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THE INNER APERTURE OF A CONVEX SET

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It is a standard fact that the asymptotic cone $O(C)$ of a convex set C in $\boldsymbol{R}^n$ is the polar of the barrier cone $B(C)$. In the present note we show that the inner aperture $P(C)$ of C may be obtained from $B(C)$ in a similar manner. We use this result to study relations between $O(C)$ and $P(C)$, and to give a short proof of D. G. Larman's characterization of inner apertures.

1. With each nonempty convex set C in $\boldsymbol{R}^n$ one can associate certain convex cones which give information about the "nature of unboundedness" of C. Among such cones are the barrier cone

$$B(C):= \{y \in \boldsymbol{R}^n \mid \sup_{x \in C} \langle x, y\rangle < +\infty\} ,$$

the asymptotic cone

$$O(C):= \{y \in \boldsymbol{R}^n \mid \exists x \in C \, \forall \lambda \geqq 0 \colon x + \lambda y \in C\} ,$$

and the inner aperture

$$P(C):= \{y \in \boldsymbol{R}^n \mid \exists x \in C \colon \sup_{\lambda \geqq 0} \operatorname{dist}(x + \lambda y, \operatorname{bd} C\} = +\infty\} .$$

The barrier cone and the asymptotic cone are well established concepts. (See, e.g., [3]. Note that $O(C)$ in our notation is $0^+ \operatorname{cl} C$ in the notation of [3].) The inner aperture was introduced by D. G. Larman [2], see also [1]. (Note that $\mathscr{P}(C)$ in the notation of [2] is $P(C) \cup \{o\}$ in the notation of the present note. Also, $P(C)$ in the notation of the present note may differ from $P(C)$ in the notation of [1] when $\operatorname{aff} C \neq \boldsymbol{R}^n$.) In the following we shall use that

$$P(C) = \{y \in \boldsymbol{R}^n \mid \forall x \in \boldsymbol{R}^n \exists \lambda \geqq 0 \colon x + \lambda y \in C\} , \tag{1}$$

cf. [1]. We find this description of $P(C)$ in terms of the linear structure easier to handle.

In the present note we shall study relations between $B(C)$, $O(C)$ and $P(C)$. In § 2 we shall introduce a new polar operation $A \mapsto A^{\Delta}$, closely related to the usual polar operation $A \mapsto A^{\circ}$, such that

$$B(C)^{\Delta} = P(C) .$$

Based on this fact (which we find interesting in itself) and the standard fact that

$$B(C)^{\circ} = O(C) , \tag{2}$$

(see, e.g., [3]) we shall next examine relations between $O(C)$ and $P(C)$. Finally, in § 3 we shall use the insight obtained in § 2 to give a proof of the main result of [2], a characterization of those convex cones that are inner apertures.

By a convex cone we mean a convex set A such that $\lambda x \in A$ for all $x \in A$ and all $\lambda > 0$; note that o is not required to be in A. For any nonempty convex set C, the sets $B(C)$, $O(C)$ and $P(C)$ are in fact convex cones. One has $o \in B(C)$ and $o \in O(C)$. Also, one has $o \notin P(C)$ unless $C = \boldsymbol{R}^n$; in that case $P(C) = \boldsymbol{R}^n$.

2. For any set A in $\boldsymbol{R}^n$ we let

$$A := \{y \in \boldsymbol{R}^n \mid \forall x \in A \backslash \{o\}: \langle x, y \rangle < 0\} .$$

Geometrically speaking, A^{Δ} is the intersection of all open halfspaces

$$\{y \in \boldsymbol{R}^n \mid \langle x, y \rangle < 0\} ,$$

where x runs through $A \backslash \{o\}$. In particular, it is a convex cone (not containing o unless $A \subset \{o\}$). Also, $A^{\Delta} = (\operatorname{cone} A)^{\Delta}$ where $\operatorname{cone} A$ denotes the convex cone generated by A.

Furthermore, we let

$$A^{o} := \{y \in \boldsymbol{R}^n \mid \forall x \in A: \langle x, y \rangle \leqq 0\} .$$

This is a standard polar operation. Note that A^{o} is a closed convex cone, $A^{o} = (\operatorname{cl}(\operatorname{cone} A))^{o}$ and $A^{oo} = \operatorname{cl}(\operatorname{cone} A)$.

