SCHRÖDINGER AND DIRAC OPERATORS WITH SINGULAR POTENTIALS AND HYPERBOLIC EQUATIONS

PAUL R. CHERNOFF

In an earlier paper we employed techniques from the theory of hyperbolic partial differential equations to deduce the self-adjointness of Dirac and Schrödinger operators with smooth potentials. The present paper applies these techniques to operators with singular potentials.

1. **Introduction.** There is a physical idea in back of our arguments which is especially clear in the case of operators of Dirac type. The Dirac operator $H = D + V$, a first-order partial differential operator, is the quantum Hamiltonian governing the dynamics of a relativistic particle in an external electromagnetic field. Intuitively, we expect that H (with domain C_0^∞) should be essentially self-adjoint if the time evolution is determined by the formal differential expression alone—that is, no boundary conditions are needed to tell the particle how to be scattered when it hits a singularity. First of all, we require that the underlying "physical space" be a complete Riemannian manifold so that no finite points are missing. Moreover, in a relativistic system waves propagate at the speed of light; hence compactly supported data are not propagated to infinity in a finite time, and thus no boundary conditions at infinity are required. Finally, suppose that the potential term V is locally well-behaved in a sense which we will make precise later—roughly that everywhere V is locally equal to a potential that requires no special boundary conditions. Then we expect that H is essentially self-adjoint. That is, by exploiting the finite propagation speed of the Dirac equation, we can patch together local good behavior to deduce global good behavior. This is the main result of §2.

In §§3 and 4 we apply analogous ideas to second-order Schrödinger operators, by considering the associated wave equations. Although the underlying ideas are similar to the Dirac case, there are a number of technical complications, some of which are dealt with in the preliminary material in §3. The conclusion is, roughly, that a Schrödinger operator with a locally well-behaved potential, which does not decrease too rapidly at infinity, is essentially self-adjoint; this global condition on the potential is needed because the nonrelativistic Schrödinger equation has infinite velocity of propagation.

(There is a large literature devoted to conditions which imply the essential self-adjointness of formally symmetric partial differential

operators. Some recent general references are the books by Schechter [11], Faris [4], and Reed and Simon [10, vol. II].)

2. **Dirac operators.** We work in the setting of our paper [2]. Let M be a complete Riemannian manifold and ξ a Hermitian vector bundle over M. Form the Hilbert space $H = L^2(\xi)$ of square integrable sections of ξ. Denote by $C_0^\infty(\xi)$ the C^∞ sections of ξ with compact support. We want to discuss perturbations of a Dirac-type operator $D: C_0^\infty(\xi) \to C_0^\infty(\xi)$, a first-order differential operator which we assume is symmetric. For $x \in M$ let $c(x)$ be the local propagation speed associated with D (for the precise definition see [2]; roughly, $c(x)$ is the largest eigenvalue of the symbol of D). Fix a point $0 \in M$ and let B_r be the ball in M with center 0 and radius r. Define

$$c(r) = \sup \{c(x): x \in B_r\} \ .$$

Assume that the integral $\int dr/c(r)$ diverges. (This is certainly true if $c(r)$ is constant, as it is for the usual Dirac operator in physics.) Then it was shown in [2, Theorems 1.3 and 2.2] that D, with domain $C_0^\infty(\xi)$, is essentially self-adjoint. Moreover, if $f \in C^\infty(\xi)$ is given, then the differential equation

$$(1) \qquad\qquad \frac{du}{dt} = iDu(t), \, u(0) = f$$

has a global solution (namely $u(t) = e^{it\bar{D}}f$) with $u(t) \in C_0^\infty(\xi)$ for all t. Moreover influence in (1) propagates no faster than the "speed of light." That is, if supp $f \subseteq B_r \subseteq B_R$, and $c = c(R)$, then supp $u(t) \subseteq B_{r+c(t)}$ so long as $r + c(t) \leqq R$.

We will deal with perturbations of D by zeroth-order operators V. Such an operator maps sections of ξ to sections of ξ and is of the form $(Vf)(x) = V(x) \cdot f(x)$ where $V(x) \in \mathrm{Hom}\,(\xi_x, \xi_x)$; in local coordinates, V is a matrix-valued function. We require that $V(x)$ be symmetric for all x, and that $x \mapsto V(x)$ be measurable. We say that V is locally pth power integrable, and write $V \in L_{loc}^p$, provided that $\|V(x)\|^p$ is locally integrable.

Now we introduce three classes of perturbing potentials V:

1. $\mathscr{V}(D)$ is the class of all "D-good" potentials $V: V \in L_{loc}^2$ and $D + V$ is essentially self-adjoint on $C_0^\infty(\xi)$. (*Note.* we require that $V \in L_{loc}^2$ in order that $C_0^\infty(\xi)$ be contained in the domain of V as an operator on $L^2(\xi)$.)

2. $\mathscr{V}_{loc}(D)$ is the class of all V such that for each $p \in M$, there exists a member $V_p \in \mathscr{V}(D)$ with $V(x) \equiv V_p(x)$ on some neighborhood of p.

3. $\mathscr{V}_{loc^*}(D)$ is the class of all V such that, for every compact

$K \subsetneq M$, there is a $V_K \in \mathscr{V}(D)$ with $V \equiv V_p$ on some neighborhood of K.

Obviously $\mathscr{V}(D) \subsetneq \mathscr{V}_{\mathrm{loc}*}(D) \subsetneq \mathscr{V}_{\mathrm{loc}}(D)$. Our main result says that in fact these classes coincide.

THEOREM 2.1. *With the above definitions,* $\mathscr{V}(D) = \mathscr{V}_{\mathrm{loc}}(D)$. *(That is, essential self-adjointness of* $D + V$ *depends only on the local behavior of* V.*)*

Proof. Let $V \in \mathscr{V}_{\mathrm{loc}}(D)$ be given. We write L for $D + V$ with domain $\mathscr{D}(L) = C_0^\infty(\xi)$, and $\bar{L}$ for the closure of L. Then $L \subsetneq \bar{L} \subsetneq L^*$; $\bar{L}$ is symmetric. We shall show that $\bar{L}$ is self-adjoint.

LEMMA 2.2. *If* $u \in \mathscr{D}(\bar{L})$ *and* $\phi \in C_0^\infty(M)$, *then* $\phi \cdot u \in \mathscr{D}(\bar{L})$; *similarly, if* $u \in \mathscr{D}(L^*)$ *then* $\phi \cdot u \in \mathscr{D}(L^*)$.

Proof. Suppose that $u \in \mathscr{D}(L^*)$. This means that $u \in L^2(\xi)$ and that as a distribution $Du + Vu = L^*u \in L^2(\xi)$. Now

$$(D + V)(\phi u) = D(\phi u) + \phi V u$$
$$= \phi Du + (D_1\phi) \cdot u + \phi V u \, ;$$

that is,

$$(2) \qquad (D + V)(\phi u) = \phi L^* u + (D_1\phi) \cdot u$$

where D_1 is a certain first-order differential operator whose coefficients are related to those of D. The right side of (2) is clearly a member of $L^2(\xi)$, and hence $\phi \cdot u \in \mathscr{D}(L^*)$.

If in addition $u \in \mathscr{D}(\bar{L})$, then there is a sequence $\{u_n\}_1^\infty$ in $\mathscr{D}(L)$ with $u_n \to u$ and $Lu_n \to \bar{L}u$. Then clearly $\phi \cdot u_n \in \mathscr{D}(L)$, $\phi \cdot u_n \to \phi \cdot u$, and

$$(3) \qquad L(\phi \cdot u_n) = \phi L(u_n) + (D_1\phi) \cdot u_n \, .$$

The right side of (3) converges to $\phi \bar{L}u + (D_1\phi) \cdot u = L^*(\phi u) \in L^2(\xi)$. Hence, by the definition of the closure of an operator, $\phi \cdot u \in \mathscr{D}(\bar{L})$ and we have the formula $\bar{L}(\phi u) = \phi \bar{L}u = (D_1\phi) \cdot u$.

