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TOTAL POSITIVITY AND THE EXACT n-WIDTH OF CERTAIN SETS IN L^1

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In this paper we obtain the exact value of the L^1 n-width, both in the sense of Kolmogorov and Gel'fand, and characterize optimal subspaces for the set

$$\mathscr{K}_r = \left\{\sum_{j=1}^{r} a_j k_j(t) + \int_0^1 K(t, s)h(s)ds \colon (a_1, \cdots, a_r) \in \boldsymbol{R}^r,\ \|h\|_1 \leqq 1\right\},$$

under certain total positivity assumptions on

$$\{k_1(t), \cdots, k_r(t), K(t, s)\}.$$

A matrix analogue is also described.

1. Introduction. Let X be a normed linear space, $\mathscr{A}$ a subset of X, and X_n any n-dimensional linear subspace of X. Then the n-width of $\mathscr{A}$ relative to X, in the sense of Kolmogorov, is defined to be

$$d_n(\mathscr{A}; X) = \inf_{X_n} \sup_{x \in \mathscr{A}} \inf_{y \in X_n} \|x - y\|.$$

X_n is called an optimal subspace for $\mathscr{A}$ provided that

$$d_n(\mathscr{A}; X) = \delta(\mathscr{A}; X_n) = \sup_{x \in \mathscr{A}} \inf_{y \in X_n} \|x - y\|.$$

The n-width of $\mathscr{A}$ relative to X, in the sense of Gel'fand, is defined as

$$d^n(\mathscr{A}; X) = \inf_{L_n} \sup_{x \in \mathscr{A} \cap L_n} \|x\|,$$

where L_n is any subspace of X of codimension n. If

$$d^n(\mathscr{A}; X) = \sup_{x \in \mathscr{A} \cap L_n} \|x\|,$$

then L_n is an optimal subspace for the Gel'fand n-width of $\mathscr{A}$.

A typical choice for $\mathscr{A}$ is the image of the unit ball under a compact mapping K of X into itself,

$$\mathscr{K} = \{Kx \colon \|x\| \leqq 1\}.$$

When X is a Hilbert space then it is possible to obtain an exact value for $d_n(\mathscr{K}; X)$. This fact originated with the methods used in Kolmogorov's seminal paper [4]. For $X = L^\infty[0, 1]$, we computed (in [6]) the n-widths of $\mathscr{K}$ when K is an integral operator determined by a totally positive kernel.

In this paper, we obtain the exact value of the L^1 n-width, both in the sense of Kolmogorov and Gel'fand, for such $\mathscr{K}$ (translated by some finite dimensional subspace) and identify respective optimal subspaces.

For a general statement of the apparent duality between the Gel'fand and Kolmogorov n-widths, see Ioffe and Tikhomirov [1]. Our problem requires, in addition, certain extensions and modifications of our results in [6]. Let us note that Tikhomirov, in his doctoral dissertation summary [10], indicated that the L^1 n-width of the Sobolev space (see Corollary 2.1) could be computed on the basis of the methods employed in [9]. This latter paper serves as our original motivation for the present work.

Finally, we remark that we have made an effort to make §2 of this paper self-contained, so that it may be read independently of [6] and [7].

2. L^1-widths. Let $X = L^1[0, 1]$, $Y = C[0, 1]$, and suppose $\|\cdot\|_1$ and $\|\cdot\|_\infty$ denote the usual L^1 and L^∞ norms on $[0, 1]$. We use $\|\cdot\|_1$ as the norm on both X and Y and hence Y is a dense subset of X. Later we will find it more convenient to consider the Gel'fand width of $\mathscr{K}_r$ relative to Y.

Given functions $k_1(t), \cdots, k_r(t)$ defined and continuous on $[0, 1]$, and a kernel $K(t, s)$ jointly continuous in $t, s \in [0, 1]$, we define

$$\mathscr{K}_r = \left\{\sum_{j=1}^{r} a_j k_j(t) + \int_0^1 K(t, s)h(s)ds \colon (a_1, \cdots, a_r) \in \boldsymbol{R}^r, \|h\|_1 \leqq 1\right\}.$$

In this section, we compute the n-widths of $\mathscr{K}_r$. Since the closure of $\mathscr{K}_r$ in X is

$$\bar{\mathscr{K}}_r = \left\{\sum_{j=1}^{r} a_j k_j(t) + \int_0^1 K(t, s)d\lambda(s) \colon (a_1, \cdots, a_r) \in \boldsymbol{R}^r, \|\lambda\| \leqq 1\right\},$$

where $\|\lambda\|$=total variation of λ on $[0, 1]$, the n-widths of $\mathscr{K}_r$ and $\bar{\mathscr{K}}_r$ are the same.

We require, in what follows, that the following properties hold.

I.

$$K\begin{pmatrix}1, \cdots, r, x_1, \cdots, x_m \\ y_1, \cdots, y_r, y_{r+1}, \cdots, y_{r+m}\end{pmatrix} = \begin{vmatrix} k_1(y_1) & \cdots\cdots & k_1(y_{r+m}) \\ \vdots & & \vdots \\ k_r(y_1) & \cdots\cdots & k_r(y_{r+m}) \\ K(y_1, x_1) & \cdots & K(y_{r+m}, x_1) \\ \vdots & & \vdots \\ K(y_1, x_m) & \cdots & K(y_{r+m}, x_m) \end{vmatrix} \geqq 0$$

for any points $0 < x_1 < \cdots < x_m < 1$, $0 < y_1 < \cdots < y_{r+m} < 1$, and integer $m \geqq 0$. Furthermore, we require that for any fixed set of x-points (y-points), the above determinant is not identically zero for all y-points (x-points).

II. For any $0 < y_1 < \cdots < y_r < 1$,

$$K\begin{pmatrix} 1, & \cdots, & r \\ y_1, & \cdots, & y_r \end{pmatrix} > 0 .$$

Thus we see that our conditions imply that the set of functions $\{k_1(t), \cdots, k_r(t)\}$ is a Chebyshev on $(0, 1)$, while for every $0 < x_1 < \cdots < x_m < 1$, $\{k_1(t), \cdots, k_r(t), K(t, x_1), \cdots, K(t, x_m)\}$ is a weak Chebyshev system.

For fixed $\zeta = (\zeta_1, \cdots, \zeta_n)$, $\zeta_0 = 0 < \zeta_1 < \cdots < \zeta_n < \zeta_{n+1} = 1$, $n \geqq r$, we introduce the auxiliary kernel

$$J(t, s; \zeta) = \frac{K\begin{pmatrix} 1, & \cdots, & r, & s \\ \zeta_1, & \cdots, & \zeta_r, & t \end{pmatrix}}{K\begin{pmatrix} 1, & \cdots, & r \\ \zeta_1, & \cdots, & \zeta_r \end{pmatrix}} ,$$

and the function $h_\zeta(t) = (-1)^j$, $\zeta_j \leqq t < \zeta_{j+1}$, $j = 0, 1, \cdots, n$.

