PACIFIC JOURNAL OF MATHEMATICS
Vol. 70, No 2, 1977

QUASI-AFFINE TRANSFORMS OF SUBNORMAL OPERATORS

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For an operator T which is a quasi-affine transform of a subnormal operator S, we show that: (1) if S^* has no point spectrum and $f\colon \lambda \mapsto (T-\lambda)^{-1}x$ is defined on an open set Ω, then there is a dense subset Ω_0 of Ω such that $f \mid \Omega_0$ is analytic; and (2) if Σ is a spectral set of T and Q is a peak set of $R(\Sigma)$, then the spectral manifold $X_T(Q)$ is a reducing subspace of T and Q is a spectral set of $T \mid X_T(Q)$.

1. Introduction. We generalize results of Putnam [5] and [6] which concern local spectral properties of subnormal operators to quasi-affine transforms of subnormal operators.

Before we proceed, we fix some notation and terminology. All operators are assumed to be linear, bounded and defined on Hilbert spaces. For an operator T, we write $\sigma(T)$ for the spectrum of T. For an operator T defined on $\mathcal{H}$ and a closed set F in the complex plane $\mathbf{C}$, we write $\mathcal{X}_T(F)$ for those x in $\mathcal{H}$ such that there exists a vector-valued analytic function f from $\mathbf{C}\backslash F$ into $\mathcal{H}$ satisfying $(T-\lambda)f(\lambda)=x$ for all $\lambda \in \mathbf{C}\backslash F$. An operator T has the single-valued extension property if whenever g is a vector-valued analytic function defined on an open set in $\mathbf{C}$ with $(T-\lambda)g(\lambda)\equiv 0$ then $g(\lambda)\equiv 0$. (See Colojoară and Foiaş [1].) By a quasi-affinity we mean a (bounded linear) mapping $W\colon \mathcal{H}\to\mathcal{K}$ between two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ which is one-one and has its range dense in $\mathcal{K}$. An operator T defined on $\mathcal{H}$ is said to be a quasi-affine transform of an operator S defined on $\mathcal{K}$ if there is a quasi-affinity $W\colon \mathcal{H}\to\mathcal{K}$ such that $SW = WT$.

Suppose we have $NW_0 = W_0T$, where N is a normal operator defined on $\mathcal{K}_0$, T is an operator on $\mathcal{H}$ and $W_0\colon \mathcal{H}\to\mathcal{K}_0$ is one-one. Let $\mathcal{K}$ be the closure of the range of W_0 and $W\colon \mathcal{H}\to\mathcal{K}$ be the map which has the same value as W_0 at each point in $\mathcal{H}$. Then $\mathcal{K}$ is invariant under N and $SW = WT$ where S is the subnormal operator defined by restricting N to $\mathcal{K}$. Therefore T is a quasi-affine transform of a subnormal operator. Conversely, suppose T is a quasi-affine transform oof a subnormal operator S. Let W be a quasi-affinity such that $SW = WT$ and N be a normal extension of S. Then $NW_0 = W_0T$ where W_0 is the one-one mapping which takes the same value as W at each point. Thus, an operator T is a quasi-affine transform of a subnormal operator if and only if there is a one-one mapping intertwining T and a normal operator.

2. Simple properties.

PROPOSITION 1. *If T is a quasi-affine transform of a subnormal operator, then T has the single-valued extension property.*

Proof. Let N be a normal operator, W_0 be a one-one map such that $NW_0 = W_0T$. Suppose g is a vector-valued analytic function defined on an open set such that $(T-\lambda)g(\lambda) \equiv 0$. Then we have $(N-\lambda)W_0g(\lambda) = W_0(T-\lambda)g(\lambda) = 0$ for all λ. Since normal operators have the single-valued extension property, $W_0g(\lambda) = 0$ for all λ. Since W_0 is one-one, we have $g = 0$.

