PACIFIC JOURNAL OF MATHEMATICS
Vol. 69, No. 1, 1977

CERTAIN HYPOTHESES CONCERNING L-FUNCTIONS

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Some conditional results are discussed concerning Dirichlet L-functions. In particular, a method is introduced which on the one hand gives a new proof of a result of Wolke concerning the least prime quadratic residue, and on the other hand, gives a result on the least quadratic non-residue which does not seem to follow from previously known arguments.

Let q be an odd prime, $\chi(n)$ the Legendre symbol (n/q) and $L(s, \chi)$ the Dirichlet L-function pertaining to χ. Let Y_+ (respectively Y_-) denote the least prime p such that $\chi(p) = 1$ (respectively -1). Various results are known connecting together:

(A) zero-free regions for $L(s, \chi)$

(B) the magnitude of $L(1, \chi)$

(C) the magnitudes of Y_+ and Y_-.

Roughly speaking, a statement about any of these implies a corresponding statement about the subsequent ones.

In the case of (A) $\Rightarrow$ (B), we have the following theorem of Littlewood [8].

(1) Assume $L(s, \chi) \neq 0$ for $\sigma = \mathrm{Re}\, s > 1 - \theta(q)$. There exist positive absolute constants c_1 and c_2 such that

$$\frac{c_1\theta(q)}{\log\log q} < L(1, \chi) < \frac{c_2 \log\log q}{\theta(q)} .$$

Actually, Littlewood proves this only for $\theta(q) = 1/2$ (the Extended Riemann Hypothesis) but his method extends easily (as remarked by Elliott [4] for the lower bound) to give the stated result. A brief sketch of the method is given in Lemma 11.

In the case (A) $\Rightarrow$ (C), there is the result of Rodosskii [9]:

(2) Let $\psi > e$ and assume $L(s, \chi) \neq 0$ for $\sigma > 1 - \psi/\log q$. There is a positive constant c such that

$$Y_- \ll q^{c \log \psi/\psi} .$$

(Actually, Rodosskii's assumption is somewhat weaker, postulating a zero-free region only up to a certain height.)

In the case of the Extended Riemann Hypothesis ($\psi = 1/2 \log q$), we have the slightly stronger result of Ankeny [1]

$$Y_- \ll \log^2 q$$

and a similar result for Y_+.

We shall be primarily concerned with results of the type (B) $\Rightarrow$ (C). The first result of this nature seems to be that of Linnik and Renyi [6]:

(3) Given $\varepsilon > 0$, there exists $c(\varepsilon)$ such that, if

$$L(1, \chi) < c(\varepsilon) \log q\ , \quad \text{then} \quad Y_- < q^\varepsilon\ .$$

The idea of the proof of (3) may be summarized as follows. Let $X = q^2$ and $a(m) = \sum_{d|m} \chi(d)$. As is well-known, $\sum_{m \leqq X} a(m) \sim XL(1, \chi)$ and, on the other hand, since $a\ (m) \geqq 0$, this sum is $\geqq \sum_{m \leqq X\ p|m \to p \leqq Y_-} \tau(m)$ where $\tau(m)$ is, as usual, the number of divisors of m. If $Y_- \geqq q^\varepsilon$ for fixed ε, this latter sum can be given a lower bound of the form $f(\varepsilon) X \log Y$ (cf. 3.4.8 of [5]), and this in turn forces a lower bound on $L(1, \chi)$.

More recently, Wolke [11], improving an earlier theorem of Elliott [4], gave a somewhat analogous result for Y_+.

(4) $\log Y_+ \ll (\log q / L(1, \chi))^{1/2}$.

Hence, if $L(1, \chi) > (t(q)/\log q)$, then $Y_+ \leqq q^{c(t(q))^{-1/2}}$.

