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A SCHEME FOR APPROXIMATING BOUNDED ANALYTIC FUNCTIONS ON CERTAIN SUBSETS OF THE UNIT DISC

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We denote by H^∞ the space of all bounded analytic functions in the unit disc $D = \{z: |z| > 1\}$. We consider a relatively closed subset S of D with the following property: If $f \in H^\infty$ and its restriction $f|_S$ to S is uniformly continuous, there exists a bounded sequence of polynomials $\{p_n\}$ such that

(i) $p_n \to f$ uniformly on compact subsets of D.

(ii) $p_n \to f$ uniformly on S.

A set S with this property, is called a Mergelyan set for H^∞. The object of this paper is to give a new and constructive proof of the following result:

THEOREM. Let S be a relatively closed subset of D. Assume that the clusterpoints of S on the unit circle $T = \{z: |z| = 1\}$ which are not in the nontangential closure of S, has zero linear measure. Then S is a Mergelyan set for H^∞.

A point $\eta \in T$ is said to be in the nontangential closure of S if η is a limit point of $S \cap D(\lambda, \eta)$ for some $\lambda > 0$ where $D(\lambda, \eta) = \{z \in D: |z - \eta| < \lambda(1 - |z|)\}$.

The theorem above solves a problem raised by J. Detraz [1]. (She also proved the converse of the theorem.)

The first proof of it (due to A. M. Davie) was based on functional analysis, and is rather short and elegant. See [5] for details. But from a constructive point of view, the proof is not very satisfactory. So the main reason for introducing this new and longer proof, is that it gives a rather explicit construction of the approximating polynomials $\{p_n\}_{n=1}^\infty$.

This theorem is part of a more general problem in simultaneous approximation introduced to me by Professor L. Rubel. (See [4], [5], and [6].)

Proof of the theorem. We denote by S^2 the extended complex plane. If $B \subset S^2$, and g is a function defined on B, then we define $||g||_B = \sup\{|g(z)|, z \in B\}$. Various absolute constants will be denoted by $A_1, A_2, \cdots$.

Let f and S be as in the above theorem. We assume $||f||_D \leqq 1$.

If $E \subset T$ is measurable, $|E|$ denotes its linear measure. Let $D(\lambda, \rho)$ be as above. By a well known result of Fatou, ([3], page 34) there is a set $F \subset T$, with $|F| = 0$, such that

$$(1) \qquad \lim_{\substack{z \to \rho \\ z \in D(\lambda, \rho)}} f(z) \overset{\text{def}}{=} f^*(\rho)$$

exists for all $\rho \in T \backslash F$ and all $\lambda > 0$. Let E denote the limitpoints of S on T. By (1) and our hypothesis on S, the continuous extension of f to $S \cup E$, coincides with f^* almost everywhere on E. (With respect to linear measure.)

By approximating as in §1 in [7], we can also assume that f^* is continuous on $T \backslash E$. But then f^* (considered as a function on T) is continuous on $T \backslash (E \backslash B)$, where B denotes the interior of E relative to T.

The approximation of f is now done in two steps. If $|E \backslash B| = 0$, the first step can be omitted. So we assume $|E \backslash B| > 0$.

The idea is to modify Vitushkin's scheme for rational approximation (see [2], page 210) to be suitable for our problem. We shall construct functions f_n, $n = 1, 2, \cdots$ such that $g = f - \sum_1^\infty f_n$ has the following two properties if $\varepsilon > 0$ is given in advance:

(2) $\|f - g\|_S < \varepsilon$.

(3) g is holomorphic in D and continuous in $D \cup T \backslash E_0$, where $E_0 \subset E \backslash B$ is compact and $|E_0| = 0$. Also $\|g\|_D \leqq A_1$.

Before we construct $\{f_n\}_{n=1}^\infty$, let us indicate the second and last step in the proof. Denote by $A(D)$, all uniformly continuous analytic functions in D. We choose functions a and b from $A(D)$ such that there is a neighborhood V of E_0 in C with the following property

$$(4) \qquad \begin{cases} |a - f| < \varepsilon \cdot 2^{-1} \quad \text{on} \quad S \cap V\,. \\ \max\{\|a\|_{D'}\, \|b\|_D\} \leqq 2\,. \\ b = 0 \quad \text{on} \quad E_0 \quad \text{and if} \quad z \in S, \quad \text{then} \\ |a(z) - f(z)| > 2\varepsilon \Rightarrow |1 - b(z)| < \varepsilon\,. \end{cases}$$

The construction of a and b can be done as in [3], page 80. We partion E_0 into finitely many subsets $E_1, \cdots, E_n$ with pairwise disjoint neighborhoods $W_1, \cdots, W_n$, such that there are complex numbers λ_j $1 \leqq j \leqq n$ with $|f - \lambda_j| < \varepsilon 2^{-2}$ on each E_j. For each j we argue as on page 80 in [3] and find $b_j \in A(D)$ with $b_j = 0$ on E_j and $\operatorname{Re} b_j < 0$ on $D \backslash E_j$. Then we define

$$a = \sum_{1}^{n} \lambda_j e^{Nb_j}$$

where N is a sufficiently large number. Then $|a - f| < \varepsilon 2^{-1}$ in some neighborhood V of E_0, and we choose $b \in A(D)$ with $b = 0$ on E_0, $||b||_D \leqq 1$ and $|1 - b| < \varepsilon$ on $D \backslash V$.