LEMMA 2.1. *For any constants* $c_1, \cdots, c_{n-r}$, *the function*

$$g(s) = \int_0^1 h_\zeta(t) J(t, s; \zeta) dt - \sum_{i=1}^{n-r} c_i J(\zeta_{i+r}, s; \zeta)$$

has at most $n - r$ *distinct zeros in* $(0, 1)$.

Proof. Suppose to the contrary that there exists a g which has $n - r + 1$ zeros $0 < z_1 < \cdots < z_{n-r+1} < 1$. Since the functions $k_1(t), \cdots, k_r(t), K(t, z_1), \cdots, K(t, z_{n-r+1})$ form a weak Chebyshev system of dimension $n + 1$, there exists a nontrivial function

$$f(t) = \sum_{j=1}^{r} a_j k_j(t) + \sum_{j=1}^{n-r+1} b_j K(t, z_j)$$

which has (weak) sign changes at $\zeta_1, \cdots, \zeta_n$, i.e., $(-1)^j f(t) \geqq 0$, $\zeta_j \leqq t \leqq \zeta_{j+1}$, $j = 0, 1, \cdots, n$. Thus f necessarily vanishes at $\zeta_1, \cdots, \zeta_n$. Therefore

$$\begin{aligned} f(t) &= \frac{1}{K\begin{pmatrix} 1, & \cdots, & r \\ \zeta_1, & \cdots, & \zeta_r \end{pmatrix}} \begin{vmatrix} k_1(\zeta_1) & \cdots & k_1(\zeta_r) & k_1(t) \\ \vdots & & \vdots & \vdots \\ k_r(\zeta_1) & \cdots & k_r(\zeta_r) & k_r(t) \\ f(\zeta_1) & \cdots & f(\zeta_r) & f(t) \end{vmatrix} \\ &= \sum_{l=1}^{n-r+1} b_l J(t, z_l; \zeta) , \end{aligned}$$

and we have

$$\begin{aligned}\int_0^1 |f(t)|\,dt &= \sum_{j=0}^{n}(-1)^j\int_{\zeta_j}^{\zeta_{j+1}} f(t)dt - \sum_{j=1}^{n-r} c_j f(\zeta_{j+r}) \\ &= \sum_{l=1}^{n-r+1} b_l\Big[\sum_{j=0}^{n}(-1)^j\int_{\zeta_j}^{\zeta_{j+1}} J(t, z_l; \zeta)dt - \sum_{j=1}^{n-r} c_j J(\zeta_{j+r}, z_l; \zeta)\Big] \\ &= \sum_{l=1}^{n-r+1} b_l g(z_l) = 0 \,.\end{aligned}$$

This contradiction proves the lemma.

The following result is of central importance in this section.

THEOREM 2.1. *Given any integer* $n \geqq r$, *there exist points* $0 = \xi_0 < \xi_1 < \cdots < \xi_n < \xi_{n+1} = 1$ *such that the function*

$$g_{n,r}(s) = \int_0^1 h_\xi(t)K(t, s)dt$$

equioscillates $n - r + 1$ *times on* $[0, 1]$, *that is,*

$$g_{n,r}(\eta_i) = (-1)^i\sigma\|g_{n,r}\|_\infty\,, \quad i = 1, \cdots, n - r + 1\,, \tag{2.1}$$

for some points $0 \leqq \eta_1 < \cdots < \eta_{n-r+1} \leqq 1$ *and* $\sigma = +1$ *or* -1, *fixed, and furthermore*

$$(h_\xi, k_i) = \int_0^1 h_\xi(t)k_i(t)dt = 0\,, \qquad i = 1, \cdots, r\,. \tag{2.2}$$

Proof. Our proof of Theorem 2.1 applies a technique used in [8]. Set

$$S^n = \Big\{z = (z_1, \cdots, z_{n+1}) \colon \sum_{i=1}^{n+1}|z_i| = 1\Big\}\,,$$

and define $\xi_0(z) = 0$, $\xi_j(z) = \sum_{k=1}^{j}|z_k|$, $j = 1, \cdots, n + 1$. Let

$$G(s; z) = \sum_{j=0}^{n}(\operatorname{sgn} z_{j+1})\int_{\xi_j(z)}^{\xi_{j+1}(z)} K(t, s)dt\,.$$

Let $\{u_i(s)\}_{i=1}^{n-r}$ be any Chebyshev system on $[0, 1]$, and for each $z \in S^n$, let $\sum_{i=1}^{n-r} c_i(z)u_i(s)$ denote the best L^∞ approximation to $G(s; z)$ from the Chebyshev system $\{u_i(s)\}_{i=1}^{n-r}$. Define $T(z) = (T_1(z), \cdots, T_n(z))$ by

$$T_i(z) = \begin{cases}\displaystyle\sum_{j=0}^{n}(\operatorname{sgn} z_{j+1})\int_{\xi_j(z)}^{\xi_{j+1}(z)} k_i(t)dt\,, & i = 1, \cdots, r \\ c_{i-r}(z)\,, & i = r + 1, \cdots, n\,.\end{cases}$$

It is easily seen that $T(z)$ is a continuous odd mapping of S^n into $\boldsymbol{R}^n$. Thus, by the Borsuk Antipodality Theorem (cf. [5]), there exists a $z^* \in S^n$ for which $T_i(z^*)=0, i=1, \cdots, n$. Furthermore, since $\{u_i(s)\}_{i=1}^{n-r}$ is a Chebyshev system, $G(s; z^*)$ must equioscillate on at least $n - r + 1$ points, unless $G(s; z^*) \equiv 0$. Since $T_i(z^*) = 0, i = 1, \cdots, r$,

$$G(s; z^*) = \int_0^1 h_\zeta(t)J(t, s; \zeta)dt$$

for some $\{\zeta_i\}_1^m$ with $m = S^-(z^*) \leqq n$ (see Definition 3.2). Therefore, by Lemma 2.1, $G(s; z^*)$ has at most $S^-(z^*) - r$ zeros. This means that $G(s; z^*)$ cannot vanish identically. Therefore $G(s; z^*)$ equioscillates exactly $n - r + 1$ times on $[0, 1]$, and $S^-(z^*) = n$, i.e., $z_j^* z_{j+1}^* < 0, j = 1, \cdots, n$. The function $g_{n,r}(s) = G(s; z^*)$ satisfies the requirement of the theorem and thus the theorem is proven.