Although it is somewhat hidden in Wolke's proof, the ideas involved are essentially the same as for (3). Wolke, following Elliott, based his proof on the expression

$$\sum_{m \leqq X} \left(1 - \frac{m}{X}\right) \sum_{d|m} \mu^2(d) \chi(d)$$

used by Linnik and A. I. Vinogradov [7] in their unconditional (but ineffective) proof that $Y_+ \ll q^{1/4+\varepsilon}$. However, with little additional effort, the proof can be based on $\sum_{m \leqq X} a(m)$. The same incidentally can be said about the result of Linnik and Vinogradov. Indeed, with $X = q^{1/4+\varepsilon}$ their Theorem 2 gives, for some $\delta > 0$,

$$\sum_{m \leqq X} a(m) = XL(1, \chi) + O(X^{1-\delta})\ .$$

The assumption $\chi(p) = -1$ for $p \leqq X$ gives $\sum_{m \leqq X} a(m) = \sum_{m^2 \leqq X} 1 \leqq X^{1/2}$, and, using Siegel's theorem, a contradiction ensues.

Returning to (4) we see that, if one may take $t(q) \to \infty$ as $q \to \infty$, then one has the estimate $Y_+ \ll q^\varepsilon$. In this sense, the result is an exact analogue of (3).

A further consequence is that, for given $c > 0$, there exists $c' > 0$ such that if $Y_+ > c'$, then $L(1, \chi) > c \log q$.

The exponent $-1/2$ of $t(q)$ should be difficult to improve, since if it could be replaced by $\alpha < -1/2$, it would follow from the trivial estimate $Y_+ \geqq 2$ that $L(1, \chi) = o(\log q)$. Moreover, if for some $\beta(q) \to 0$ as $q \to \infty$, the assumption $L(1, \chi) > t(q)\beta(q)/\log q$ were sufficient, then one could easily deduce that the estimate $Y_+ \ll q^{\varepsilon}$ holds apart from those q for which $L(s, \chi)$ has an exceptional zero.

The result (3) seems somewhat weak in comparison with (4). By analogy, one might expect:

(5) CONJECTURE. $\log Y_- \ll (L(1, \chi) \log q)^{1/2}$.

Hence, if $L(1, \chi) < \log q/t(q)$, then $Y_- \leqq q^{c(t(q))^{-1/2}}$.

From the point of view of the proof of (3) as outlined above, the stronger result (5) is not obtained because the function $f(\varepsilon)$ tends to zero with ε, far too rapidly. Moreover, in view of the trivial bound $Y_- \geqq 2$, (5) would immediately give $L(1, \chi) \gg 1/\log q$, so that one must expect it to be very difficult.

Taking an even more hopeful view one has:

(6) CONJECTURE. $\log Y_- \ll L(1, \chi) (\log \log q)^{1+\varepsilon}$ which almost (but not quite) corresponds to taking -1 as the power of $t(q)$ in (5).

If one assumes the Extended Riemann Hypothesis, then for $\varepsilon \geqq 1$, (6) follows easily from (1) and (2).

The results of the above type can be examined by a slightly different approach (than that used in proving (3) and (4)) which can be outlined as follows.

Let $g(m)$ and $h(m)$ be totally multiplicative functions having absolute value $\leqq 1$. One might expect that if $\sum_{m \leqq X} g(m)/m$ is in some sense small, then so is $\sum_{m \leqq X\, p | m \to p \leqq Y} g(m)/m$ and conversely. Hence if $h(m) = g(m)$ for all $m \leqq Y$, then an upper bound for $\sum_{m \leqq X} g(m)/m$ should lead to an upper bound for $\sum_{m \leqq X} h(m)/m$. For example, letting $g(m) = \lambda(m)$ (the Liouville function), the prime number theorem gives an upper bound for the former sum. Choosing $Y < Y_+$ and $h(m) = \chi(m)$, we expect an upper bound for $\sum_{m \leqq X} \chi(m)/m$ and thus, if X is not too small, for $L(1, \chi)$. In similar fashion one can apply this method to the problem of estimating Y_-. What is somewhat surprising is that, whereas in the case of Y_+, the method gives precisely the same result as did the previous one (i.e., a new proof of (4)), in the case of Y_-, a different type of result is obtained, which is more in the nature of (5) (although of course weaker), and which, as far as one can see, is not easliy shown to be equivalent to (3).