Note that we used only the local square-integrability of V in the proof.

COROLLARY 2.3. *If* $u \in \mathscr{D}(\bar{L})$ *has compact support contained in an open set* Ω, *then there is a sequence* $\{u_n\}_1^\infty$ *in* $\mathscr{D}(L)$ *with* supp $u_n \subseteq \Omega$ *such that* $u_n \to u$ *and* $Lu_n \to \bar{L}u$.

Proof. Choose a C^∞ function ϕ with support in Ω and $\phi \equiv 1$ on a neighborhood of the support of u. As in the proof of Lemma 2.2,

take a sequence $\{v_n\}_1^\infty$ in $\mathscr{D}(L)$ with $v_n \to u$ and $Lv_n \to \bar{L}u$. Finally, let $u_n = \phi \cdot v_n$. Then $u_n \in \mathscr{D}(L)$, supp $u_n \subseteq \Omega$, and the proof of Lemma 2.2 shows that $u_n \to \phi \cdot u = u$ while $Lu_n \to \bar{L}(\phi \cdot u) = \bar{L}u$.

LEMMA 2.4. *Suppose* $W \in \mathscr{V}(D)$. *Define* $N = \overline{D+W}$. *Then if* $f \in \mathscr{D}(N)$ *and* f *has compact support, the differential equation*

$$(4) \qquad\qquad \frac{du}{dt} = iNu(t) , \quad u(0) = f$$

has a unique solution, namely $u(t) = e^{itN}f$. *Moreover for each* t, $u(t)$ *has compact support; if* supp $f \subseteq B_r \subseteq B_R$ *and* $c = c(R)$ *then* $u(t)$ *is supported in* $B_{r+c(t)}$ *for* $r + c(t) \leqq R$.

Proof. All the conclusions are immediate except for the assertion about the support of $u(t)$. For the latter we consider the Trotter product formula:

$$(5) \qquad\qquad u(t) = \lim_{n\to\infty} (e^{it\bar{D}/n} e^{itW/n})^n f .$$

Because f is supported in B_r and $e^{itW/n}$ is merely a multiplication operator, it follows that $e^{itW/n}f$ is supported in B_r. But then $e^{it\bar{D}/n} e^{itW/n}f$ is supported in $B_{r+c|t|/n}$. Iterating this process n times, we see that $(e^{it\bar{D}/n} e^{itW/n})^n f$ is supported in $B_{r+nc|t|/n} = B_{r+c|t|}$. Hence so is the limit vector $u(t)$.

We come next to the heart of argument: The proof of an analogue of Lemma 2.4 for the given potential $V \in \mathscr{V}_{\mathrm{loc}}(D)$.

LEMMA 2.5. *Fix* $V \in \mathscr{V}_{\mathrm{loc}}(D)$. *Let* f *be a compactly supported member of* $\mathscr{D}(\bar{L}) = \mathscr{D}(\overline{D+V})$. *Then the differential equation*

$$(6) \qquad\qquad \frac{du}{dt} = i\bar{L}u(t) , \quad u(0) = f$$

has a unique global solution $u(t)$, $-\infty < t < \infty$. *Moreover* $u(t)$ *is compactly supported for all* t.

Proof. The uniqueness follows in the usual way from the symmetry of $\bar{L}$.

For the existence proof, we suppose that f is supported in B_r. We shall show that there exists a solution of (6) for $0 \leqq t \leqq 1/c(r+2)$ with support in B_{r+1}. Proceeding inductively, we can then construct a solution for all $t \geqq 0$ because the series $1/c(r+2) + 1/c(r+3) + \cdots$ is divergent. Similarly we get a solution for all negative t.

We turn to the first step in the construction. Since B_{r+2} is com-

pact (by the completeness of M) and $V \in \mathcal{V}_{\text{loc}}(D)$ we may find a positive number 2ε with the following property: for each $p \in B_{r+2}$, if $\Omega_{p,2\varepsilon}$ is the 2ε-ball with center p, there is a $V_p \in \mathcal{V}(D)$ which coincides with V on $\Omega_{p,2\varepsilon}$. By compactness a finite number of the corresponding ε-balls $\{\Omega_{p,\varepsilon}\}_1^m$ cover B_{r+2}.

Let $\{\theta_p\}_1^m$ be a corresponding set of C_0^∞ functions with θ_p supported in $\Omega_{p,\varepsilon}$ and $\sum_{p=1}^m \theta_p \equiv 1$ on B_{r+1}.

Returning to the initial vector f, define $f_p = \theta_p \cdot f$ for $1 \leq p \leq m$. Note that f_p is supported in $\Omega_{p,\varepsilon}$ and that $f_p \in \mathcal{D}(\bar{L})$ by Lemma 2.2. Moreover, if we define $L_p = D + V_p$, we have the relations (in the sense of distributions)

$$\bar{L}f_p = Df_p + Vf_p = Df_p + V_p f_p = L_p^* f_p = \bar{L}_p f_p \, ,$$

where the last equality follows from the self-adjointness of $\bar{L}_p$. In particular, $f_p \in \mathcal{D}(\bar{L}_p)$.

Now let $\tau = \varepsilon/c(r+2)$. For $0 \leq t \leq \tau$, define

$$u(t) = \sum_{p=1}^m e^{itL_p} f_p = \sum_{p=1}^m u_p(t) \, .$$

Then $u(0) = \sum f_p = f$, $u(t)$ is differentiable, and we have the formula

$$(7) \qquad\qquad \frac{d}{dt} u(t) = \sum_{p=1}^m i\bar{L}_p u_p(t) \, .$$

Moreover, Lemma 2.4 implies that $u_p(t)$ is supported in $\Omega_{p,2\varepsilon}$ for $0 \leq t \leq \tau$. It then follows from Corollary 2.3 that $\bar{L}_p u_p(t) = \bar{L} u_p(t)$. Reason: because $u_p(t)$ is supported in $\Omega_{p,2\varepsilon}$, there exist C^∞ functions v_n supported in $\Omega_{p,2\varepsilon}$ with $v_n \to u_p(t)$ and $L_p v_n \to \bar{L}_p u_p(t)$. But since v_n is supported in $\Omega_{p,2\varepsilon}$ we have $L_p v_n = L v_n$. We can conclude that $u_p(t) \in \mathcal{D}(\bar{L})$ and that $\bar{L} u_p(t) = \bar{L}_p u_p(t)$.

Thus, from (7), we have, for $0 \leq t \leq \tau$,

$$\frac{d}{dt} u(t) = \sum_{p=1}^m i\bar{L} u_p(t) = i\bar{L} \sum_{p=1}^m u_p(t) = i\bar{L} u_p(t) \, .$$

That is, the differential equation (6) is satisfied, $0 \leq t \leq \tau$. Furthermore, $u(\tau) \in \mathcal{D}(\bar{L})$ and is obviously supported in $B_{r+c\tau}$, $c = c(r+2)$. This is so because, for all p, f_p is supported in B_r; hence by Lemma 2.4 (with $W = V_p$) $u_p(t)$ is supported in B_{r+ct}, $0 \leq t \leq \tau$. And so $u(t) = \sum u_p(t)$ is supported in B_{r+ct}.

Now we start the process over again, assuming that $r + c\tau < r + 1$, with $u(\tau)$ as the new initial data. We thus extend the solution $u(t)$ throughout the interval $\tau \leq t \leq 2\tau$. This may be continued for n steps, until $r + nc\tau \geq r + 1$ (the final step is modified by shortening

τ if necessary so as not to overshoot the mark). Of course the crucial fact that makes this argument work is that τ does not depend on the initial data.

Completion of the proof of Theorem 2.1. We can now deduce that $\bar{L}$ is self-adjoint by means of the "invariance of domain" method (cf. [2, Lemma 2.1]). Let $\mathscr{D}$ consist of the compactly supported members of $\mathscr{D}(\bar{L})$. We have just shown that for every $f \in \mathscr{D}$ there is a solution $u(t)$ of the differential equation (6) with $u(0) = f$ and $u(t) \in \mathscr{D}$ for all t. Moreover $||u(t)||^2$ is constant because $\bar{L}$ is symmetric.