We leave it to the reader to verify that, in Theorem 2.1, $\sigma = (-1)^{r+1}$.

For the remainder of our discussion we set $J(t, s; \xi) = J(t, s)$ and $h_\xi(t) = h_0(t)$, where $\xi = (\xi_1, \cdots, \xi_n)$ as defined in Theorem 2.1.

LEMMA 2.2. *The function $g_{n,r}$ of Theorem 2.1 has exactly $n - r$ distinct zeros in $(0, 1)$, at $0 < \tau_1 < \cdots < \tau_{n-r} < 1$ say, and*

$$K\begin{pmatrix} 1, \cdots, r, \tau_1, \cdots, \tau_{n-r} \\ \xi_1, \cdots\cdots\cdots\cdots, \xi_n \end{pmatrix} > 0 . \tag{2.3}$$

Proof. From Theorem 2.1 and Lemma 2.1, $g_{n,r}$ has exactly $n - r$ distinct zeros in $(0, 1)$, since the orthogonality conditions (2.2) imply that

$$g_{n,r}(s) = \sum_{j=0}^{n} (-1)^j \int_{\xi_j}^{\xi_{j+1}} J(t, s)dt .$$

We prove (2.3) by contradiction. If (2.3) fails, then there is a nontrivial function

$$u(s) = \sum_{j=1}^{n-r} c_j J(\xi_{j+r}, s)$$

which vanishes at $\tau_1, \cdots, \tau_{n-r}$. Our assumptions (Property I) imply that $J(\xi_{r+1}, s), \cdots, J(\xi_n, s)$ are linearly independent. Hence there is a $\tau_0 \in (0, 1)\backslash\{\tau_1, \cdots, \tau_{n-r}\}$ with $u(\tau_0) \neq 0$. Thus we may choose a constant c such that the function $g_{n,r}(s) - cu(s)$ vanishes $n - r + 1$ times at $\tau_0, \tau_1, \cdots, \tau_{n-r}$. However, this conclusion contradicts Lemma 2.1. Hence (2.3) is valid.

In the computation of the n-width of $\mathscr{K}_r$, the following proposition plays a crucial role.

Let $B = (b_{ij})$ be the $n + 1 \times n + 1$ matrix defined as

$$(2.4) \qquad b_{ij} = \begin{cases} \int_{\xi_{j-1}}^{\xi_j} k_i(t)dt \,, & i = 1, \cdots, r;\ j = 1, 2, \cdots, n+1 \\ \int_{j-1}^{\xi_j} K(t, \eta_{i-r})dt \,, & i = r+1, \cdots, n+1;\ j = 1, 2, \cdots, n+1 \,, \end{cases}$$

where the $\{\eta_i\}_{i=1}^{n-r+1}$ are points of equioscillation of $g_{n,r}$ (see (2.1)).

PROPOSITION 2.1. *The matrix B is invertible, and furthermore, for any $\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_n)$ satisfying*

$$\sum_{j=0}^{n} \alpha_j \int_{\xi_j}^{\xi_{j+1}} k_i(t)dt = 0 \,, \quad i = 1, \cdots, r \,,$$

we have

$$\max_{1 \leqq i \leqq n-r+1} \left| \sum_{j=0}^{n} \alpha_j \int_{\xi_j}^{\xi_{j+1}} K(t, \eta_i)dt \right| \geqq \|g_{n,r}\|_\infty \|\alpha\|_\infty \,,$$

where $\|\alpha\|_\infty = \max_{0 \leqq i \leqq n} |\alpha_i|$.

We precede the proof of Proposition 2.1 with the following lemma.

The minors of a matrix $A = (a_{ij})$ are denoted by

$$A\begin{pmatrix} i_1, \cdots, i_k \\ j_1, \cdots, j_k \end{pmatrix} = \begin{vmatrix} a_{i_1 j_1} & \cdots & a_{i_1 j_k} \\ \vdots & & \vdots \\ a_{i_k j_1} & \cdots & a_{i_k j_k} \end{vmatrix} .$$

LEMMA 2.3. *Let $n \geqq r$ and $A = (a_{ij})$ be an $n + 1 \times n$ matrix such that*

(a) $A\begin{pmatrix} 1, \cdots, r \\ 1, \cdots, r \end{pmatrix} > 0$,

and

(b) $A\begin{pmatrix} 1, \cdots, r, i_1, \cdots, i_k \\ j_1, \cdots\cdots\cdots, j_{r+k} \end{pmatrix} \geqq 0$

for all $r + 1 \leqq i_1 < \cdots < i_k \leqq n + 1$, $1 \leqq j_1 < \cdots < j_{r+k} \leqq n$. Then there exists a nontrivial vector $\alpha \in \boldsymbol{R}^{n+1}$ such that $\alpha_j(-1)^j \geqq 0$, $j = r + 1, \cdots, n + 1$, and

$$\alpha A = 0 \,.$$

Proof. For $r = 0$, our hypothesis means that A is totally positive. Hence there exists a sequence $A_N \to A$ as $N \to \infty$, such that A_N is strictly positive, [2], that is, all the minors of A_N are strictly positive. Let

$$\alpha_j^N = (-1)^i A_N\begin{pmatrix}1, \cdots, j-1, j+1, \cdots, n+1 \\ 1, \quad \cdot \quad \cdot \quad \cdot \quad, \quad n\end{pmatrix}$$

then $\alpha^N A_N = 0$ and $\beta^N = \alpha^N/\|\alpha^N\|_\infty$, $\|\alpha^N\|_\infty = \max\{|\alpha_j^N|: 1 \leqq j \leqq n+1\}$, has a subsequence $\{\beta^{N'}\}$ converging to an $\alpha \in \boldsymbol{R}^{n+1}$ which satisfies the demands of the lemma.

For $r > 0$, we define a new matrix $C = (c_{ij})$,

$$c_{ij} = \frac{A\begin{pmatrix}1, \cdots, r, i \\ 1, \cdots, r, j\end{pmatrix}}{A\begin{pmatrix}1, \cdots, r \\ 1, \cdots, r\end{pmatrix}}, \quad r+1 \leqq i \leqq n+1, r+1 \leqq j \leqq n .$$

Then by Sylvester's determinant identity

$$C\begin{pmatrix}i_1, \cdots, i_k \\ j_1, \cdots, j_k\end{pmatrix} = \frac{A\begin{pmatrix}1, \cdots, r, i_1, \cdots, i_k \\ 1, \cdots, r, j_1, \cdots, j_k\end{pmatrix}}{A\begin{pmatrix}1, \cdots, r \\ 1, \cdots, r\end{pmatrix}} .$$

Hence all the minors of C are nonnegative and by the above remarks there is a nontrivial $\delta \in \boldsymbol{R}^{n-r+1}$ such that

$$\delta C = 0, \delta_j(-1)^{j+r} \geqq 0 , \quad j = 1, \cdots, n-r+1 .$$

Let $\alpha = (\alpha_1, \cdots, \alpha_r, \delta_1, \cdots, \delta_{n-r+1})$ where $\alpha_1, \cdots, \alpha_r$ are chosen so that

$$\sum_{j=1}^{r} \alpha_j A_{ji} = -\sum_{j=r+1}^{n+1} \delta_{j-r} A_{ji} , \qquad i = 1, \cdots, r .$$

Then it is easily verified that α satisfies the requirements of the lemma. The proof is complete.