(7) LEMMA. *For* $Y > 3/2$,

$$\sum_{\substack{m \leqq X \\ p \mid m \to p \leqq Y}} \frac{\lambda(m)}{m} \ll \frac{1}{\log Y} .$$

Proof. Letting $u = \log X/\log Y$, we distinguish two cases.

Case I. $u < \exp(c\sqrt{\log Y})$. Hence c is a positive constant associated with the error term in the prime number theorm. (Actually, the more refined versions of the prime number theorem allow us to take c to be any positive constant, but this is unimportant for our purpose.) In this case, the result is a consequence of the results of Levin and Fainleib (Theorem 3.2.4 of [5]) and we even have, uniformly for $1 + \varepsilon \leqq u < \exp(c\sqrt{\log Y})$,

$$\sum_{\substack{m \leqq X \\ p \mid m \to p \leqq Y}} \frac{\lambda(m)}{m} = \frac{\pi^2 \omega(u)}{6 \log Y} + O\Big(\frac{1}{\log^2 Y}\Big) ,$$

where $\omega(u)$ is the continuous function defined by $w(u) = u^{-1}$ for $1 < u \leqq 2$ and $(u\omega(u))' = \omega(u-1)$ for $u > 2$. (For $u \leqq 1 + \varepsilon$, we do not get an asymptotic formula, but the upper bound is straight-forward.)

Case II. $u \geqq \exp(c\sqrt{\log Y})$.

Let $\varphi(X, Y) = \sum_{m \leqq X\ p \mid m \to p > Y} 1$ and $\psi(X, Y) = \sum_{m \leqq X\ p \mid m \to p \leqq Y} 1$. We have,

$$\sum_{\substack{m \leqq X \\ p \mid m \to p \leqq Y}} \frac{\lambda(m)}{m} = \sum_{\substack{m \leqq X \\ p \mid m \to p \leqq Y}} \frac{1}{m} \sum_{\delta d^2 = m} \mu(\delta) .$$

Changing the order of summation gives

$$\frac{\pi^2}{6} \sum_{\substack{\delta \leqq X \\ p \mid \delta \to p \leqq Y}} \frac{\mu(\delta)}{\delta} + O\Big(\frac{1}{Y} \sum_{\substack{\delta \leqq X \\ p \mid \delta \to p \leqq Y}} \frac{1}{\delta}\Big) + O\Big(X^{-1/2} \sum_{\substack{\delta \leqq X \\ p \mid \delta \to p \leqq Y}} \frac{1}{\delta^{1/2}}\Big) .$$

Applying partial summation and well-known upper bounds for $\psi(X, Y)$ (e.g., 1.6 and 1.7 of [3]) we find that (for case II) the error terms are $O(1/\log^2 Y)$. Combining the above with the Legendre formula

$$\varphi(X, Y) = X \sum_{\substack{d \leqq X \\ p \mid d \to p \leqq Y}} \frac{\mu(d)}{d} + O(\psi(X, Y)) ,$$

we have

$$\sum_{\substack{m \leqq X \\ p \mid m \to p \leqq Y}} \frac{\lambda(m)}{m} = \frac{\pi^2}{6X} \varphi(X, Y) + O\Big(\frac{1}{\log^2 Y}\Big) .$$

Using the known estimates for $\varphi(X, Y)$ available either from the sieve or from the work of de Bruijn [2], the result follows.