Denote by A the restriction of $\bar{L}$ to $\mathscr{D}$. Suppose that $A^*\psi = \pm i\psi$. We claim that ψ must be 0. (This will establish that A is essentially self-adjoint and hence that $\bar{L}$ is self-adjoint.) Consider the case $A^*\psi = i\psi$. Given $f \in \mathscr{D}$ let $u(t)$ be the corresponding solution of (6). Define

$$F(t) = (u(t), \psi) .$$

Then $F(t)$ is bounded, $-\infty < t < \infty$, and we have

$$F'(t) = (i\bar{L}u(t), \psi) = (iu(t), A^*\psi) = (iu(t), i\psi) = F(t) .$$

Hence $F(t) = F(0)e^t$. Since F is bounded we must have $F(0) = 0$; that is, $(f, \psi) = 0$. Since $\mathscr{D}$ is dense, ψ must be 0. A similar argument takes care of the case: $A^*\psi = -i\psi$.

Since Theorem 2.1 shows that only the local behavior of V affects the essential self-adjointness of $D + V$, it follows immediately that if $V \in \mathscr{V}(D)$ and W is locally bounded, then $V + W \in \mathscr{V}(D)$. For explicit local conditions that guarantee that V belongs to $\mathscr{V}(D)$ see [5] or [6].

3. **Abstract wave equations.** In this section we collect some results on second-order operator differential equations which we will later apply to study Schrödinger operators by means of the allied wave equations.

We begin by recalling some standard facts. If $S \geq a > 0$ is a semibounded self-adjoint operator on a Hilbert space $\mathscr{H}$, we can solve the second-order equation $d^2u/dt^2 = -Su(t)$ by transforming it into a first-order system: $du/dt = v, dv/dt = -Su$. More precisely, we form the Hilbert space $H = \mathscr{D}(S^{1/2}) \oplus \mathscr{H}$, with norm $||(u, v)||^2 = ||S^{1/2}u||^2 + ||v||^2$, and the operator

$$(1) \qquad\qquad A = \begin{bmatrix} 0 & I \\ -S & 0 \end{bmatrix}$$

with domain $\mathscr{D}(A) = \mathscr{D}(S) \oplus \mathscr{D}(S^{1/2})$. Then A is a skew-adjoint operator on H and generates the one-parameter unitary group

$$(2) \qquad e^{tA} = \begin{bmatrix} \cos t\sqrt{S} & \dfrac{\sin t\sqrt{S}}{\sqrt{S}} \\ -\sqrt{S}\sin t\sqrt{S} & \cos t\sqrt{S} \end{bmatrix}$$

If $\psi(t) = e^{tA}\psi(0)$ has components $(u(t), v(t))$ then $u(t) = \cos t\sqrt{S}f + \sin t\sqrt{S}/\sqrt{S}g$, where $f = u(0)$, $g = v(0)$; if $f \in \mathscr{D}(S)$ and $g \in \mathscr{D}(S^{1/2})$ then $u(t)$ satisfies the second-order equation $\ddot{u}(t) = -Su(t)$ with initial data $u(0) = f$, $\dot{u}(0) = g$.

More generally, for any real number α, let $\mathscr{H}_\alpha$ be the completion of $\mathscr{H}$ in the norm $\|S^\alpha f\|$. Note that $\mathscr{H}_{1/2}$ is just $\mathscr{D}(S^{1/2})$ with the graph norm. We have the inclusions

$$\mathscr{H}_{1/2} \subseteqq \mathscr{H} \subseteqq \mathscr{H}_{-1/2},$$

and S extends by continuity to an isometry of $\mathscr{H}_{1/2}$ with $\mathscr{H}_{-1/2}$. Let K be the Hilbert space $\mathscr{H} \oplus \mathscr{H}_{-1/2}$. Then the matrix formula (1) defines a skew-adjoint operator A_1 on K, with $\mathscr{D}(A_1) = \mathscr{D}(S^{1/2}) \oplus \mathscr{H}$, and the group generated by A_1 is given by formula (2). This alternative conversion of the second-order equation into a first-order system will be useful later for technical reasons.

Now let C be another operator on $\mathscr{H}$ which is bounded relative to S with relative bound less than 1: that is, $\mathscr{D}(S) \subseteqq \mathscr{D}(C)$ and there are constants $\gamma < 1$ and $k < \infty$ with $\|Cf\| \leqq \gamma\|Sf\| + k\|f\|$ for all $f \in \mathscr{D}(S)$. Consider the differential equation

$$(3) \qquad \ddot{u}(t) = -Su(t) - Cu(t).$$

We write this as a first-order system on $H = \mathscr{D}(S^{1/2}) \oplus \mathscr{H}$ as follows:

$$\dot{u} = v$$
$$\dot{v} = -Su - Cu.$$

That is, if $\psi = (u, v)$,

$$\frac{d\psi}{dt} = A\psi + B\psi$$

where A is defined by (1) and $B = \begin{bmatrix} 0 & 0 \\ -C & 0 \end{bmatrix}$. Unfortunately B is a rather bad perturbation of A even though C is a good perturbation of S; for instance, B does not generate a semigroup on H. Nevertheless, the following result is true.

PROPOSITION 3.1. *Let S be a positive self-adjoint operator on*

$\mathscr{H}$. *Let C be a closed symmetric operator on $\mathscr{H}$ which is bounded relative to S with relative bound less than 1. Then the operator L defined by the relation*

$$(4) \qquad L = \begin{bmatrix} 0 & 0 \\ -S-C & I \end{bmatrix},$$

with domain $\mathscr{D}(L) = \mathscr{D}(S) \oplus \mathscr{D}(S^{1/2})$, generates a (C_0) group on $H = \mathscr{D}(S^{1/2}) \oplus \mathscr{H}$.

Proof. First, note that by the Kato-Rellich theorem the operator $T = S + C$ is self-adjoint with domain $\mathscr{D}(T) = \mathscr{D}(S)$. In addition, T is semibounded. Indeed, since C has relative bound $\leqq \gamma < 1$, there is a constant k such that

$$(5) \qquad \|Cf\| \leqq \gamma\|(S+k)f\|$$

for all $f \in \mathscr{D}(S)$. Hence, by the monotonicity of the square root on operators (cf. [7, Th. V. 4.12] we have

$$(6) \qquad (|C|f, f) \leqq \gamma((S+k)f, f)$$

for all $f \in \mathscr{D}(S)$. Hence we have the inequalities

$$(7) \qquad S + C \geqq S - |C| = (1-\gamma)S + \gamma S - |C| \geqq -\gamma k .$$

Also, the operator $\begin{bmatrix} 0 & 0 \\ \alpha & 0 \end{bmatrix}$, where α is any constant, is a bounded perturbation on H. Accordingly, by adding a suitable constant $\alpha \geqq 1 + \gamma k$ to C, we may arrange that $S + C$ be $\geqq 1$ without affecting the issue of whether L is a generator.

But then our preliminary remarks show that equation (3) $(\ddot{u} = -Tu)$ can be solved on the space $\mathscr{D}(T^{1/2}) \oplus \mathscr{H}$, and that L is a generator on this space with domain $\mathscr{D}(T) \oplus \mathscr{D}(T^{1/2})$. So all we have to do is prove that $\mathscr{D}(T^{1/2}) \oplus \mathscr{H}$ coincides with H—i.e., that $\mathscr{D}(T^{1/2}) = \mathscr{D}(S^{1/2})$. (The graph norms will then automatically be equivalent, by the closed graph theorem.)