Observe that vector $\alpha = (\alpha_1, \cdots, \alpha_{n+1})$ constructed above has the property that

$$\sum_{j=r+1}^{n+1} |\alpha_j| > 0 .$$

On the basis of this lemma, we now prove Proposition 2.1.

Proof. According to Lemma 2.3 and Properties I and II, there exists an $n+1 \times n+1$ matrix E such that $EB = D$, $E_{ij}(-1)^{i+j} \geqq 0$, $i = 1, \cdots, n+1$, $j = r+1, \cdots, n+1$ and $\sum_{j=r+1}^{n+1} |E_{ij}| > 0$, $i = 1, \cdots, n+1$, where $D = \operatorname{diag}\{d_1, \cdots, d_{n+1}\}$.

Let

$$e = (\underbrace{0, \cdots, 0}_{r}, (-1)^r, \cdots, (-1)^n)$$

and

$$\gamma = (1, -1, \cdots, (-1)^n) .$$

Then according to Theorem 2.1, $B\gamma = \|g_{n,r}\|_\infty e$. Hence

$$\begin{aligned} d_k &= (-1)^{k-1}(D\gamma)_k = (-1)^{k-1}\|g_{n,r}\|_\infty(Ee)_k \\ &= \|g_{n,r}\|_\infty \sum_{j=r+1}^{n+1} |E_{kj}| > 0 , \qquad k = 1, \cdots, n+1 . \end{aligned}$$

We conclude that B is invertible and $B^{-1} = D^{-1}E$.

Now, if $\alpha \in \boldsymbol{R}^{n+1}$ satisfies $(B\alpha)_i = 0$, $i = 1, \cdots, r$ then $\alpha_k = \sum_{j=r+1}^{n+1} B_{kj}^{-1}(B\alpha)_j$ and therefore

$$\begin{aligned} \|\alpha\|_\infty &\leqq \|B\alpha\|_\infty \max\left\{\sum_{j=r+1}^{n+1} |B_{kj}^{-1}| : 1 \leqq k \leqq n+1\right\} \\ &= \|B\alpha\|_\infty \max\{(-1)^{k-1}(B^{-1}e)_k : 1 \leqq k \leqq n+1\} \\ &= \|B\alpha\|_\infty / \|g_{n,r}\|_\infty . \end{aligned}$$

The proposition is proven.

We are now prepared to prove our main results.

Let X_n^0 denote the linear space spanned by the functions $k_1(t), \cdots, k_r(t)$, $K(t, \tau_1), \cdots, K(t, \tau_{n-r})$,

$$X_n^0 = [k_1, \cdots, k_r, K(\cdot, \tau_1), \cdots, K(\cdot, \tau_{n-r})]$$

and suppose S is the linear mapping from $C[0, 1]$ onto X_n^0, defined by the interpolation conditions

$$(Sf)(\xi_i) = f(\xi_i) , \quad i = 1, \cdots, n, f \in C[0, 1] .$$

From Lemma 2.2, this is a well-defined linear map. We recall that

$$d_n(\mathscr{K}_r; X) = \inf_{X_n} \sup_{f \in \mathscr{K}_r} \inf_{g \in X_n} \|f - g\|_1 ,$$

where X_n is any n-dimensional subspace of $X = L^1[0, 1]$.

THEOREM 2.2.

$$d_n(\mathscr{K}_r; X) = \begin{cases} \infty , & n < r \\ \|g_{n,r}\|_\infty , & n \geqq r \end{cases}$$

and for $n \geqq r$, X_n^0 is an optimal subspace for the n-width of $\mathscr{K}_r$. Furthermore, when $n \geqq r$,

$$d_n(\mathscr{K}_r; X) = \sup_{f \in \mathscr{K}_r} \|f - Sf\|_1 .$$

Proof. If Q_r is the subspace spanned by $k_1, \cdots, k_r$, then since $Q_r \subseteqq \mathscr{K}_r$, $d_n(\mathscr{K}_r; X) = \infty$ when $n < r$. Now, let us suppose $n \geqq r$. We will first prove a lower bound for the n-width.

Let X_n be any n-dimensional subspace of X, and define the characteristic functions

$$\chi_j(t) = \begin{cases} 1\,, & \xi_j < t \leqq \xi_{j+1} \\ 0\,, & \text{otherwise} \end{cases}$$

for $j = 0, 1, \cdots, n$. In computing the n-width of $\mathscr{K}_r$, it is sufficient to consider only those subspaces containing Q_r. Choose

$$\beta = (\beta_0, \beta_1, \cdots, \beta_n)$$

such that $\|\beta\|_\infty = 1$, and $h_\beta = \sum_{j=0}^n \beta_j \chi_j$ is orthogonal to X_n. Since $\|h_\beta\|_\infty = \|\beta\|_\infty = 1$, and $f \mapsto (f, h_\beta) = \int_0^1 f(t)h_\beta(t)dt$ is a norm one linear functional which annihilates X_n, we conclude that

$$\delta(\mathscr{K}_r; X_n) \geqq \sup\{|(f, h_\beta)|: f \in \mathscr{K}_r\}\ .$$

Since $Q_r \subseteqq X_n$, we have

$$\begin{aligned}\sup_{f \in \mathscr{K}_r} |(f, h_\beta)| &= \left\| \int_0^1 K(t, s)h_\beta(t)dt \right\|_\infty = \left\| \sum_{j=0}^n \beta_j \int_{\xi_j}^{\xi_{j+1}} K(t, s)\,dt \right\|_\infty \\ &\geqq \|g_{n,r}\|_\infty \|\beta\|_\infty = \|g_{n,r}\|_\infty\ .\end{aligned}$$