Consider now $h = b(g - a) + a$. Then $h \in A(D)$ and since $g - h = (g - a)(1 - b)$, we have from (2), (3), and (4) above that $||g - h||_S < \max\{(A_1 + 2)\varepsilon, 5\varepsilon\}$. But then $||f - h||_S < A_2\varepsilon$, and we also have $||h||_D \leqq A_3$. Since $h \in A(D)$, this function is easy to approximate uniformly on D by polynomials, and this completes the second step.

We turn to the construction of $\{f_n\}$.

Let $\varepsilon > 0$ be given. We choose pairwise disjoint discs $\Delta_n = \Delta(w_n, \delta_n)$, $n = 1, 2, \cdots$ and constants t, M, and λ with the following properties

(5) $w_n \in E \backslash B$ and there are numbers η_n with $\delta_n < \eta_n < 2\delta_n < t$ such that $[(\Delta(w_n, \eta_n) \backslash \Delta(w_n, \delta_n)) \cap T] \subset T \backslash (E \backslash B)$.

(6) $|E \backslash (B \cup (\bigcup_n \Delta_n))| = 0$.

(7) $\sup\{|f(z) - f(\rho)|,\ z, \rho \in \Delta(w_n, 2\delta_n) \cap D(\lambda, w_n)\} < \varepsilon$.

(8) $\sup\{|f(z) - f(\rho)|,\ z, \rho \in S,\ |z - \rho| \leqq Mt\} < \varepsilon$.

More details about how t, λ, and M depend on ε, will be given below. The important thing is that they are given in advance, before $\{\Delta_k\}$ is constructed. The existence of $\{\Delta_k\}$ satisfying (5)-(8) follows from Fatou's result (1) about nontangential limits mentioned above, and the fact that $E \backslash B$ is totally disconnected. To obtain (8) we use that $f|_S$ is uniformly continuous.

We now choose numbers r_n with $\delta_n < r_n < \eta_n$, and smooth functions φ_n with support in $\Delta(w_n, r_n)$ such that we also have for all n

$$(9)\qquad \begin{aligned} &0 \leqq \varphi_n \leqq 1\,, \quad \varphi_n \equiv 1 \ \text{ near } \ \Delta_n, \ \text{ and} \\ &\left\|\frac{\partial \varphi_n}{\partial \bar{z}}\right\|_C \leqq A_2(\eta_n - \delta_n)^{-1}\,. \end{aligned}$$

We extend f to $\boldsymbol{C}$ by the equation $f(z) = f(1/\bar{z})$. Define functions G_n, $n = 1, 2, \cdots$ by

$$(*)\qquad \begin{aligned} G_n(\rho) = T_{\varphi_n} f(\rho) &= \frac{1}{\pi} \iint \frac{f(z) - f(\rho)}{z - \rho} \frac{\partial \varphi_n}{\partial \bar{z}} dx dy \\ &= \varphi_n(\rho) \cdot f(\rho) + \frac{1}{\pi} \iint \frac{f(z)}{z - \rho} \frac{\partial \varphi_n}{\partial \bar{z}} dx dy\,. \end{aligned}$$

For basic properties of the T_φ-operator we refer to [2], page

30. These properties gives that G_n is analytic in D and continuous wherever f is. Also $f - G_n$ is analytic near $\Delta_n \cap (E\backslash B)$. Since $||f||_C \leqq 1$, (9) shows that

(10) $$||G_n||_C \leqq A_3 .$$

We also claim that the constants t, M, and λ can be chosen so that

(11) $$||G_n||_S < A_4\varepsilon .$$

To see this, note first that since G_n is analytic in $S^2\backslash\Delta(w_n, t)$ and $G_n(\infty) = 0$, an easy application of Schwartz lemma shows that $|G_n(\rho)| < A(M-1)^{-1} < \varepsilon$ if M is sufficiently large, and $|\rho - w_n| > Mt$. This does not contradict (8) if t is sufficiently small. If we use (8), we see that if $p \in S$ and $|\rho - w_n| \leqq Mt$, then we can estimate the first integral in (*) so that

(12) $$|G_n(\rho)| \leqq A_5\varepsilon + K(\lambda)$$

where $K(\lambda)$ is a constant tending to zero if $\lambda \to \infty$. It is easy to see that we can choose λ depending on ε, but not on n, such that $K(\lambda) < \varepsilon$. (The constant $K(\lambda)$ comes from integration over those $z \in \Delta(w_n, r_n)$ where $|f(z) - f(\rho)| > 2\varepsilon$).

We can now write f_n as

$$f_n = G_n - H_n$$

where H_n is a rational function with poles only in $\Delta(w_n, \delta_n)\backslash\bar{D}$ such that

$$||H_n||_S \leqq ||G_n||_S \leqq \varepsilon A_6$$
$$||H_n||_D \leqq ||G_n||_D \leqq A_7$$
$$||H_n - G_n||_{S^2\backslash\Delta(w_n,\eta_n)} < \varepsilon 2^{-n} .$$

The existence of $\{H_n\}$ is not quite trivial. If we map $(S^2\backslash\Delta(w_n, r_n)) \cup D$ conformally onto D, $S \cup (S^2\backslash\Delta(w_n, \eta_n))$ is mapped to a Farrel set (see [5] for definition) in D, and the existence of H_n, $n = 1, 2, \cdots$, now follows from the main result in [5]. This completes the proof. In fact, the set E_0 in (3) above, is simply $(T\backslash(B \cup (\bigcup_n \Delta_n))$.

Concluding remark. It is easy to verify that this proof applies to more general planar domains.

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