Since $\mathscr{D}(S) = \mathscr{D}(T)$ and $T \geqq 1$, there is a constant δ such that, for all $f \in \mathscr{D}(T) = \mathscr{D}(S)$,

$$\|Sf\| \leqq \delta\|Tf\| .$$

Then by another application of [7, Th. V. 4.12] we have

$$(Sf, f) \leqq \delta(Tf, f)$$

for all such vectors f. Similarly there is a constant δ' such that

$$(Tf, f) \leqq \delta'(Sf, f) .$$

Hence the two inner products (Sf, g) and (Tf, g) are equivalent on $\mathscr{D}(S) = \mathscr{D}(T)$. But $\mathscr{D}(S^{1/2})$ is just the completion with respect to the first inner product, while $\mathscr{D}(T^{1/2})$ is the completion with respect to the second inner product. This proves that $\mathscr{D}(S^{1/2}) = \mathscr{D}(T^{1/2})$.

We also have a result about approximation of solutions to (3).

COROLLARY 3.2. *Let S and C be operators as in Proposition 3.1. Define $\chi_n(\lambda) = \lambda$ if $|\lambda| \leqq n$, $= 0$ otherwise; and consider the "cutoff" perturbations $C_n = \chi_n(C)$.*

On $\boldsymbol{H} = \mathscr{D}(S^{1/2}) \oplus \mathscr{H}$, let e^{tL_n} be the (C_0) group generated by the operator

$$L_n = \begin{bmatrix} 0 & 1 \\ -S - C_n & 0 \end{bmatrix},$$

and let L be as in equation (4). Then as $n \to \infty$

$$e^{tL_n} \longrightarrow e^{tL}$$

in the strong operator topology on $\boldsymbol{H}$.

Proof. Since $\|C_n f\| \leqq \|Cf\|$ for all $f \in \mathscr{D}(S)$, the estimates (5), (6), and (7) of Proposition 3.1 all hold with C_n replacing C. Thus if we add the constant $\alpha = \gamma k + 1$ to all C_n's and C, we may assume without loss of generality that $S + C \geqq 1$ and $S + C_n \geqq 1$ for all n.

Write $T = S + C$ and $T_n = S + C_n$. Then $L = \begin{bmatrix} 0 & I \\ -T & 0 \end{bmatrix}$, and $L_n = \begin{bmatrix} 0 & I \\ T_n & 0 \end{bmatrix}$. By Proposition 3.1, we know that $\mathscr{D}(L_n) = \mathscr{D}(L) = \mathscr{D}(S) \oplus \mathscr{D}(S^{1/2})$. Moreover it is trivial that $L_n \psi \to L \psi$ for any $\psi \in \mathscr{D}(L)$. Hence the Trotter-Kato theorem [7, Th. IX. 2.16] will show that $e^{tL_n} \to e^{tL}$ provided that we verify that these groups are uniformly bounded on $\boldsymbol{H}$ with a bound independent of n.

We know that e^{tL_n} is a group of isometries relative to the norm induced from $\mathscr{D}(T_n^{1/2}) \oplus \mathscr{H}$. We denote this norm by $\|\cdot\|_n$, while we denote the norm from $\mathscr{D}(S^{1/2}) \oplus \mathscr{H}$ by $\|\cdot\|$. The proof will be finished if we can show that there are constants α, β in $(0, \infty)$ such that

(8) $$\alpha\|\cdot\| \leqq \|\cdot\|_n \leqq \beta\|\cdot\| ;$$

for then we will have the estimate $\|e^{tL_n}\| \leqq \beta/\alpha$ for all real t and all $n = 1, 2, \cdots$.

Because of our normalization, it follows from (7), with C replaced by C_n, that

$$T_n = S + C_n \geqq (1 - \gamma)S + 1 .$$

Similarly, if λ is large enough, we have

$$C_n \leqq \lambda S$$

for all n.   Hence there are constants α, β independent of n such that

(9) $\alpha^2 S \leqq T_n \leqq \beta^2 S$.

Now (8) is an immediate consequence of (9).

We turn next to some simple facts about time-dependent equations.

PROPOSITION 3.3.   *Let e^{tA} be a (C_0) group on the Hilbert space H, and let $B(t)$ be a norm-continuous function from the reals to the bounded operators on H.   Assume also that, for each t, the operator $B(t)$ maps $\mathscr{D}(A)$ into itself, and that $t \mapsto B(t)$ is continuous as a map from the reals into the bounded operators on $\mathscr{D}(A)$ (equipped with the graph norm).*
Then, for each $f \in \mathscr{D}(A)$, the differential equation

(10) $\dfrac{d\psi}{dt} = A\psi(t) + B(t)\psi(t), \ \psi(0) = f$

has a unique solution $\psi(t)$.

Proof.   If $\psi(t)$ satisfies (10) then it satisfies the integral equation

(11) $\psi(t) = e^{tA}f + \displaystyle\int_0^t e^{(t-\tau)A} B(\tau)\psi(\tau)d\tau$.

Uniqueness of solutions to (11) follows in the usual way (Gronwall's inequality).   As for *existence*:   The additional hypothesis on $B(t)$ shows that (11) makes sense as an integral equation on $\mathscr{D}(A)$, and the usual iteration method shows that it has a solution $\psi(t) \in \mathscr{D}(A)$.

Since $\psi(t)$ satisfies (11) and $\psi(\tau) \in \mathscr{D}(A)$, we have $B(\tau)\psi(\tau) \in \mathscr{D}(A)$ and so we may differentiate under the integral sign in (11) to deduce that $\psi(t)$ is a solution of the original differential equation (10).

The next proposition deals with second-order equations.

PROPOSITION 3.4.   *Suppose now that S is a semibounded self-adjoint operator on a Hilbert space $\mathscr{H}$, and that $\beta(t)$ is a differentiable real-valued function on the real line.   Consider the time-dependent wave equation*

(12) $\begin{aligned} d^2u/dt^2 &= -Su(t) - \beta(t)u(t) \\ u(0) &= f \in \mathscr{D}(S), \ \dot{u}(0) = 0 \end{aligned}$.

This equation has a global solution $u(t)$. Moreover we have the "energy" formula

$$(13) \qquad \frac{d}{dt}\{\|\dot{u}(t)\|^2 + ((S + \beta(t))u(t), u(t))\} = \beta'(t)\|u(t)\|^2 .$$

Proof. By adding a constant to S and subtracting it from $\beta(t)$, we may assume that $S \geqq 1$. Then we know that the operator $A = \begin{bmatrix} 0 & 1 \\ -S & 0 \end{bmatrix}$ is skew-adjonit on the Hilbert space $\boldsymbol{H} = \mathscr{D}(S^{1/2}) \oplus \mathscr{H}$, with domain $\mathscr{D}(A) = \mathscr{D}(S) \oplus \mathscr{D}(S^{1/2})$. A generates a (C_0) group of operators e^{tA} on $\boldsymbol{H}$. Next, consider the operator $B(t)$ on $\boldsymbol{H}$ defined by

$$B(t) = \begin{bmatrix} 0 & 0 \\ -\beta(t) & 0 \end{bmatrix}.$$

Obviously $t \mapsto B(t)$ is continuous in operator norm. Moreover, it is straightforward to verify that $B(t)$ maps $\mathscr{D}(A)$ into itself and $t \to B(t)$ is norm-continuous on the space $\mathscr{D}(A)$.

By Proposition 3.3 it follows that the equation

$$(14) \qquad \frac{d\psi}{dt} = A\psi + B(t)\psi , \quad \psi(0) = \begin{bmatrix} f \\ 0 \end{bmatrix}$$

has a unique solution. If we write $\psi(t) = \begin{bmatrix} u(t) \\ \dot{u}(t) \end{bmatrix}$ then $u(t)$ satisfies the wave equation (10).