The last inequality follows from Proposition 2.1, and the orthogonality conditions $(k_i, h_\beta) = 0$, $i = 1, \cdots, r$. Thus we have shown that $\delta(\mathscr{K}_r; X_n) \geqq \|g_{n,r}\|_\infty$ for all n-dimensional subspaces X_n of X which contain Q_r. Hence,

$$\|g_{n,r}\|_\infty \leqq d_n(\mathscr{K}_r; X)\ .$$

We will now show that

$$\sup_{f \in \mathscr{K}_r} \|f - Sf\|_1 \leqq \|g_{n,r}\|_\infty\ .$$

To this end, observe that

$$f(t) - Sf(t) = \int_0^1 \frac{K\begin{pmatrix} 1, \cdots, r, \tau_1, \cdots, \tau_{n-r}, s \\ \xi_1, \quad \cdot \quad \cdot \quad \cdot \quad, \xi_n, t \end{pmatrix}}{K\begin{pmatrix} 1, \cdots, r, \tau_1, \cdots, \tau_{n-r} \\ \xi_1, \quad \cdot \quad \cdot \quad \cdot \quad, \xi_n \end{pmatrix}} g(s)ds$$

for some $g \in X$, $\|g\|_1 \leqq 1$. Thus,

$$\|f - Sf\|_1 = \max_{\|h\|_\infty \leq 1} |(h, f - Sf)|$$

$$\leq \max \left\{ \left| \int_0^1 \frac{K\begin{pmatrix} 1, \cdots, r, \tau_1, \cdots, \tau_{n-r}, s \\ \xi_1, \quad \cdot \quad \cdot \quad \cdot \quad, \xi_n, t \end{pmatrix}}{K\begin{pmatrix} 1, \cdots, r, \tau_1, \cdots, \tau_{n-r} \\ \xi_1, \quad \cdot \quad \cdot \quad \cdot \quad, \xi_n \end{pmatrix}} h(t)dt \right| : \right.$$

$$\left. 0 \leq s \leq 1, \|h\|_\infty \leq 1 \right\}$$

$$= \max \left\{ \left| \int_0^1 \frac{K\begin{pmatrix} 1, \cdots, r, \tau_1, \cdots, \tau_{n-r}, s \\ \xi_1, \quad \cdot \quad \cdot \quad \cdot \quad, \xi_n, t \end{pmatrix}}{K\begin{pmatrix} 1, \cdots, r, \tau_1, \cdots, \tau_{n-r} \\ \xi_1, \quad \cdot \quad \cdot \quad \cdot \quad, \xi_n \end{pmatrix}} h_0(t)dt \right| : 0 \leq s \leq 1 \right\}$$

where $h_0(t) = (-1)^j$, $\xi_j < t \leq \xi_{j+1}$, $j = 0, 1, \cdots, n$. Since $(h_0, k_i) = 0$, $i = 1, \cdots, r$, and $(h_0, K(\cdot, \tau_i)) = 0$, $i = 1, \cdots, n - r$, it follows that

$$\int_0^1 \frac{K\begin{pmatrix} 1, \cdots, r, \tau_1, \cdots, \tau_{n-r}, s \\ \xi_1, \quad \cdot \quad \cdot \quad \cdot \quad, \xi_n, t \end{pmatrix}}{K\begin{pmatrix} 1, \cdots, r, \tau_1, \cdots, \tau_{n-r} \\ \xi_1, \quad \cdot \quad \cdot \quad \cdot \quad, \xi_n \end{pmatrix}} h_0(t)dt = \int_0^1 K(t, s)h_0(t)dt = g_{n,r}(s) .$$

Thus, $\sup_{f \in \mathscr{K}_r} \|f - Sf\|_1 \leq \|g_{n,r}\|_\infty$ and since necessarily

$$\delta(\mathscr{K}_r; X_n^0) \leq \sup_{f \in \mathscr{K}_r} \|f - Sf\|_1 ,$$

we obtain

$$\|g_{n,r}\|_\infty = d_n(\mathscr{K}_r; X) = \delta(\mathscr{K}_r; X_n^0) = \sup_{f \in \mathscr{K}_r} \|f - Sf\|_1 .$$

Thus the theorem is proven.

Let us observe that Theorem 2.2 also expresses the fact that simply interpolating $f \in \mathscr{K}_r$ by means of the function Sf is as good as approximating $\mathscr{K}_r$ in the L^1-norm by any fixed n-dimensional subspace of $L^1[0, 1]$. Clearly, then S represents an optimal linear method for approximating $\mathscr{K}_r$ (see also [7]).

Recall the Gel'fand width

$$d^n(\mathscr{K}_r; Y) = \inf_{L_n} \sup_{f \in \mathscr{K}_r \cap L_n} \|f\|_1 ,$$

where L_n is any subspace of $Y = C[0, 1]$ of codimension n.

THEOREM 2.3.

$$d^n(\mathscr{K}_r; Y) = \begin{cases} \infty , & n < r \\ \|g_{n,r}\|_\infty , & n \geqq r \end{cases}$$

and for $n \geqq r$,

$$L_n^0 = \{f: f \in C[0, 1], f(\xi_i) = 0, i = 1, \cdots, n\}$$

is an optimal subspace for $\mathscr{K}_r$.

Proof. As previously, it is easily shown that $d^n(\mathscr{K}_r; Y) = \infty$ if $n < r$. For $n \geqq r$, we have $\sup_{f \in \mathscr{K}_r} \|f - Sf\|_1 = \|g_{n,r}\|_\infty$. Thus $\sup_{f \in \mathscr{K}_r \cap L_n^0} \|f\|_1 \leqq \|g_{n,r}\|_\infty$, and $d^n(\mathscr{K}_r; Y) \leqq \|g_{n,r}\|_\infty$.

To prove the reverse inequality we now suppose that L_n is any subspace of Y of codimension n. Choose the vector $\gamma = (\gamma_1, \cdots, \gamma_{n+1})$, $\gamma \not\equiv 0$, such that the function

$$F(t) = \sum_{j=1}^{r} \gamma_j k_j(t) + \sum_{j=1}^{n-r+1} \gamma_{j+r} K(t, \eta_j)$$

is in L_n. We may normalize γ so that $\sum_{j=r+1}^{n+1} |\gamma_j| = 1$, since otherwise L_n would contain a nonzero element of Q_r and thus $\sup_{f \in \mathscr{K}_r \cap L_n} \|f\|_1 = \infty$. In addition, we define the vector $\delta = (\delta_0, \cdots, \delta_n)$ such that $h_\delta = \sum_{i=0}^{n} \delta_i \chi_i$ satisfies

$$\begin{aligned} &(h_\delta, k_i) = 0 , \quad i = 1, \cdots, r \\ &(h_\delta, K(\cdot, \eta_i)) = (\operatorname{sgn} \gamma_{i+r}) \|g_{n,r}\|_\infty , \quad i = 1, \cdots, n - r + 1 . \end{aligned}$$