LEMMA 1. (See Colojoară and Foiaş [1] Proposition 3.8.) *If T is an operator on $\mathcal{H}$ with the single-valued extension property and F is a closed set in $\mathbf{C}$ such that $\mathcal{X}_T(F)$ is closed, then we have $\sigma(T \mid \mathcal{X}_T(F)) \subset F$. In particular, if $\mathcal{X}_T(F) = \mathcal{H}$, then $\sigma(T) \subset F$.*

PROPOSITION 2. *If T is a quasi-affine transform of the subnormal operator S and N is the minimal normal extension of S, then $\sigma(N) \subset \sigma(S) \subset \sigma(T)$.*

Proof. That $\sigma(N) \subset \sigma(S)$ is well-known. Suppose $W: \mathcal{H} \to \mathcal{K}$ is a quasi-affinity such that $SW = WT$. Then $W\mathcal{H} = W\mathcal{X}_T(\sigma(T)) \subset \mathcal{X}_S(\sigma(T))$. Since WH is dense in $\mathcal{K}$ and $\mathcal{X}_S(\sigma(T))$ is closed (see Radjabalipour [7]), $\mathcal{X}_S(\sigma(T)) = \mathcal{K}$. By the above lemma $\sigma(S) \subset \sigma(T)$.

REMARK 1. Using the same argument as above we can show that if T is a quasi-affine transform of the hyponormal operator S, then $\sigma(S) \subset \sigma(T)$.

REMARK 2. Let S be a subnormal operator on $\mathcal{H}$ and N be the minimal normal extension of S on $\mathcal{K}$. Then $S^*P = PN^*$, where P is the projection from $\mathcal{K}$ onto $\mathcal{H}$. Therefore we have $\mathcal{H} = P\mathcal{K} = P\mathcal{X}_{N^*}(\sigma(N^*)) \subset \mathcal{X}_{S^*}(\sigma(N^*))$. If S^* has the single-valued extension property, then, by Lemma 1, $\sigma(S^*) \subset \sigma(N^*)$ and hence $\sigma(S) = \sigma(N)$.

EXAMPLE. Let S be the unilateral shift. Then its minimal normal extension is the bilateral shift, denoted by U. Note $\sigma(U) =$ the unit circle $\neq$ the unit disk $= \sigma(S)$. Hence, from the above remark, S^* does not have the single-valued extension property. For a construction of a nonzero analytic function g such that $(S^* - \lambda)g(\lambda) \equiv 0$, see Colojoară and Foiaş [1] p. 10.

It is well-known that a completely subnormal operator does not have a nontrivial invariant subspace on which the operator is normal. The same holds for operators which are quasi-affine transforms of completely subnormal operators.

PROPOSITION 3. *If T is a quasi-affine transform of a completely subnormal operator S, then T has no nontrivial invariant subspace $\mathcal{M}$ such that $T \mid \mathcal{M}$ is normal.*

Proof. Let W_0 be a quasi-affinity and $SW_0 = W_0T$. Suppose $\mathcal{M}$ is an invariant subspace of T such that $T \mid \mathcal{M}$ is normal. Let $\mathcal{N}$ be the closure of $W_0\mathcal{M}$ and $W_1: \mathcal{M} \to \mathcal{N}$ be defined by restricting W_0 to $\mathcal{M}$. Then $\mathcal{N}$ is an invariant subspace of S and hence $S \mid \mathcal{N}$ is subnormal. Also $(S \mid \mathcal{N})W_1 = W_1(T \mid \mathcal{M})$. Therefore $S \mid \mathcal{N}$ is normal. (See e.g. Radjavi and Rosenthal [8].) Since S is subnormal, $\mathcal{N}$ is reducing for S. Since we assume that S is completely subnormal, we have $\mathcal{N} = \{0\}$. Hence $\mathcal{M} = \{0\}$.