Moreover, we have the relation

$$\|\dot{u}(t)\|^2 + (Su(t), u(t)) = \|\psi(t)\|^2$$

by the definition of the norm in $\boldsymbol{H}$. Hence

$$\frac{d}{dt}\{\|\dot{u}(t)\|^2 + ((S + \beta(t))u(t), u(t))\}$$

$$(15) \qquad = \frac{d}{dt}\{\|\psi(t)\|^2 + \beta(t)\|u(t)\|^2\}$$

$$= 2Re(\psi, \dot{\psi}) + 2\beta(t)Re(u, \dot{u}) + \beta'(t)\|u(t)\|^2 .$$

From the differential equation (14), the formula $\psi = \begin{bmatrix} u \\ \dot{u} \end{bmatrix}$, and the definition of the inner product in $\boldsymbol{H}$, we get the relation

$$(\psi, \dot{\psi}) = (S^{1/2}u, S^{1/2}\dot{u}) + (\dot{u}, -Su - \beta(t)u)$$

$$= (S^{1/2}u, S^{1/2}\dot{u}) - (S^{1/2}\dot{u}, S^{1/2}u) - \beta(t)(\dot{u}, u) .$$

Here we have used the fact that $\dot{u}(t) \in \mathscr{D}(S^{1/2})$. It follows that $Re(\psi, \dot{\psi}) = -\beta(t)Re(\dot{u}, u)$, so that the right-hand side of (15) reduces

to $\beta'(t)\|u(t)\|^2$. That is, equation (13) holds.

Finally, we have a strong uniqueness theorem.

PROPOSITION 3.5. *Let T be a semibounded symmetric operator on $\mathscr{H}$. Let $\beta(t)$ be a continuous real-valued function on the real line. Suppose that $u(t)$, $0 \leq t \leq t_0$, is a solution of the differential equation*

(16)
$$d^2u/dt^2 = -Tu(f) - \beta(t)u(t)$$
$$u(0) = \dot{u}(0) = 0 .$$

Then $u(t)$ is identically 0.

Proof. As in Proposition 3.4, we may assume without loss of generality that $T \geq 1$. Let S be the Friedrichs extension of T. Form the Hilbert space $K = \mathscr{H} \oplus \mathscr{H}_{-1/2}$ as discussed in the third paragraph of this section, and consider on K the operator $A_1 = \begin{bmatrix} 0 & I \\ -S & 0 \end{bmatrix}$ and the corresponding one-parameter group e^{tA_1}. Also define an operator $B_1(t)$ on K by the formula

$$B_1(t) = \begin{bmatrix} 0 & 0 \\ -\beta(t) & 0 \end{bmatrix}.$$

Then $t \mapsto B_1(t)$ is a norm-continuous operator-valued function on K.

Consider the curve $\psi(t) = \begin{bmatrix} u(t) \\ \dot{u}(t) \end{bmatrix}$. It is easy to see that $\psi(t)$ is a differentiable curve in K which satisfies the differential equation

$$\dot{\psi}(t) = A_1\psi(t) + B(t)\psi(t) , \quad \psi(0) = 0 .$$

The usual uniqueness proof (reduction to an integral equation plus Gronwall's inequality) then shows that $\psi(t)$ is identically 0.

Note. In Proposition 3.5 it was necessary to introduce the space K because it is not evident *a priori* that $\psi(t)$ is differentiable as a curve in the space $H = (S^{1/2}) \oplus \mathscr{H}$.

4. **Schrödinger operators.** We begin by recalling some of the results of [2] concerning second-order operators with smooth coefficients. Suppose that M is a complete Riemannian manifold and that Λ is a symmetric, negative, second-order elliptic operator on M. Let $c(x)$ denote the local propagation speed associated with Λ; roughly, $c(x)$ is the square root of the largest eigenvalue of the symbol of Λ. For example, if $\Lambda = \Delta$, the Laplace-Beltrami operator, then $c(x)$ is identically

1. Fix a point $0 \in M$, and let B_r denote the ball of radius r and center 0. Note that $\bar{B}_r$ is compact because M is complete. Define $c(r) = \sup\{c(x): x \in B_r\}$. As with the Dirac-type operators discussed in §2, we assume that the integral $\int^{\infty} dr/c(r)$ diverges. Then it was shown in [2] that Λ, with domain $C_0^{\infty}(M)$, is essentially self-adjoint. Moreover—the point of greatest interest here—the wave equation $\ddot{u}(t) = \Lambda u(t)$ has the following propagation property: if the initial data $u(0)$, $\dot{u}(0)$ have compact support then so does $u(t)$ for all t.

We shall state a fairly precise form of this result, which says that data supported in B_r propagates with instantaneous velocity at most $c(r)$. For $r \geqq 0$ we define $\theta(r) = \int_0^r ds/c(s)$. The assumption about the divergence of the integral to ∞ shows that θ is a diffeomorphism of $[0, \infty)$ with itself. Moreover, if $R < r$ then $\theta(r) - \theta(R) = t$ gives a lower bound for the time during which the influence of data supported in B_R remains inside B_r. Hence, given $t > 0$ and R, if $u(0)$ and $\dot{u}(0)$ are supported in B_R then $u(t)$ will be supported in B_r, where $r = \theta^{-1}(t + \theta(R))$. Let us write $t*R$ for $\theta^{-1}(t + \theta(R))$. Thus data supported in B_R generates a solution supported at time t within B_{t*R}. (Note the identity $t_1*(t_2*R) = (t_1 + t_2)*R$, whose interpretation in terms of support propagation is obvious.)

We are concerned with the Schrödinger equation $1/i(\partial u/\partial t) = -\Lambda u + Vu$, where V is a potential on M which may be rather singular. We assume henceforth that in local coordinates Λ has the form $\Lambda u = \sum_{ij} \partial_i(a_{ij}\partial_j u)$, i.e., $\Lambda u = \operatorname{div}(a \operatorname{grad} u)$, where a is a smooth matrix-valued function. (This means that we are neglecting "magnetic fields" which would appear as lower-order terms in Λ. Actually, at the cost of some technical complication, our arguments could cope with arbitrary *smooth* magnetic fields, but we shall stick to the simpler case in writing out the proofs.) We seek conditions on V which guarantee that the operator $-\Lambda + V$ is essentially self-adjoint on the domain $C_0^{\infty}(M)$ We shall use the strategy of [2]; namely, we shall relate the self-adjointness question for the operator $T = -\Lambda + V$ to that of existence of solutions to the wave equation $\ddot{u}(t) = -Tu(t)$. (A related approach has been explored by Berezanskii [1], who relies upon a uniqueness theorem for the adjoint wave equation $\ddot{u}(t) = -T^*u(t)$ rather than an existence theorem for the original equation. Our method seems to yield stronger results.) We will impose local conditions on the potential V to guarantee that $-\Lambda + V$ is "locally" essentially self-adjoint, together with a global semiboundedness condition, of the form $-\Lambda + V \geqq -(a + b\theta(r)^2)$, which is the analogue of a classical condition that insures that particle trajectories do not reach infinity in a finite time. Here r is the function $r(x) = d(x, 0)$, the distance to the "origin" $0 \in M$. Note that in case $\Lambda = \Delta$ we have

$c(r) = 1$ so that $\theta(r) = r$ and our condition reduces to the well known condition $-\varDelta + V \geqq -(a + br^2)$.

We turn first to the local problem. There are a number of techniques in the literature which we might use. We shall employ the following three known theorems, which we list for the reader's convenience.

THEOREM A ((Faris) [3, Thm. 4.1], [4, Thm. 8.3]). *Let $\mathscr{H} = L^2(m, \mu)$. Let $e \in \mathscr{H}, e > 0$ a.e. Let $H_0 \geqq 0$ be a self-adjoint operator acting in $\mathscr{H}$ with $H_0 e = 0$ Assume that $(H_0 + c)^{-1}$ is positivity-preserving for all $c > 0$. Let $U \geqq 0$ be a real function in $L^2(M, e^2\mu)$.*

Let $H = H_0 + U$. Then H is essentially self-adjoint on $\mathscr{D}(H) = \mathscr{D}(H_0) \cap \mathscr{D}(U)$. Moreover $(H + c)^{-1}L_\infty(e)$ is a domain of essential self-adjointness (a "core") for H.