In the following, when A is a convex cone we denote by $\operatorname{rel\,int} A$ the interior of A relative to the subspace $\operatorname{span} A$ generated by A.

LEMMA. *For any convex cone A in $\boldsymbol{R}^n$ one has*:

(a) $A^{o} = (\operatorname{int} A)^{\Delta} \cup \{o\}$.

(b) $\operatorname{int} A^{o} = (\operatorname{cl} A)^{\Delta}$.

(c) $\operatorname{rel\,int} A^{o} = ((\operatorname{cl} A) \backslash V)^{\Delta}$, *where V denotes the largest linear subspace contained in* $\operatorname{cl} A$.

Proof. Statement (a) is an easy consequence of the definitions.

To prove (c), note first that $V = U^{\perp}$ where $U := \operatorname{span} A^{o}$. Consider $y \in \operatorname{rel\,int} A^{o}$, and let $x \in (\operatorname{cl} A) \backslash V$. Then there is $u \in U$ such that $\langle x, u \rangle > 0$, and there is $\lambda > 0$ such that $y + \lambda u \in A^{o}$. Since $A^{o} = (\operatorname{cl} A)^{o}$ we obtain $\langle x, y \rangle < \langle x, y + \lambda u \rangle \leqq 0$, showing that $y \in ((\operatorname{cl} A) \backslash V)^{\Delta}$. Conversely, consider $y \notin \operatorname{rel\,int} A^{o}$. Then by a standard separation theorem there is $x \neq o$ such that $\langle x, z \rangle < 0 \leqq \langle x, y \rangle$ for all $z \in \operatorname{rel\,int} A^{o}$. The first inequality shows that

$$x \in (\operatorname{rel\,int} A^{o})^{o} = A^{oo} = \operatorname{cl} A ,$$

and it also shows that

$$x \notin (\operatorname{span}(\operatorname{rel\,int} A^{\circ}))^{\perp} = U^{\perp} = V ,$$

whence $x \in (\operatorname{cl} A)\backslash V$. But then the second inequality shows that $y \notin ((\operatorname{cl} A)\backslash V)$.

To prove (b) note that when $\operatorname{int} A^{\circ} \neq \emptyset$, then $U = \boldsymbol{R}^n$ and so $V = \{o\}$. Since $(\operatorname{cl} A)^{\Delta} = ((\operatorname{cl} A)\backslash\{o\})^{\Delta}$, we see that (b) follows from (c) when $\operatorname{int} A \neq \emptyset$. On the other hand, if $\operatorname{int} A^{\circ} = \emptyset$, then $U \neq \boldsymbol{R}^n$ and so V contains a line. But then clearly $(\operatorname{cl} A)^{\Delta} = \emptyset$, and hence (b) also holds when $\operatorname{int} A^{\circ} = \emptyset$.

We shall use the Lemma to obtain:

PROPOSITION. *For any convex cone A in $\boldsymbol{R}^n$ one has*:

(d) $\operatorname{int} A^{\circ} \subset A^{\Delta} \subset A^{\circ}$.

(e) $A^{\circ} = A^{\Delta} \cup \{o\}$ *if and only if* $A\backslash\{o\}$ *is open.*

(f) $\operatorname{int} A^{\circ} = A^{\Delta}$ *if and only if every exposed face* $(\neq\{o\})$ *of* $\operatorname{cl} A$ *contains a ray of A; this holds in particular if A is closed.*

(g) $\operatorname{rel\,int} A^{\circ} \subset A^{\Delta}$ *if and only if* $A^{\Delta} \neq \emptyset$.

Proof. The first inclusion in (d) follows from (b), the second is obvious (and is in fact contained in (a)). Statement (e) follows from (a), and (f) follows from (b). To prove the if part of (g), let $y \in A^{\Delta}$. Then $\langle x, y\rangle < 0$ for all $x \in A\backslash\{o\}$, and since by (d) we have $A^{\Delta} \subset A^{\circ} \subset V^{\perp}$, it follows that no point of $A\backslash\{o\}$ can be in V, i.e., $A\backslash V = A\backslash\{o\}$. But then we have $((\operatorname{cl} A)\backslash V)^{\Delta} \subset (A\backslash V)^{\Delta} = (A\backslash\{o\})^{\Delta} = A^{\Delta}$, and therefore the desired inclusion follows from (c). The only if part of (g) is trivial.