3. Spectral manifolds.

PROPOSITION 4. *If T is an operator on $\mathcal{H}$ which is a quasi-affine transform of a subnormal operator S, S^* has no point spectrum, $x \in \mathcal{H}$, Ω is an open set in $\mathbf{C}$ and $f: \Omega \to \mathcal{H}$ is a bounded function such that $(T - \lambda)f(\lambda) = x$ for all λ, then f is analytic.*

Proof. Let N be the minimal normal extension for S and $\mathcal{K}$ be the underlying Hilbert space of N. Let W_0 be a one-one mapping such that $NW_0 = W_0T$. Since S^* has no point spectrum, it is easy to show that N also has no point spectrum. (From $NW_0 = W_0T$ and the fact that W_0 is one-one we see that the point spectrum of T is empty.) For $\lambda \in \Omega$, we have

$$(N - \lambda)W_0f(\lambda) = W_0(T - \lambda)f(\lambda) = W_0x.$$

By Putnam [5], $\lambda \to W_0f(\lambda)$ is analytic. Hence, for $y \in \mathcal{K}$, the function $\lambda \to (f(\lambda), W_0{}^*y) = (W_0f(\lambda), y)$ is analytic. Since W_0 is one-one, the range of $W_0{}^*$ is dense and hence $\lambda \to (f(\lambda), x)$ is analytic for each x in a dense subset of $\mathcal{H}$. By the boundedness of f, we can show that $\lambda \to (f(\lambda), x)$ is analytic for each x in $\mathcal{H}$. Therefore f is analytic.

For the next proposition we need a technical lemma.

LEMMA 2. *Suppose that Ω is an open set in $\mathbf{C}$, $f: \Omega \to \mathcal{H}$ is a vector-valued function and D is a dense subset of $\mathcal{H}$ such that $\lambda \to (f(\lambda), x)$ is analytic for $x \in D$. Then there is an open dense subset Ω_0 of Ω on which f is analytic.*

Proof. It suffices to show that, for every nonempty open subset U of Ω, there is a nonempty open subset of U on which f is bounded. Fix a nonempty open set U in Ω. First we show that, for every positive integer n, the set

$$F_n = \{\lambda \in U: \|f(\lambda)\| \leqq n\}$$

is relatively closed in U. Let $\lambda_0 \in U$ be in the closure of F_n. Since, for $x \in D$, $\lambda \to (f(\lambda), x)$ is continuous and $|(f(\lambda), x)| \leqq n\|x\|$ for $\lambda \in F_n$, we have $|(f(\lambda_0), x)| \leqq n\|x\|$ for $x \in D$. Since D is dense, $\|f(\lambda_0)\| \leqq n$. Therefore $\lambda_0 \in F_n$. Now, $U = \bigcup_{n=1}^{\infty} F_n$. By the Baire Category Theorem, there is some n such that the interior of F_n is nonempty. The proof is complete.

PROPOSITION 5. *If T is an operator on $\mathcal{H}$ which is a quasi-affine transform of a subnormal operator S, S^* has no point spectrum, $x \in \mathcal{H}$, Ω is an open set in $\mathbf{C}$ and $f: \Omega \to \mathcal{H}$ is a function such that $(T - \lambda)f(\lambda) = x$ for all $\lambda \in \Omega$, then there is a dense open subset Ω_0 of Ω such that $f \mid \Omega_0$ is analytic.*

Proof. The argument makes use of Lemma 2. It is a slight modification of that of Proposition 4, and hence is left to the reader.

COROLLARY. *If T on $\mathcal{H}$ is a quasi-affine transform of a subnormal operator S on $\mathcal{K}$, Ω is a nonempty open subset of $\sigma(S)$ and $\cap \{(T - \lambda)\mathcal{H} : \lambda \in \Omega\} \neq \{0\}$, then T has a nontrivial invariant subspace.*