THEOREM B ((Davies-Faris) [3, Thm. 4.4], [10, Thm. X. 31]). *Let H_0 be a positive self-adjoint operator on $L^2(M, \mu)$ such that $(H_0 + c)^{-1}$ is positivity-preserving. Let $U \geqq 0$ be a multiplication operator. Suppose that $H = H_0 + U$ is essentially self-adjoint. Let W be a multiplication operator which is bounded relative to H_0. Then W is also H-bounded. Indeed, if we have*

$$\| W \psi \| \leqq a \| (H_0 + b)\psi \| , \quad \psi \in \mathscr{D}(H_0)$$

then we have the estimate

$$\| W \psi \| \leqq a \| (H + b)\psi \| , \quad \psi \in \mathscr{D}(H)$$

with the same constants a, b.

THEOREM C. *Let M be compact Riemannian manifold. Let $\varLambda$ be a second-order elliptic operator on M. Let d be the dimension of M. Following [12], we say that p is "d-canonical" if $p = 2$ for $d \leqq 3$; $p > 2$ if $d = 4$; $p = d/2$ if $d \geqq 5$.*

Suppose p is d-canonical and $W \in L^p(M)$. Then the multiplication operator W is bounded relative to $\varLambda$ with relative bound 0.

Proof. In case $\varLambda = \varDelta$ on $\boldsymbol{R}^d$, this follows from Sobolev inequalities (cf. [10, Thms. X. 16, X. 30, X. 21]; the case $p = 2$, $d = 3$ is a famous result of Kato).

In the manifold case the conclusion follows from the "flat" case by a straightforward argument using a partition of unity to patch together inequalities in local charts.

PROPOSITION 4.1. *Let M be a compact Riemannian manifold,*

dimension d. Let Λ be a second-order elliptic operator on M: $\Lambda u =$ div (a grad u), a smooth. Let $U \geqq 0$ belong to $L^2(M)$, and let $W \geqq 0$ belong to $L^p(M)$, where p is d-canonical. Write $V = U - W$. Let $H = -\Lambda + V$. Then H is semibounded and essentially self-adjoint on $C^\infty(M)$.

Proof. We shall apply Theorem A, with $H_0 = -\bar{\Lambda}$. The first step is to check that if $c > 0$ then $(-\bar{\Lambda} + c)^{-1}$ is positivity-preserving. This follows from the classical maximum principle. (Indeed, suppose that $\phi \in C^\infty(M)$, $\phi \geqq 0$, and $(-\bar{\Lambda} + c)\psi = \phi$. We must show $\psi \geqq 0$. By elliptic regularity, ψ is C^∞. If ψ is negative anywhere then ψ attains a negative minimum at some point $p \in M$. Then $\psi(p) < 0$ but $\Lambda\psi(p) \geqq 0$, so $\phi(p) = -\Lambda\psi(p) + c\psi(p) < 0$, a contradiction. Thus $(-\bar{\Lambda} + c)^{-1}$ is positivity-preserving on the positive cone in $C^\infty(M)$, which is dense in the positive cone of $L^2(M)$.)

Now define $e(x) \equiv 1$. Then $e > 0$ everywhere, $e \in L^2(M)$, and $\Lambda e = 0$. Moreover $U \in L^2(M, e^2\mu) = L^2(M)$. Hence, by Theorem A, it follows that $-\bar{\Lambda} + U = H_0 + U$ is essentially self-adjoint on $\mathscr{D}(\bar{\Lambda}) \cap \mathscr{D}(U)$.

By Theorem C, the multiplication operator W is bounded relative to H_0 with relative bound 0. Hence, by Theorem B, W is bounded relative to $H_1 = H_0 + U$ with relative bound 0. It follows that $H_1 - W = H_0 + U - W$ is semibounded and essentially self-adjoint on $\mathscr{D}(H_1) \cap \mathscr{D}(W)$. Moreover any core for H_1 is a core for $H_1 - W$.

The proof will be finished if we show that $C^\infty(M)$ is a core for H_1. Now, by Theorem A, $(H_1 + c)^{-1}L^\infty(M)$ is a core for H_1 Hence the larger space $\mathscr{M} = L^\infty(M) \cap \mathscr{D}(-\bar{\Lambda})$ is a core for H_1. To prove that $C^\infty(M)$ is a core for H_1, we have to show that a given $f \in \mathscr{M}$ may be approximated by a sequence $f_n \in C^\infty(M)$ so that $H_1 f_n \rightarrow H_1 f$, at least weakly. In fact we shall construct such a sequence with $\|f_n\|_\infty \leqq \|f\|_\infty$ and $\Lambda f_n \rightarrow \bar{\Lambda}f$, $Uf_n \overset{w}{\rightarrow} Uf$.

We simply take $f_n = e^{\bar{\Lambda}/n}f$. Then $\|f_n\|_\infty \leqq \|f\|_\infty$ by the maximum principle; $f_n \in C^\infty(M)$ by the smoothing properties of the heat equation; $f_n \rightarrow f$ in L^2; and $\Lambda f_n \rightarrow \bar{\Lambda}f$ in L^2. As for Uf_n, note that $\|Uf_n\|_2 \leqq \|U\|_2\|f_n\|_\infty \leqq \|U\|_2\|f\|_\infty$, so $\{Uf_n\}_1^\infty$ is a bounded sequence in L^2. Moreover it is obvious that $Uf_n \rightarrow Uf$ in measure, so Uf is the unique adherent point in L^2 of the sequence $\{Uf_n\}$. Accordingly $Uf_n \overset{w}{\rightarrow} Uf$.

COROLLARY 4.2. *Let M be a compact Riemannian manifold. Let Λ and V be as in Proposition 4.1. Write $H = -\overline{\Lambda + V}$. Suppose that $f \in \mathscr{D}(H)$ has support in the ball B_r. If $s > r$ there is a sequence of C^∞ functions f_n with supports in B_s such that $f_n \rightarrow f$ and $(-\Lambda + V)f_n \rightarrow Hf$ in L^2.*

Proof. By Proposition 4.1, $C^\infty(M)$ is a core for H, as well as for $H_1 = [-\varDelta + U]^-$; moreover, since W is H_1-bounded, $\mathscr{D}(H_1) = \mathscr{D}(H)$. Accordingly, there is a sequence of C^∞ functions $\{g_n\}_1^\infty$ such that $g_n \to f$, $(-\varDelta + U)g_n \to H_1 f$, and $Wg_n \to Wf$; in particular $(-\varDelta + V)g_n \to Hf$.

Choose a C^∞ function ϕ such that ϕ is supported in the ball B_s and ϕ is identically 1 in a neighborhood of the support of f: thus $\phi f = f$ and $\phi H f = H f$. Define $f_n = \phi g_n$. Clearly f_n is C^∞, supported in B_s, and $f_n \to f$. It remains to show that $(-\varDelta + V)f_n \to Hf$. (Actually we will show only weak convergence; but then suitable convex combinations of the Hf_n will converge strongly.)

We have the formula

$$(1) \qquad (-\varDelta + V)f_n = \phi(-\varDelta + V)g_n - g_n \varDelta \phi - 2(a\nabla\phi)\cdot\nabla g_n .$$

The first two terms on the right converge in norm to $\phi H f = H f$ and $-f\varDelta\phi = 0$ respectively. We must show that the remaining term converges to 0 at least weakly. It obviously converges to 0 in the sense of distributions, so it is enough to show that the sequence of L^2 norms $\|(a\nabla\phi)\cdot\nabla g_n\|$ is uniformly bounded. This amounts to showing that the sequence $\|\nabla g_n\|$ is bounded.

Now, because of the inequality $-\varDelta \leqq \text{const.} -\varDelta$, we have

$$
\begin{aligned}
(2) \qquad \|\nabla g_n\|^2 &= (-\varDelta g_n, g_n) \leqq c(-\varDelta g_n, g_n) \\
&\leqq c((-\varDelta + U)g_n, g_n) .
\end{aligned}
$$

The right side of (2) converges to $c(H_1 f, f)$. Hence the sequence $\|\nabla g_n\|$ is bounded.