This is possible since the matrix B of (2.4) was shown in Proposition 2.1 to be invertible. Now, $\|h_\delta\|_\infty = \|\delta\|_\infty$ and $F \in L_n \cap \mathscr{K}_r$, since $\sum_{j=r+1}^{n+1} |\gamma_j| = 1$. Thus,

$$\sup_{f \in \mathscr{K}_r \cap L_n} \|f\|_1 \geqq \|F\|_1 \geqq \frac{|(F, h_\delta)|}{\|\delta\|_\infty} .$$

From Proposition 2.1, $\|\delta\|_\infty \leqq 1$ and thus

$$\begin{aligned} \sup_{f \in \mathscr{K}_r \cap L_n} \|f\|_1 &\geqq |(F, h_\delta)| \\ &= \Big(\sum_{j=r+1}^{n+1} |\gamma_j|\Big) \|g_{n,r}\|_\infty \\ &= \|g_{n,r}\|_\infty . \end{aligned}$$

The proof is complete.

Let $r \geqq 2$, and $W^{r,1} = \{f: f^{(r-1)}$ absolutely continuous on $[0, 1]$, $\|f^{(r)}\|_1 \leqq 1\}$. Then, as a corollary to Theorems 2.2 and 2.3, we have

COROLLARY 2.1.

$$d^n(W^{r,1}; C) = d_n(W^{r,1}; L^1) = \begin{cases} \infty \ , & n < r \\ \|P_{n,r}\|_\infty \ , & n \geqq r \end{cases}$$

where $P_{n,r}$ is a perfect spline

$$P_{n,r}(s) = \sum_{j=0}^{n} \frac{(-1)^j}{(r-1)!} \int_{\xi_j}^{\xi_{j+1}} (t-s)_+^{r-1} dt$$

$0 = \xi_0 < \xi_1 < \cdots < \xi_n < \xi_{n+1} = 1$ *such that* $P_{n,r}^{(i)}(0) = P_{n,r}^{(i)}(1) = 0$, $i = 0, 1, \cdots, r-1$, *which equioscillates at* $n - r + 1$ *points of* $(0, 1)$. *The subspace*

$$X_n^0 = [1, t, \cdots, t^{r-1}, (t-\tau_1)_+^{r-1}, \cdots, (t-\tau_{n-r})_+^{r-1}] \ ,$$

($t_+^{r-1} = t^{r-1}$, $t \geqq 0$, *zero elsewhere*) *where* $\tau_1, \cdots, \tau_{n-r}$ *are the unique zeros of* $P_{n,r}$, *is optimal for the Kolmogorov n-width of* $W^{r,1}$, *while* $L_n^0 = \{f : f(\xi_i) = 0,\ i = 1, \cdots, n,\ f \in C[0, 1]\}$ *is optimal for the Gel'fand n-width of* $W^{r,1}$.

Proof. This result follows by specializing Theorems 2.2 and 2.3 to the choice $k_i(t) = t^{i-1}$, $i = 1, \cdots, r$, and $K(t, s) = 1/(r-1)!\,(t-s)_+^{r-1}$. The fact that this choice satisfies Properties I and II is a well-known property of spline functions, see [2].

When $n = r$, then $P_{r,r}$ is explicitly given by

$$P_{r,r}(s) = \frac{1}{(r-1)!} \int_0^1 [\operatorname{sgn} T'_{r+1}(t)](t-s)_+^{r-1} dt \ ,$$

where T_{r+1} is the $(r+1)$st Chebyshev polynomial on $[0, 1]$. This is a consequence of a classical result of Bernstein on best L^1 approximation by polynomials.

In Corollary 2.1, we assumed $r \geqq 2$ in order to satisfy the continuity assumption on $K(t, s)$. However, it can be easily verified that the result remains valid for the case $r = 1$.

For additional examples of $\{k_i(t)\}_{i=1}^r$ and $K(t, s)$ satisfying Properties I and II, see [6].

3. *n*-widths in l^1. The purpose of this section is to briefly describe a matrix version of Theorems 2.2 and 2.3, complementing work done in [6]. The results are stated for the most part without proof. However, proofs may be reconstructed based on the analysis of §2.

It is convenient to begin by recalling some definitions and properties of totally positive matrices. We adhere to the notation in [6].

DEFINITION 3.1. An $N \times M$ matrix A is said to be totally positive of order l (TP_l) if

$$A\begin{pmatrix} i_1, \cdots, i_k \\ j_1, \cdots, j_k \end{pmatrix} = \begin{vmatrix} a_{i_1j_1} & \cdots & a_{i_1j_k} \\ \vdots & & \vdots \\ a_{i_kj_1} & \cdots & a_{i_kj_k} \end{vmatrix} \geqq 0 \tag{3.1}$$

for all $1 \leqq i_1 < \cdots < i_k \leqq N$, $1 \leqq j_1 < \cdots < j_k \leqq M$, and $k = 1, \cdots, l$. A is said to be strictly totally positive of order $l(STP_l)$ if strict inequality holds in (3.1).

DEFINITION 3.2. Let $x = (x_1, \cdots, x_l)$ be a real vector of l components.

(i) $S^-(x)$ denotes the number of actual sign changes in the sequence $x_1, \cdots, x_l$ with zero terms discarded.

(ii) $S^+(x)$ counts the maximum number of sign changes in the sequence $x_1, \cdots, x_l$ where zero terms are assigned values $+1$ or -1, arbitrarily.

For example, $S^-(-1, 0, 1, -1, 0, -1) = 2$, and

$$S^+(-1, 0, 1, -1, 0, -1) = 4 .$$

THEOREM 3.1. *If A is an $N \times M$ matrix which is STP_{n+1}, and if x is any nontrivial M-vector such that $S^-(x) \leqq n$, then*

(i) $S^+(Ax) \leqq S^-(x)$

(ii) *If $S^+(Ax) = S^-(x)$ then the first* (and last) *component of Ax* (if zero, then the sign given in determining $S^+(Ax)$), *agrees in sign with the first* (and last) *nonzero component of x.*

The above theorem is to be found in Karlin [2, p. 223] in a slightly different form. The complete statement of the above theorem is found in Karlin and Pinkus [3].

DEFINITION 3.3. Given $0 = j_0 < j_1 < \cdots < j_l < j_{l+1} = M + 1$, and a vector $x \in \boldsymbol{R}^M$, we say that x alternates between $j_1, \cdots, j_l$ provided that there exists a sign σ, $\sigma^2 = 1$, such that $x_k = (-1)^{i-1}\sigma$, $j_{i-1} < k < j_i$, $i = 1, \cdots, l + 1$. (Note that no requirement is placed on the components $x_{j_1}, \cdots, x_{j_l}$.) When $\sigma = 1$ we will say that x alternates with positive orientation.