Proof. Suppose $SW = WT$ with W as a quasi-affinity. If the point spectrum of S^* is nonempty, from $W^*S^* = T^*W^*$ we see that the point spectrum of T^* is also nonempty and hence T has an invariant subspace. Therefore we may assume that the point spectrum of S^* is empty. Let x be a nonzero vector in $\cap \{(T - \lambda)\mathcal{H}: \lambda \in \Omega\}$. By Proposition 5, there is a nonempty open set Ω_0 in Ω such that $x \in \mathcal{X}_T(\mathbf{C}\backslash\Omega_0)$. Let $\mathcal{M}$ be the closure of $\mathcal{X}_T(\mathbf{C}\backslash\Omega_0)$. Then $\mathcal{M} \neq \{0\}$. By Radjabalipour [7], $\mathcal{X}_S(\mathbf{C}\backslash\Omega_0)$ is closed. Since $\mathbf{C}\backslash\Omega_0 \not\subset \sigma(S)$, by Lemma 1, $\mathcal{X}_S(\mathbf{C}\backslash\Omega_0) \neq \mathcal{K}$. Now $W_0\mathcal{M} \subset \mathcal{X}_S(\mathbf{C}\backslash\Omega_0)$. Hence $\mathcal{M} \neq \mathcal{H}$.

REMARK. In view of Stampfli and Wadhwa [**12**], Proposition 4 still holds if we merely assume that T is a quasi-affine transform of a hyponormal operator without point spectrum.

4. Peak sets. The following theorem is a generalization of Theorem 1 in Putnam [**6**]:

THEOREM. *Let T (defined on $\mathcal{H}$) be a quasi-affine transform of a subnormal operator. Let Σ be a spectral set of T and Q be a peak set of $R(\Sigma)$ (the uniform closure of rational function with poles off Σ). Then there is a projection $F(Q)$ on $\mathcal{H}$ such that $F(Q)\mathcal{H} = \mathcal{K}_T(Q)$ and $F(Q)$ is in the weakly closed inverse-closed algebra generated by T. Furthermore, $T \mid F(Q)\mathcal{H}$ and $T \mid (I - F(Q))\mathcal{H}$ are quasi-affine transforms of subnormal operators and Q is a spectral set for $T \mid F(Q)\mathcal{H}$.*

Proof. Suppose $N = \int \lambda dE_\lambda$ on $\mathcal{K}_0$ is a normal operator, W_0 is a one-one mapping and $NW_0 = W_0T$. Since Σ is a spectral set of T, $g(T)$ is defined for $g \in R(\Sigma)$ and $\|g(T)\| \leqq \sup\{|g(\lambda)|: \lambda \in \Sigma\}$. Furthermore, it is straightforward to show that $g(N)W_0 = W_0g(T)$ for $g \in R(\Sigma)$. Let f be a peak function of Q, i.e., $f = 1$ on Q and $|f(\lambda)| < 1$ for $\lambda \notin Q$. Then

$$\|f(T)^n\| \leqq \sup\{|f(\lambda)^n|: \lambda \in \Sigma\} \leqq 1$$

for each n. Hence $\{f(T)^n: n = 1, 2, \ldots\}$ has a weakly convergent subsequence, say, $w\text{-}\lim f(T)^{n_i} = F(Q)$. Since $\{f^n: n = 1, 2, \ldots\}$ converges pointwisely to the characteristic function of Q and $f(N)^nW_0 = W_0f(T)^n$ for all n, we have $E(Q)W_0 = W_0F(Q)$. Since W is one-one and $W_0F(Q)^2 = E(Q)^2W_0 = E(Q)W_0 = W_0F(Q)$, we have $F(Q)^2 = F(Q)$. Since $\|F(Q)\| \leqq 1$, we see that $F(Q)$ is a projection. From the definition of $F(Q)$ we see that $F(Q)$ is in the weakly closed inverse-closed algebra generated by T.

For convenience, we write $T_1 = T \mid F(Q)\mathcal{H}$, $N_1 = T \mid E(Q)\mathcal{K}_0$ and W_1: $F(Q)\mathcal{H} \to E(Q)\mathcal{K}_0$ for the restriction of W_0 to $F(Q)\mathcal{H}$. We have $N_1W_1 = W_1T_1$. Note that W_1 is one-one, N_1 is normal and $\sigma(N_1) \subset Q$.