We now consider the wave equation

$$(3) \qquad \frac{d^2 u}{dt^2} = (\varDelta - V - \beta(t))u(t) .$$

Our aim is to show that under suitable hypotheses on V, equation (3) has global solutions. We first turn to the case of compact manifolds M, where existence of solutions is immediate from our earlier work on essential self-adjointness of $-\varDelta + V$. Here our intent is to establish support properties of solutions to (3). We treat the time-independent case first.

PROPOSITION 4.3. *Let M be compact Riemannian manifold. Let $\varDelta$ and V be as in Proposition 4.1. Write H for $\overline{-\varDelta + V}$. Consider the differential equation*

$$(4) \qquad d^2u/dt^2 = -Hu(t) , \quad u(0), \dot{u}(0) \in C^\infty(M) .$$

By the results of §3 we know that (4) has a global solution u(t).

Claim: This solution has the same support properties as the "free" equation (V = 0). That is, if the initial data are supported in the ball B_r, then, for all $t \geq 0$, $u(t)$ is supported in the ball B_{t+r}.

Proof. Basically this follows from the Trotter product formula as in the analogous case of Dirac operators in §2. First, suppose that V is bounded. Without loss of generality we may assume $-\Lambda \geq 1$. Consider, on the space $\boldsymbol{H} = \mathscr{D}(\sqrt{-\Lambda}) \oplus \mathscr{H}$, the operators

$$A = \begin{bmatrix} 0 & I \\ \Lambda & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ -V & 0 \end{bmatrix}.$$

Note that B is a bounded operator on $\boldsymbol{H}$. If $\psi(t) = \begin{bmatrix} u(t) \\ \dot{u}(t) \end{bmatrix}$ where u is the unique solution to (4) we have $\psi(t) = e^{t(A+B)}\psi(0)$. Moreover, by the Trotter product formula,

$$(5) \qquad \psi(t) = \lim_{n \to \infty} (e^{t/nA}e^{t/nB})^n \psi(0).$$

Now $e^{t/nB}$ does not expand support, while if ϕ is supported in B_r, $e^{t/nA}\phi$ is supported in $B_{t/n*R}$. Hence the right side of (5) is supported in $B_{t/n*\cdots*t/n*r} = B_{t+r}$. The same is true of the limit $\psi(t)$.

Now consider the general case: $V = U - W$, v, $W \geq 0$, $U \in L^2(M)$, $W \in L^p(M)$. Then $H = \overline{-\Lambda + V}$ is semibounded, and the solution to (4) is given by

$$u(t) = \cos(t\sqrt{H})u(0) + \frac{\sin(t\sqrt{H})}{\sqrt{H}}\dot{u}(0).$$

Consider the cut-off potentials: $V_n(x) = V(x)$ if $|V(x)| \leq n$, $V_n(x) = 0$ otherwise. Let $H_n = \overline{-\Lambda + V_n}$. The operators H_n are uniformly bounded below, and $H_n \to H$ pointwise on $\mathscr{D}(\Lambda) \cap \mathscr{D}(V)$, a core for H. Hence by [10, Th. VIII. 20] it follows that $\cos t\sqrt{H_n} \to \cos t\sqrt{H}$ and

$$\sin(t\sqrt{H_n})/\sqrt{H_n} \longrightarrow \sin(t\sqrt{H})/\sqrt{H}$$

in the strong operator topology. That is, if $u_n(t)$ is the solution to (4) with H replaced by H_n, then $u_n(t) \to u(t)$ for all t. By the first part of the proof we know that $u_n(t)$ is supported in B_{t+r}. Hence $u(t)$ is supported in B_{t+r}.

Next we establish the same result for the time-dependent case.

PROPOSITION 4.4. *Let M be a compact Riemannian manifold. Let Λ and V be as in Proposition 4.1. Let $\beta(t)$ be a differentiable*

real function of t. Write $H = -\overline{A + V}$. Then the equation

(6) $d^2u/dt^2 = -Hu(t) - \beta(t)u(t)$, $u(0), \dot{u}(0) \in C^\infty(M)$

has a unique global solution $u(t)$. This solution satisfies the energy equation

(7) $d/dt\{\| \dot{u}(t) \|^2 + ((H + \beta(t))u(t), u(t))\} = \beta'(t) \| u(t) \|^2$.

Moreover if the initial data are supported in the ball B_r, then, for $t \geqq 0$, $u(t)$ is supported in B_{t*r}.

Proof. Existence and uniqueness of the solution $u(t)$, as well as the energy equation (7), all follow from the results of §3. We must establish the support properties of $u(t)$.

Without loss of generality we may assume $H \geq 1$. Then consider, on the space $\boldsymbol{H} = \mathscr{D}(H^{1/2}) \oplus \mathscr{H}$, the operators

$$A = \begin{bmatrix} 0 & I \\ -H & 0 \end{bmatrix} \quad \text{and} \quad B(t) = \begin{bmatrix} 0 & 0 \\ -\beta(t) & 0 \end{bmatrix} .$$

If $\psi(t) = \begin{bmatrix} u(t) \\ \dot{u}(t) \end{bmatrix}$ we know that ψ satisfies the integral equation

(8) $\psi(t) = e^{tA}\psi(0) + \displaystyle\int_0^t e^{(t-\tau)A}B(\tau)\psi(\tau)d\tau$.

Moreover, the series expansion from iterating (8) converges because $\| B(\tau) \|$ is bounded on any bounded interval, so that we have the formula

(9) $\psi(t) = e^{tA}\psi(0) + \displaystyle\sum_{n=1}^\infty \int_{\varDelta_m} e^{(t-\tau_1)A}B(\tau_1)e^{(\tau_1-\tau_2)A} \cdots e^{\tau_n A}\psi(0)d\tau_1 \cdots d\tau_n$

where $\varDelta_n$ is the region $0 \leqq \tau_n \leqq \tau_{n-1} \leqq \cdots \leqq \tau_1 \leqq t$. It is straightforward to check that each term in this series is supported in B_{t*r} if $\psi(0)$ is supported in B_r. This follows from the fact that the operators $B(\tau)$ do not expand support, while, by Proposition 4.3, the operators $e^{(t-\tau)A}$ expand support from B_r to $B_{(t-\tau)*r}$.

We can now extend the above results to the case of a complete (not necessarily compact) Riemannian manifold M.

THEOREM 4.5. *Let M be a complete Riemannian manifold, dimension d. Let A be a second-order elliptic differential operator on M of the form $Au = \nabla \cdot (a\nabla u)$. Define the propagation speed $c(r)$ associated with M as indicated above, and assume that the integral $\displaystyle\int^\infty dr/c(r)$ diverges.*

Let V be a real function on M of the form $V = U - W$ where U, $W \geqq 0$, $U \in L^2_{loc}(M)$, $V \in L^p_{loc}(M)$, and p is M-canonical.

Let $H = \overline{-\varLambda + V}$.

Let $\beta(t)$ be a real differentiable function of t.

Then the differential equation

(10) $d^2u/dt^2 = -Hu(t) - \beta(t)u(t)$, $u(0), \dot{u}(0) \in C^\infty_0(M)$

has a global solution $u(t)$. This solution satisfies the energy equation (7) above. Moreover, $u(t)$ is compactly supported for all t; if the initial data are supported in the ball B_r, then $u(t)$ is supported in B_{t+r}.

Finally, let H_0 denote the closure of the restriction of H to $C^\infty_0(M)$. Then, for all t, the solution $u(t)$ belongs to $\mathscr{D}(H_0)$ (and of course $H_0 u(t) = Hu(t)$).

Proof. The idea of the proof is to use the preceding theorems to construct local solutions of (10) on an increasing sequence of compact subsets of M. The finite propagation speed guarantees that these fit together to define a global solution. The details follow.