REMARK. If follows from (d) and (e) that $A^{\Delta} \cup \{o\}$ is closed if and only if $A\backslash\{o\}$ is open, and it follows from (d) and (f) that A^{Δ} is open if and only if every face of $\operatorname{cl} A$ contains a ray of A.

Next we shall prove:

THEOREM 1. *For any nonempty convex set C in $\boldsymbol{R}^n$ one has* $B(C)^{\Delta} = P(C)$.

Proof. Let $y \in B(C)^{\Delta}$. Suppose that there exists $x \in \boldsymbol{R}^n$ such that $x + \lambda y \notin C$ for all $\lambda \geqq 0$. A standard separation theorem then yields the existence of $z \in \boldsymbol{R}^n\backslash\{o\}$ and $\alpha \in \boldsymbol{R}$ such that

$$\langle x + \lambda y, z\rangle \geqq \alpha \geqq \langle u, z\rangle$$

for all $\lambda \geqq 0$ and all $u \in C$. Now, the first inequality implies $\langle y, z\rangle \geqq 0$, whence $z \notin B(C)$, whereas the second inequality shows that $z \in B(C)$. Thus we have obtained a contradiction, and (1) then shows that $y \in P(C)$.

Conversely, let $y \notin B(C)^{\Delta}$. Then there exists $z \in B(C) \backslash \{o\}$ such that $\langle y, z \rangle \geqq 0$. Let $\alpha \in \boldsymbol{R}$ be such that $\langle u, z \rangle < \alpha$ for all $u \in C$. Let x by any point in $\boldsymbol{R}^n$ with $\langle x, z \rangle \geqq \alpha$. Then $\langle x + \lambda y, z \rangle \geqq \alpha$ for all $\lambda \geqq 0$, whence $x + \lambda y \notin C$ for all $\lambda \geqq 0$. By (1) we see that $y \notin P(C)$.

Combining now the Proposition, Theorem 1 and (2) we obtain:

COROLLARY. *For any nonempty convex set C in $\boldsymbol{R}^n$ one has:*

(h) $\operatorname{int} O(C) \subset P(C) \subset O(C)$.

(i) $O(C) = P(C) \cup \{o\}$ *if and only if* $B(C) \backslash \{o\}$ *is open.*

(j) $\operatorname{int} O(C) = P(C)$ *if and only if every exposed face* $(\neq \{o\})$ *of* $\operatorname{cl} B(C)$ *contains a ray of* $B(C)$; *this holds in particular if* $B(C)$ *is closed.*

(k) $\operatorname{rel\,int} O(C) \subset P(C)$ *if and only if* $P(C) \neq \varnothing$.

REMARKS. Statement (h) is of course also a direct consequence of the definitions. When C is a cone, then $B(C) = C^{\circ}$, and therefore $O(C) = C^{\circ\circ} = \operatorname{cl} C$ and $P(C) = \operatorname{int}(\operatorname{cl} C) = \operatorname{int} C$, the latter by (j); this is also a fairly obvious consequence of the definitions.

3. Every asymptotic cone is closed, and every closed convex cone is its own asymptotic cone. D. G. Larman [2] characterized those convex cones that are inner apertures. In our terminology his theorem is essentially as follows:

THEOREM 2 (D. G. Larman). *A nonempty convex cone K in $\boldsymbol{R}^n$ is the inner aperture of some convex set C if and only if there is an F_σ set A in the unit sphere S of $\boldsymbol{R}^n$ such that $K = A^{\Delta}$.*