Let q be a rational function with poles off Σ. Let C be an arbitrary compact set in $\mathbf{C}$ disjoint from Q. Then, when n is large enough, we have

$$\|q(T)f(T)^n\| \leqq \sup\{|q(\lambda)f(\lambda)^n|: \lambda \in \Sigma\backslash C\}.$$

Hence we have $\|q(T)F(Q)\| \leqq \sup\{|q(\lambda)|: \lambda \in \Sigma\backslash C\}$. Since C is arbitrary, we have

$$(*)\qquad \|q(T_1)\| = \|q(T)F(Q)\| \leqq \sup\{|q(\lambda)|: \lambda \in Q\}.$$

Next, suppose r is a rational function with poles off Q. Since Q is a peak set of $R(\Sigma)$, for every connected component Ω of $\mathbf{C}\backslash Q$, we have $\Omega \not\subset \Sigma$. (Otherwise, $f-1$ would be a nonzero continuous function which is analytic on Ω and zero on $\partial\Omega$, contradicting the maximal modulus principle.) By Rudin **[10]** Theorem 13.9, there is a sequence $\{q_n\}$ of rational functions with poles off Σ such that $\sup\{|q_n(\lambda) - r(\lambda)|: \lambda \in Q\} \to 0$ as $n \to \infty$. Hence, by (*),

$$\|q_n(T_1) - q_m(T_1)\| \leqq \sup\{|q_n(\lambda) - q_m(\lambda)|: \lambda \in Q\} \to 0$$

as $n,m \to \infty$. Therefore $\{q_n(T_1): n = 1,2,\dots\}$ is convergent in the norm topology, to T_r say. It is easy to see that $\|T_r\| \leqq \sup\{|r(\lambda)|: \lambda \in Q\}$, $r(N_1)W_1 = W_1T_r$ and T_r is in the inverse-closed, uniformly closed algebra generated by T_1. In particular, if $\mu \notin Q$ and r is taken to be the function $\lambda \to (\lambda - \mu)^{-1}$, then $(N_1 - \mu)^{-1}W_1 = W_1T_r$ and

$$W_1 = (N_1 - \mu)^{-1}(N_1 - \mu)W_1 = (N_1 - \mu)^{-1}W_1(T_1 - \mu) = W_1T_r(T_1 - \mu).$$

Since W_1 is one-one, we have $T_r(T_1 - \mu) = I$. Therefore $T_1 - \mu$ is invertible. We have shown that $\sigma(T_1) \subset Q$. Now it is easy to see that, for general r, $T_r = r(T_1)$. Hence Q is a spectral set for T_1.

Since $\sigma(T_1) \subset Q$, we have $F(Q)\mathscr{H} \subset \mathscr{X}_T(Q)$. Conversely, suppose $x \in \mathscr{X}_T(Q)$. Then there is an analytic vector-valued function $f: \mathbf{C}\backslash Q \to \mathscr{H}$ such that $(T-\lambda)f(\lambda) = x$ for all λ. Hence, for $\lambda \notin Q$, $(N-\lambda)W_0f(\lambda) = W_0(T-\lambda)f(\lambda) = W_0x$. Therefore $W_0x \in \mathscr{X}_N(Q) = E(Q)\mathscr{H}_0$. Now $W_0F(Q)x = E(Q)W_0x = W_0x$. Since W_0 is one-one, $F(Q)x = x$, or $x \in F(Q)\mathscr{H}$. Therefore $F(Q)\mathscr{H} = \mathscr{X}_T(Q)$. The proof is complete.