Suppose that $u(0), \dot{u}(0)$ are C^∞ functions supported in the ball B_r. For each positive integer n let $\varOmega_n$ be a compact subset of M with smooth boundary which contains $\bar{B}_{n+r}$ in its interior. Finally, let M_n be a compact Riemannian manifold (without boundary) in which $\varOmega_n$ is isometrically embedded. (For example, let M_n be the double covering of a neighborhood of $\varOmega_n$; it is easy to see via a partition of unity argument that the Riemannian metric on $\varOmega_n$ extends to a Riemannian metric on M_n. The precise choice of M_n is immaterial.)

We extend the operator $\varLambda$ on $\varOmega_n$ to be elliptic on M_n; call the extension $\varLambda_n$. The functions U, W, and V may be extended to M_n by defining them to be 0 outside $\varOmega_n$. Since $\varOmega_n$ has compact closure, we have $U_n \in L^2(M_n)$, $W_n \in L^p(M_n)$. Finally the functions $u(0)$, $\dot{u}(0)$ are extended by setting them equal to 0 outside $\varOmega_n$.

We define $H_n = \overline{-\varLambda_n + V_n}$. It follows from Proposition 4.4 that the wave equation (10), with H_n replacing H, has a unique global solution $u_n(t)$ in $L^2(M_n)$, and the energy equation (7) is satisfied. Moreover, for $|t| < n$, $u_n(t)$ is supported in the ball B_{n+r}, so for this range of t we identify $u_n(t)$ with a function in $L^2(M)$. It follows from Corollary 4.2 that, for $|t| < n$, $u_n(t) \in \mathscr{D}(H_0)$, and $H_0 u_n(t) = H_n u_n(t)$.

Now suppose $m < n$, and $|t| < m$. From the uniqueness of solutions to the wave equation on M_m it is clear that $u_m(t) \equiv u_n(t)$, $|t| < m$. Hence the functions $\{u_n\}^\infty_{n=1}$ fit together to define a global function

$u(t) \in L^2(M)$ which has all the asserted properties.

We now come to the main theorem.

THEOREM 4.6. *Let M be a complete Riemannian manifold. Let Λ be a second-order elliptic differential operator as in Theorem 4.5. We assume the integral $\int_0^\infty dr/c(r)$ diverges. Then define $r: M \to \mathbf{R}^+$ by $r(x) = d(x, 0)$, where 0 is the fixed origin in M, and set $\theta(r) = \theta(r(x)) = \int_0^{r(x)} ds/c(s)$. Suppose that $V = U - W$ as in Theorem 4.5. Define $H_1 = -\Lambda + V$ on the domain $C_0^\infty(M)$. Assume that for some constants a, b we have the inequality*

$$(11) \qquad\qquad H_1 = -\Lambda + V \geqq -(a + b\theta(r)^2) \,.$$

Then H_1 is essentially self-adjoint.

Proof. We shall employ a simplified version of an argument of Kato [9].

First note that by using a change of scale if necessary, we can assume that the constant b in (11) equals 1. Also, the constant a can be absorbed into the potential V, so we may assume that $a = 0$.

We have to show that, for some positive λ, the equations $H_0^* \psi = \pm i\lambda \psi$ have no nonzero solutions. It turns out that $\lambda = 18$ is a convenient value. So suppose $H_0^* \psi = -i \cdot 18 \psi$.

Given $f \in C_0^\infty(M)$, we know by Theorem 4.5 with $\beta(t) = 4t^2$ that there is a global solution $u(t)$ of the equation

$$(12) \qquad d^2u/dt^2 = -(H_0 + 4t^2)u(t) \,, \quad u(0) = f \,, \quad \dot{u}(0) = 0 \,.$$

Moreover, this solution obeys the energy equation

$$(13) \qquad \frac{d}{dt}\{\| \dot{u}(t) \|^2 + ((H_0 + 4t^2)u(t), u(t))\} = 8t \| u(t) \|^2 \,.$$

Hence we have, for $t > r$,

$$(14) \qquad \| \dot{u}(t) \|^2 + ((H_0 + 4t^2)u(t), u(t)) = C_r + \int_r^t 8s \| u(s) \|^2 ds$$

where C_r is the value of the left side when $t = r$. Now suppose that the initial value f is supported in the ball B_R. Then we know that $u(t)$ is supported in the ball B_{t*R} where $t*R = \theta^{-1}(t + \theta(R))$. Accordingly, by (11) (with $a = 0$, $b = 1$) we have

$$
(15) \qquad
\begin{aligned}
((H_0 + 4t^2)u(t), u(t)) &\geqq [4t^2 - \theta(t*R)^2] \| u(t) \|^2 \\
&= [4t^2 - (t + \theta(R))^2] \| u(t) \|^2 \\
&\geqq t^2 \| u(t) \|^2
\end{aligned}
$$

provided that $t \geqq r = 2\theta(R)$.

Equations (14) and (15) give the inequality

$$t^2 \|u(t)\|^2 \leqq C_r + \int_r^t 8s \|u(s)\|^2 ds , \quad t \geqq r = 2\theta(R)$$

from which it follows, by Gronwall's method, that $\|u(t)\| = 0(t^3)$.

Now return to our ψ with $H_0^* \psi = -18i\psi$ and define

$$F(t) = (\psi, u(t)) .$$

Then $F(0) = (\psi, f)$, $\dot{F}(0) = 0$, and

$$\ddot{F}(t) = (\psi, \ddot{u}(t)) = (\psi, -H_0 u(t)) - 4t^2 (\psi, u(t))$$
$$= -(H_0^* \psi, u(t)) - 4t^2 (\psi, u(t)) ,$$

or

$$(16) \qquad \ddot{F}(t) = (18i - 4t^2) F(t) .$$

Since F is a solution of (16) with $\dot{F}(0) = 0$, it follows that $F(t) = k(t^4 - (3/2)it^2 - (3/16))e^{it^2}$ for some constant k. But since $\|u(t)\| = 0(t^3)$ we must have $F(t) = 0(t^3)$, and thus $k = 0$. In particular $F(0) = (\psi, f) = 0$. Hence ψ is orthogonal to $C_0^\infty(M)$ and $\psi = 0$.

A similar argument works with $-18i$ replaced by $18i$. This concludes the proof.

REFERENCES

1. Yu. M. Berezanskii, *Self-adjointness of elliptic operators with singular coefficients* (in Russian), Ukrainian Math. J., **26** (1974), 579-590.
2. P. R. Chernoff, *Essential self-adjointness of powers of generators of hyperbolic equations*, J. Functional Analysis, **12** (1973), 401-414.
3. W. G. Faris, *Essential self-adjointness of operators in ordered Hilbert space*, Commun. Math. Phys., **30** (1973), 23-34.
4. ———, *Self-Adjoint Operators*, Lecture Notes in Mathematics No. 433, Springer-Verlag (1975).
5. K. Jörgens, *Perturbations of the Dirac Operator*, in "Conference on the Theory of Ordinary and Partial Differential Equations," Lecture Notes in Mathematics No. 380, Springer-Verlag (1972), 87-102.
6. H. Kalf, U.-W. Schmincke, J. Walter, and R. Wüst, *On the Spectral Theory of Schrödinger and Dirac Operators with Strongly Singular Potentials*, in "Spectral Theory and Differential Equations," Lecture Notes in Mathematics No. 448, Springer-Verlag (1975), 182-226.
7. T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, 1966.
8. T. Kato, *Schrödinger operators with singular potentials*, Israel J. Math., **13** (1972), 135-148.
9. T. Kato, *A remark to the preceding paper by Chernoff*, J. Functional Analysis, **12** (1973), 415-417.
10. M. Reed and B. Simon, *Methods of Modern Mathematical Physics, I: Functional Analysis. II: Fourier Analysis. Self-Adjointness*, Academic Press, 1973 and 1975.

11. M. Schechter, *Spectra of Partial Differential Operators*, North Holland, 1971.
12. B. Simon, *Essential self-adjointness of Schrödinger operators with positive potentials*, Math. Ann., **201** (1973), 211-220.

UNIVERSITY OF CALIFORNIA
BERKELEY, CA 94720