REMARK. In applying the Sieve of Eratosthenes, one has the formula

$$\varphi(X, Y) - X \sum_{\substack{d \leqq X \\ p \mid d \to p \leqq Y}} \frac{\mu(d)}{d} = - \sum_{\substack{d \leqq X \\ p \mid d \to p \leqq Y}} \mu(d) \left\{ \frac{X}{d} \right\}$$

and one would like to know that the right hand side is small. Unfortunately, it is only in view of the prime number theorem itself that one can say that the right hand side is, for $X^{1/2} \leqq Y \leqq X^{1-\varepsilon}$, bounded above by $c(X/\log^2 Y)$. Applying the argument of the previous lemma with λ replaced by μ, one sees that the upper bound

$$\sum_{\substack{d \leqq X \\ p \mid d \to p \leqq Y}} \mu(d) \left\{ \frac{X}{d} \right\} \ll \frac{X}{\log^2 Y}$$

holds uniformly for $1 + \varepsilon \leqq Y \leqq X^{1-\varepsilon}$.

(8) LEMMA. *Let* $3/2 \leqq Y \leqq X$. *Let* $h(m)$ *be totally multiplicative with* $|h(m)| \leqq 1$ *for all* m *and* $h(m) = \lambda(m)$ *for all* $m \leqq Y$. *Then,*

$$\sum_{m \leqq X} \frac{h(m)}{m} \ll \frac{\log X}{\log^2 Y} .$$

Proof. We have

$$\left| \sum_{d \leqq X} \frac{h(d)}{d} \right| \leqq \left| \sum_{\substack{m \leqq X \\ p \mid m \to p \leqq Y}} \frac{\lambda(m)}{m} \right| + \sum_{\substack{Y < d \leqq X/Y \\ p \mid d \to p > Y}} \frac{1}{d} \left| \sum_{\substack{m \leqq X/d \\ p \mid m \to p \leqq Y}} \frac{\lambda(m)}{m} \right|$$
$$+ \sum_{\substack{X/Y < d \leqq 2X/3 \\ p \mid d \to p > Y}} \frac{1}{d} \left| \sum_{m \leqq X/d} \frac{\lambda(m)}{m} \right| + \sum_{\substack{2X/3 < d \leqq X \\ p \mid d \to p > Y}} \frac{1}{d} .$$

The third and fourth terms are estimated by means of partial summation, and the bounds $\varphi(t, Y) \ll t/\log Y$ and $\sum_{m \leqq t} \lambda(m)/m \ll 1/\log^2 t$. Each term is thus $\ll 1/\log Y$. That the same fate befalls the first term is seen from the previous lemma. Applying that lemma and partial summation to the second term gives the bound $\log X/\log^2 Y$ as required.

REMARK. In view of the asymptotic nature of (7) it is obvious that any inprovement must stem from (8), but it seems difficult to accomplish this without some strong additional assumptions.

(9) THEOREM. $\log Y_+ \ll (\log q/L(1, \chi))^{1/2}$.

Proof. Choose $X = q^2$, $Y < Y_+$ and $h(m) = \chi(m)$ in the previous lemma and note that $\sum_{m \leqq X} \chi(m)/m = L(1, \chi)\ (1 + o(1))$ for this choice of X.

Similarly, choosing $h(m) = \lambda(m)\chi(m)$ and $Y < Y_-$, we have:

(10) THEOREM. $\sum_{m \leqq X} \lambda(m)\chi(m)/m \ll \log X/\log^2 Y_-$.

At this point, one would like to use the sum on the left as an approximation for $\zeta(2)/L(1, \chi)$. *To do this it seems necessary* (to avoid having to choose X exceedingly large) *to postulate a zero-free region for* $L(s, \chi)$.

(11) LEMMA. *Let* $c > 0$ *be given,* $s = \sigma + it$, $\tau = |t| + 2$, *and assume* $L(s, \chi)$ *has no zeros for* $\sigma > 1 - c \log\log(q\tau)/\log(q\tau)$. *Then, there exists* $c_1 > 0$ *depending only on* c, *such that, on the contour* $\sigma_0 = 1 - c \log\log(q\tau)/2\log(q\tau)$, *we have* $|\log L(s, \chi)| < c_1 \log\log(q\tau)$.

