced fibre space if and only if E admits an A_{∞} -action of ΩB .

References

1. T. Ganea, A generalization of the homology and homotopy suspension, Comment. Math. Helv., **39** (1965), 295-322.

C. S. HOO

2. T. Ganea, Induced fibrations and cofibrations, Trans. Amer. Math. Soc., 127 (1967), 442-459.

3. P. J. Hilton, Remark on loop spaces, Proc. Amer. Math. Soc., 15 (1964), 596-600.

4. J. P. Meyer, Principal fibrations, Trans. Amer. Mhth. Soc., 107 (1963), 177-185.

R. A. Nowlan, A_n actions on fibre spaces, Indiana Univ. Math. J., **21** (1971), 285-313.
G. J. Porter, H-fibrations, Quart. J. Math., **22** (1971), 23-31.

 Homomorphisms of principal fibration: applications to classification, induced fibrations, and the the extension problem, Illinois J. Math., 16 (1972), 41-60.
J. Stasheff, Extensions of H-spaces, Trans. Amer. Math. Soc., 105 (1962), 126-135.

Received December 6, 1975. This research was supported by NRC Grant A3026.

UNIVERSITY OF ALBERTA AND

OXFORD UNIVERSITY, OXFORD, ENGLAND

400