DEFINITION 3.4. A vector $y \in \boldsymbol{R}^N$ equioscillates on $i_1, \cdots, i_{l+1}$, $1 \leqq i_1 < \cdots < i_{l+1} \leqq N$, provided that there exists a σ, $\sigma^2 = 1$, such that $y_{i_k} = (-1)^k\sigma \| y \|_\infty$, $k = 1, \cdots, l + 1$.

We shall also denote by a^j the jth column vector of A. We now state an analogue of Theorem 2.1.

THEOREM 3.2. *Let A be an $N \times M$ STP_{n+1} matrix and suppose $0 \leqq r \leqq n < \min\{N, M\}$. Then there exists $j^0 = (j_1^0, \cdots, j_n^0)$, $1 \leqq j_1^0 < \cdots < j_n^0 \leqq M$, and a vector $x^0 \in \boldsymbol{R}^M$ such that*

(1) *x^0 alternates between $j_1^0, \cdots, j_n^0$*

(2) $\|x^0\|_\infty = 1$

(3) $(Ax^0)_i = 0$, $i = 1, \cdots, r$

(4) *Ax^0 equioscillates $n - r + 1$ times.*

Proof. Fix $j = (j_1, \cdots, j_n)$, and if $n > r$, define the $N - r \times M - r$ matrix $B = (b_{ij})$, $i = r + 1, \cdots, N$; $j = 1, \cdots, M$, $j \neq j_l$, $l = 1, \cdots, r$, by

$$b_{ij} = \frac{A\begin{pmatrix}1, \cdots, r, i \\ j_1, \cdots, j_r, j\end{pmatrix}}{A\begin{pmatrix}1, \cdots, r \\ j_1, \cdots, j_r\end{pmatrix}} .$$

The column vectors $b^{j_{r+1}}, \cdots, b^{j_n}$ of B form a Chebyshev system since by Sylvester's determinant identity,

$$B\begin{pmatrix}i_1, \cdots, i_{n-r} \\ j_{r+1}, \cdots, j_n\end{pmatrix} = \frac{A\begin{pmatrix}1, \cdots, r, i_1, \cdots, i_{n-r} \\ j_1, \cdots, j_r, j_{r+1}, \cdots, j_n\end{pmatrix}}{A\begin{pmatrix}1, \cdots, r \\ j_1, \cdots, j_r\end{pmatrix}} > 0$$

for $r + 1 \leqq i_1 < \cdots < i_{n-r} \leqq N$.

Let $\tilde{f}$ be the $M - r$ vector $\tilde{f} = (f_j)$, $j = 1, \cdots, M$, $j \neq j_l$, $l = 1, \cdots, r$ obtained from the M-vector f with f_{j_l}, $l = 1, \cdots, r$ deleted, where f alternates between $j_1, \cdots, j_n$ and $f_{j_l} = 0$, $l = 1, \cdots, n$.

The error $B\tilde{f} - \sum_{s=1}^{n-r} d_s b^{j_{r+s}}$, in approximating $B\tilde{f}$ by linear combinations of $b^{j_{r+1}}, \cdots, b^{j_n}$, in the l^∞-norm, necessarily equioscillates $n - r + 1$ times on some rows $r + 1 \leqq i_1 < \cdots < i_{n-r+1} \leqq N$. Now, for $n \geqq r$, we define $x_j \in \boldsymbol{R}^M$ by setting

$$\begin{aligned} (x_j)_j &= (\tilde{f})_j , & j &= 1, \cdots, M, j \notin \{j_1, \cdots, j_n\} \\ (x_j)_{j_{r+s}} &= -d_s , & s &= 1, \cdots, n - r \\ (Ax_j)_i &= 0 , & i &= 1, \cdots, r . \end{aligned}$$

Thus x_j alternates between $j_1, \cdots, j_n$, $\|x_j\|_\infty \geqq 1$, and since

$$(Ax_j)_i = \left(B\tilde{f} - \sum_{s=1}^{n-r} d_s b^{j_{r+s}}\right)_i , \quad i = r + 1, \cdots, N ,$$

Ax_j equioscillates on $i_1, \cdots, i_{n-r+1}$. We define $\boldsymbol{j}^0 = (j_1^0, \cdots, j_n^0)$ by requiring that

$$\|Ax_{j^0}\|_\infty \leqq \|Ax_j\|_\infty \tag{3.2}$$

for all j. We claim that $x^0 = x_{j^0}$ satisfies the requirements of the theorem. To prove this is the case, we must show that $|(x^0)_{j_k^0}| \leqq 1$, $k = 1, \cdots, n$.

Multiplying by -1, if necessary, we shall assume that x_j alternates with positive orientation for each j.

First, let us suppose that $j_k^0 > k$. Let l be the largest integer less than j_k^0 such that $l \neq j_i^0$, $i = 1, \cdots, k-1$. Define $j = (j_1, \cdots, j_n)$ where $\{j_1, \cdots, j_n\}$ is the set of indices $\{j_1^0, \cdots, j_{k-1}^0, j_{k+1}^0, \cdots, j_n^0, l\}$ arranged in increasing order. Since x_j alternates with positive orientation, we have $(x_j)_{j_k^0} = (-1)^k$. Now, consider the vector $Ax_j - Ax^0 = A(x_j - x^0)$. If $x_j - x^0 = 0$ then $|(x^0)_{j_k^0}| = 1$. If $x_j - x^0 \neq 0$ then, from (3.2) and the fact that Ax_j equioscillates on some $n - r + 1$ rows, and $(Ax_j)_i = 0$, $i = 1, \cdots, r$, we conclude that $S^+(Ax_j - Ax^0) \geqq n$. Since $x_j - x^0$ has, by construction, at most $n + 1$ nonzero components, that is, the components corresponding to the columns $j_1^0, \cdots, j_n^0, l$, we conclude that $S^-(x_j - x^0) \leqq n$. From Theorem 3.1, $S^+(A(x_j - x^0)) = S^-(x_j - x^0) = n$ and the sign patterns must agree.