REMARK 1. If we assume that Q, instead of being a spectral set for T, has the following property: there exists $M > 0$ such that $\|r(T)\| \leqq M\sup\{|r(\lambda)|: \lambda \in \Sigma\}$ for every rational function r with poles off Σ, then, using the same argument as in the proof of the above theorem, we can establish the existence of an idempotent operator $F(Q)$ in the weakly closed, inverse-closed algebra generated by T such that $F(Q)\mathscr{H} = \mathscr{X}_T(Q)$. Furthermore, we have

$$\|r(T \mid F(Q)\mathscr{H})\| \leqq M\sup\{|r(\lambda)|: \lambda \in Q\}$$

for every rational function r with poles off Q. Such an $F(Q)$ is unique. (Suppose F_1 and F_2 are two idempotent operators in the weakly closed, inverse-closed algebra generated by T such that $F_1\mathcal{H} = F_2\mathcal{H} = \mathcal{X}_T(Q)$. Then $F_1F_2 = F_2F_1$ is also an idempotent operator with $F_1F_2\mathcal{H} = F_1\mathcal{H}$ and $\ker F_1F_2 \subset \ker F_1$. Hence $F_1F_2 = F_1$. Similarly $F_2F_1 = F_1$. Therefore $F_1 = F_2$.)

REMARK 2. From the proof of $F(Q)\mathcal{H} \supset \mathcal{X}_T(Q)$ and in view of Putnam [**5**], we see that

$$F(Q)\mathcal{H} = \mathcal{X}_T(Q) = \bigcap \{(T-\lambda)\mathcal{H}: \lambda \notin Q\}.$$

REMARK 3. If Q_1 and Q_2 are peak sets for Σ, then we have $W_0F(Q_1 \cap Q_2) = E(Q_1 \cap Q_2)W_0 = E(Q_1)E(Q_2)W_0 = E(Q_1)W_0F(Q_2)$ $W_0F(Q_1)F(Q_2)$ and hence $F(Q_1 \cap Q_2) = F(Q_1)F(Q_2)$. In general, let $\mathcal{B}$ be the Boolean algebra generated by the family of peak sets for $R(\Sigma)$. Then F can be extended to $\mathcal{B}$ in a unique way such that:

(1) $F(B_1 \cap B_2) = F(B_1)F(B_2)$
(2) $F(B_1 \backslash B_2) = F(B_1) - F(B_1)F(B_2)$.

In fact, for $B_1 \in \mathcal{B}$, $E(B_1)W_0 = W_0F(B_1)$.

The following corollary is a generalization of a result in Conway and Olin [**4**].

COROLLARY. *Let T be a completely nonnormal contraction which is a quasi-affine transform of a subnormal operator with minimal normal extension $N = \int \lambda dE_\lambda$ on $\mathcal{K}_0$. If Z is a Borel set in $\{\lambda \in \mathbf{C}: |\lambda| = 1\}$ of arc length measure zero, then $E(Z) = 0$.*

Proof. By the inner regularity of the spectral measure E, it suffices to prove the corollary under the additional assumption that Z is closed. Since T is a contradiction, by von Neumann's well-known theorem, the closed unit disc $\Sigma = \{\lambda : |\lambda| \leqq 1\}$ is a spectral set for T. By the theorem of F. and M. Riesz (see, e.g., Hoffman [**2**], p. 32), Z is a peak set for $R(\Sigma)$. From the above theorem, we have $E(Z)W_0 = W_0F(Z)$ (W_0: $\mathcal{H} \to \mathcal{K}_0$ here is a one-one mapping implementing $NW_0 = W_0T$), and Z is a spectral set for $T \mid F(Z)\mathcal{H}$. By the Hartogs–Rosenthal Theorem, $R(Z) = C(Z)$. Therefore $T \mid F(Z)\mathcal{H}$ is normal, (by Lebow [**3**]). Since, by assumption, T is completely nonnormal, $F(Z) = 0$. Hence $E(Z)W_0 = 0$. Since N is the minimal normal extension of the subnormal operator given by restricting N to the closure of the range of W_0, $\mathscr{K}_0$ is the closure of the linear span of $\{N^{*n}x : x \in W_0\mathscr{H}, n = 1, 2, \cdots\}$. Therefore $E(Z) = 0$.

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Received September 14, 1976 and in revised form September 14, 1977. This work was partially supported by the National Science Foundation.

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