Proof. This is a slight modification of some arguments of Littlewood, and so we give only a sketch. All implied constants depend at most on c.

Step 1. For $\sigma \geqq \sigma_0$, $|L'(s, \chi)/L(s, \chi)| \ll \log(q\tau)$. For $\sigma_0 \leqq \sigma \leqq 1 + c \log\log(q\tau)/\log(q\tau)$, this is proved by the method of [8], (Lemma 5). For $\sigma > 1 + c\log\log(q\tau)/\log(q\tau)$, it follows from $|L'(s, \chi)/L(s, \chi)| < \sum_{m \geqq 1} \Lambda(m)/m^{\sigma}$ by means of partial summation.

Step 2. For $0 < \delta \leqq 1$, and $\Lambda_1(m) = \Lambda(m)/\log m$,

$$\log L(1 + it, \chi) = \sum_{m=1}^{\infty} \frac{\Lambda_1(m)\chi(m)}{m^{1+it}} e^{-\delta m} + O(\delta^{1-\sigma_0} \log(q\tau)) + O(1) .$$

This follows, using Step 1 in the argument of Theorem 14.6 of [10].

Step 3. Using Step 2 and the argument of Theorem 14.8 of [10],

$$|\log L(1 + it, \chi)| \leqq \log\log N + O(\delta^{-1} e^{-\delta N}) + O(\delta^{1-\sigma_0} \log(q\tau)) + O(1) .$$

Choosing $N = (q\tau)^{2A}$, $\delta = (q\tau)^{-A}$ where A (depending on c) is chosen so that $\delta^{1-\sigma_0} < (\log(q\tau))^{-1}$, we get

$$|\log L(1 + it, \chi)| \ll \log\log(q\tau) .$$

Using this and Step 1, (11) follows from the relation

$$\log L(\sigma_0 + it, \chi) = \log L(1 + it, \chi) - \int_{\sigma_0}^{1} \frac{L'(\sigma + it, \chi)}{L(\sigma + it, \chi)} d\sigma .$$

(12) THEOREM. *Let* $c > 0$ *be given and assume* $L(s, \chi)$ *has no zeros for* $\sigma > 1 - c \log\log(q\tau)/\log(q\tau)$. *Then,*

$$\log Y_{-} \ll (L(1, \chi) \log q)^{1/2} ,$$

where the implied constant depends on c.

Proof. From (10) and partial summation,

$$\sum_{m \leqq X}\left(1 - \frac{m}{X}\right)\frac{\lambda(m)\chi(m)}{m} \ll \frac{\log X}{\log^2 Y_-}.$$

Writing the left hand side as

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\left(1 - \frac{1}{q^{2s+2}}\right)\frac{\zeta(2s+2)}{L(s+1, \chi)}\frac{X^s ds}{s(s+1)},$$

shifting the line of integration to $\sigma = \sigma_0 - 1$, and choosing $X = q^{c_2}$ where $c_2 > 2(c_1 + 3)/c$, the previous lemma gives this to be

$$\frac{\zeta(2)}{L(1, \chi)} + O\left(\frac{1}{\log^2 q}\right).$$

Using the trivial bound $Y_- < q$, $1/L(1, \chi) \ll \log X/\log^2 Y_-$ and the result follows.

Remarks. Although we have investigated the previous method only to the extent of determining an hypothesis sufficient to establish the estimate (5), it is clear that by varying the choice of X in the above proof, different hypothetical zero-free regions will give other results of this nature.

Furthermore, by using (2) in conjunction with the lower bound in (1), we get results of type (12) immediately. They seem however to be weaker than those obtained by the above method. For example, to obtain the estimate (5) by these means it is necessary to postulate a zero-free region of width $c \log \log q/(\log q)^{1/3}$.

Conversely, using (12) together with the upper bound of (1), we get a result of the same nature as (2), but here again it is weaker than that obtained by Rodosskii's method.

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Received October 21, 1975.

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