For any set B we have $B^{\Delta} = (\operatorname{cone} B)^{\Delta}$, and if B is a closed subset of S, then $\operatorname{cone} B$ is closed. Hence, B^{Δ} is open by (f). Conversely, for any open convex cone D, the set $S \cap D^{\circ}$ is a closed subset of S with $(S \cap D^{\circ})^{\Delta} = D$; in fact, using (f) we have $(S \cap D^{\circ})^{\Delta} = (D^{\circ})^{\Delta} = \operatorname{int} D^{\circ\circ} = \operatorname{int}(\operatorname{cl} D) = D$. In conclusion, Theorem 2 is equivalent to the following:

THEOREM 3. *A nonempty convex cone K in $\boldsymbol{R}^n$ is the inner aperture of some convex set C if and only if there is a* (decreasing) *sequence $(D_i)_{i \in N}$ of open convex cones such that $K = \bigcap_{i \in N} D_i$.*

Proof. We may assume that $K \neq \boldsymbol{R}^n$.

First, assume that $K = P(C)$. Then $K = B(C)^{\Delta}$ by Theorem 1. Let $y \in K$. Then for each $i \in N$

$$A_i := \{x \in \boldsymbol{R}^n \mid \sup_{u \in C} \langle u, x\rangle \leqq -i\langle x, y\rangle\}$$

is a closed convex cone. Also, $A_i \subset A_j$ for $i < j$, and

$$B(C) = \bigcup_{i \in N} A_i .$$

Therefore, we have

$$K = B(C)^{\Delta} = (\bigcup_{i \in N} A_i)^{\Delta} = \bigcap_{i \in N} A_i' ,$$

and here $(A_i^{\Delta})_{i \in N}$ is a decreasing sequence of open convex cones, cf. (f). (Actually, $B(C)$ is the effective domain of the support function σ of C; the set A_i is the set of points x such that $\sigma(x) \leqq -i\langle x, y\rangle$.)

Conversely, assume that $(D_i)_{i \in N}$ is a sequence of open convex cones such that $K = \bigcap_{i \in N} D_i$. Clearly, we may assume that $D_i \supset D_j$ for $i < j$. Since $K \neq \boldsymbol{R}^n$, at least one of the D_i's in contained in an open halfspace, and therefore $K^{\prime} \neq \emptyset$. Let $z \in K \cap (-\operatorname{rel\,int} K^{\circ})$; the existence of z follows from a standard separation theorem. Then $z \in -K^{\Delta}$ by (g). Since each D_i is an open cone containing z, there exists an increasing sequence of positive reals λ_i such that

$$B(o, i) \subset -\lambda_i z + D_i$$

for each $i \in N$, where $B(o, i)$ denotes the closed ball centered at o with radius i. Let

$$C := \bigcap_{i \in N} (-\lambda_i z + D_i) .$$

Then C is convex, and it is clear that

$$P(C) \subset P(-\lambda_i z + D_i) = P(D_i) = D_i$$

for each $i \in \boldsymbol{N}$, whence $P(C) \subset K$. Conversely, let $y \in K$, and let x be arbitrary in $\boldsymbol{R}^n$. Then for each $i \in \boldsymbol{N}$ there is $\mu_i \geqq 0$ such that $x + \lambda y \in -\lambda_i z + D_i$ for all $\lambda \geqq \mu_i$. In order to show that $y \in P(C)$ we shall show that one may obtain $\mu_i \leqq \alpha$ for some $\alpha \in \boldsymbol{R}$ and all $i \in \boldsymbol{N}$, cf. (1). Since $y \in K$ and $-z \in K^{\Delta}$, it follows that $\langle y, z\rangle > 0$. Therefore there exists a unique $\beta \in \boldsymbol{R}$ such that $\langle x + \beta y, z\rangle = 0$, and then $\langle x + \lambda y, z\rangle > 0$ for $\lambda > \beta$. Let $j \in \boldsymbol{N}$ be such that $\|x + \beta y\| \leqq j$. Then $x + \beta y \in -\lambda_i z + D_i$ for all $i \geqq j$, and since $y \in D_i$, it follows that we actually have $x + \lambda y \in -\lambda_i z + D_i$ for all $\lambda \geqq \beta$ and all $i \geqq j$. But then any $\alpha \geqq \max\{\mu_1, \cdots, \mu_{j-1}, \beta\}$ will fulfil the requirement.

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