Since A is STP_{n+1}, $A(x_j - x^0)$ cannot have $n + 1$ zero components. Because the sign pattern in $S^+(Ax_j)$ begins with a plus and $\|Ax^0\|_\infty \leqq \|Ax_j\|_\infty$, it follows that the sign pattern in $S^+(A(x_j - x^0))$ begins with a plus. Applying Theorem 3.1(ii), we see that $\operatorname{sgn}((x_j)_{j_k^0} - (x^0)_{j_k^0}) = (-1)^k$. Since $(x_j)_{j_k^0} = (-1)^k$ we conclude that $1 > (x^0)_{j_k^0}(-1)^k$. Similarly, if $j_k^0 < M - n + k$, we let l be the smallest integer greater than j_k^0 such that $l \neq j_i^0$, $i = k+1, \cdots, n$. Then, as above, we may show that $1 > (x^0)_{j_k^0}(-1)^{k+1}$. Hence, if both $j_k^0 > k$ and $j_k^0 < M - n + k$ we obtain the desired conclusion that $|(x^0)_{j_k^0}| \leqq 1$.

In the case that $j_k^0 = k$ we have $j_i^0 = i$, $i = 1, \cdots, k-1$, and thus $\operatorname{sgn}(x^0)_{j_k^0} = (-1)^{k+1}$. However $n < M$, which implies $j_k^0 < M - n + k$. Thus by our above remarks $|(x^0)_{j_k^0}| = (-1)^{k+1}(x^0)_{j_k^0} \leqq 1$. Similarly, if $j_k^0 = M - n + k$, then $j_k^0 > k$ and $|(x^0)_{j_k^0}| = (-1)^k(x^0)_{j_k^0} \leqq 1$. Thus in all cases, we arrive at the desired conclusion.

We will now state an l^1-analogue of Theorems 2.2 and 2.3.

Let $x \in \boldsymbol{R}^M$, $\pi_r x = (0, \cdots, 0, x_{r+1}, \cdots, x_M)$, $\|x\|_1 = \sum_{j=1}^M |x_j|$, and define

$$\mathscr{A}_r = \{Ax \colon \|\pi_r x\|_1 \leqq 1\} .$$

If A is STP_{n+1}, then so is A^T = transpose of A. Thus from Theorem 3.2, there exists a vector $z^0 \in \boldsymbol{R}^N$ which alternates between some $1 \leqq j_1^0 < \cdots < j_n^0 \leqq N$, $\|z^0\|_\infty = 1$, $(A^T z^0)_j = 0$, $j = 1, \cdots, r$, and $A^T z^0$ equioscillates $n - r + 1$ times.

THEOREM 3.3. *Let A be an $N \times M$ STP_{n+1} matrix, $0 \leqq n <$*

$\min\{N, M\}$. *Then*

$$d^n(\mathscr{A}_r; l_N^1) = \begin{cases} \infty , & n < r \\ \|A^T z^0\|_\infty , & n \geqq r \end{cases}$$

and for $n \geqq r$, $L_n = \{x: x \in \boldsymbol{R}^N, (x)_{j_k^0} = 0, k = 1, \cdots, n\}$ *is an optimal subspace for* $\mathscr{A}_r$.

For a matrix version of Theorem 2.2, we observe that from Theorem 3.1, $S^+(A^T z^0) = S^-(z^0) = n$ and since $(A^T z^0)_j = 0$, $j = 1, \cdots, r$, and $A^T z^0$ equioscillates $n - r + 1$ times, $(A^T z^0)_{r+1}(A^T z^0)_M \neq 0$, and if $(A^T z^0)_i = 0$, $r + 1 < i < M$, then $(A^T z^0)_{i-1}(A^T z^0)_{i+1} < 0$. For $i = r + 1, \cdots, n$, the ith weak sign change of $A^T z^0$ "occurs at" an index k_i in one of two possible ways. Either

(a) $(A^T z^0)_{k_i - 1}(A^T z^0)_{k_i} < 0$,

or

(b) $(A^T z^0)_{k_i} = 0$, and $(A^T z^0)_{k_i - 1}(A^T z^0)_{k_i + 1} < 0$,

where $r + 1 < k_{r+1} < \cdots < k_n \leqq M$. For each i, $i = r + 1, \cdots, n$, we define an M-dimensional vector e^i as follows. If (a) holds, put

$$(e^i)_l = \begin{cases} |(A^T z^0)_l|^{-1} , & l = k_i - 1, k_i \\ 0 , & \text{otherwise} . \end{cases}$$

If (b) arises, set $(e^i)_l = \delta_{k_i l}$, $l = 1, \cdots, M$. In addition, let e^i, $i = 1, \cdots, r$ be the first r unit vectors in $\boldsymbol{R}^M$, i.e., $(e^i)_l = \delta_{il}$, $i = 1, \cdots, r$; $l = 1, \cdots, M$. Thus $(A^T z^0, e^i) = 0$, $i = 1, \cdots, n$.

Now, we define an $M \times M$ matrix P by the condition that for any $x \in \boldsymbol{R}^M$, the vector $y = Px$ is in the linear space spanned by $e^1, \cdots, e^n$, and $(Ax - APx)_{j_k^0} = 0$, $k = 1, \cdots, n$. Px exists since otherwise there exists a nonzero $y = \sum_{j=1}^n c_j e^j$ such that $(Ay)_{j_k^0} = 0$, $k = 1, \cdots, n$. Hence $S^+(Ay) \geqq n$. But, by the construction of the vectors $e^1, \cdots, e^n$, it is clear that $S^-(y) \leqq n - 1$. Applying Theorem 3.1, we arrive at a contradiction, and so Px exists.

Let $B = AP$ and note that B is an $N \times M$ matrix of rank n, whose column space is spanned by the set of vectors $\{Ae^1, \cdots, Ae^n\}$.

THEOREM 3.4. *Let A be an $N \times M$ STP_{n+1} matrix, $0 \leqq n < \min\{N, M\}$. Then,*

$$d_n(\mathscr{A}_r; l_N^1) = \begin{cases} \infty , & n < r \\ \|A^T z^0\|_\infty , & n \geqq r \end{cases}$$

and for $n \geqq r$, the linear space spanned by the set of vectors $\{Ae^1, \cdots, Ae^n\}$ is an optimal subspace for $\mathscr{A}_r$. Furthermore,

$$d_n(\mathscr{A}_r; l_N^1) = \max_{\|\pi_r x\|_1 \leqq 1} \|Ax - Bx\|_1 .$$

The proofs of Theorems 3.3 and 3.4 are similar to the proofs given in §2. We omit the details.

Finally, let us point out that we may, by a standard continuity argument, give an alternative proof of Theorem 2.1 by using Theorem 3.2, see [6] for a detailed discussion of this matter in L^∞. The advantage of this approach is that it avoids the use of Borsuk's theorem and thus is "elementary." In addition, Theorems 3.2, 3.3, and 3.4 afford us great flexibility in computing n-widths when N and/or $M = \infty$, again see [6] for a detailed description of these matters in L^∞.

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