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TRANSCENDENTAL CONSTANTS OVER THE COEFFICIENT FIELDS IN DIFFERENTIAL ELLIPTIC FUNCTION FIELDS

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Let k be a differential field of characteristic 0, and Ω be a universal extension of k. Suppose that the field of constants k_0 of k is algebraically closed. Consider the following differential polynomial of the first order over k in a single indeterminate y:

$$T(y) = (y')^2 - \lambda S(y; \kappa) ; \qquad \lambda \in k ; \qquad \lambda \neq 0 ;$$

here

$$S(y; \kappa) = y(1 - y)(1 - \kappa^2 y) ;$$
$$\kappa \in k ; \qquad \kappa^2 \neq 0, 1 ; \qquad \kappa' = 0 .$$

Take a generic point z of the general solution of T. Then, z is transcendental over k, and $k(z, z')$ is called a differential elliptic function field.

We prove the following:

THEOREM. Let $k(z, z')$ be a differential elliptic function field over k. Then, there exists a finitely generated differential extension field k^* of k such that the following three conditions are satisfied:

(i) z is transcendental over k^*;

(ii) the field of constants of k^* is the same as k_0;

(iii) there exists an element ζ of Ω such that $k^*(z, z') = k^*(\zeta, \zeta')$ and $(\zeta')^2 = 4S(\zeta; \kappa)$ with the same modulus as κ.

Matsuda [3] gave an example of a differential elliptic function field such that $k = \bar{k}$ and we can not take k as k^* (cf. [5]).

REMARK. Matsuda [3] gave a differential algebraic proof of the following theorem essentially due to Poincaré: Suppose that a differential algebraic function field K over an algebraically closed coefficient field k is free from parametric singularities. Then, K is a differential elliptic function field over k if the genus of K is 1.

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1. Two lemmas.

The following theorem is due to Kolchin [1]:

LEMMA 1. *Let Σ be a perfect differential ideal in the differential polynomial algebra $k\{y\}$, and let J be a differential polynomial* *in* $k\{y\}$ *which is not in* Σ. *Then,* Σ *has a zero* η *in* Ω *such that* $J(\eta) \neq 0$ *and the field of constants of* $k\langle\eta\rangle$ *is* k_0.

We shall prove the following:

LEMMA 2. *Let* F *be an element of* $k\{y\}$ *of the first order and let* ξ *be a zero of* F *which is transcendental over* k. *Suppose that* F *is algebraically irreducible over the algebraic closure* $\bar{k}$ *of* k, *and that the field of constants of* $k\langle\xi\rangle$ *is* k_0. *Then, there exists a nonsingular zero* η *of* F *such that* ξ *is transcendental over* $k\langle\eta\rangle$ *and the field of constants of* $k\langle\eta\rangle$ *is* k_0.

Proof. Let η be a generic point of the general solution of F over $\overline{k\langle\xi\rangle}$. Then, $\eta \notin \overline{k\langle\xi\rangle}$ and $\eta \notin \bar{k}$. Hence, $\xi \notin \overline{k\langle\eta\rangle}$. By Gourin's theorem (cf. [4, p. 49]) both ξ and η are generic points of the general solution of F over k. Hence, there exists an isomorphism of $k\langle\xi\rangle$ onto $k\langle\eta\rangle$ over k. Therefore, the field of constants of $k\langle\eta\rangle$ is k_0.

2. **Proof of Theorem.** We shall prove that there exists a nonsingular zero w of T such that z is transcendental over $k\langle w\rangle$ and the field of constants of $k\langle w\rangle$ is k_0. First we shall assume that the field of constants of $k\langle z\rangle$ contains properly k_0. Let Σ be the prime differential ideal in $k\{y\}$ associated with the general solution of T. Then, the separant $2y'$ of T does not belong to Σ. By Lemma 1, there exists a nonsingular zero w of T such that the field of constants of $k\langle w\rangle$ is k_0. Suppose that z is algebraic over $k\langle w\rangle$. Then, the field of constants of $k\langle z\rangle$ is contained in k_0, since $k\langle z\rangle \subseteq \overline{k\langle w\rangle}$. This contradicts our assumption. Hence, z is transcendental over $\overline{k\langle w\rangle}$. Secondly, let us assume that the field of constants of $k\langle z\rangle$ is the same as k_0. Then, there exists a nonsingular zero w of T such that the field of constants of $k\langle w\rangle$ is k_0 and z is transcedental over $k\langle w\rangle$ by Lemma 2, since T is algebraically irreducible over $\bar{k}$.

We shall denote $k\langle w\rangle$ by k_1. Let us define an element a of $k_1\langle z\rangle$ by

$$a = \{B(z, w) - 2\lambda^{-1}w'z'\}/A(z, w)^2 ,$$

where

$$A(y_1, y_2) = 1 - \kappa^2 y_1 y_2$$
$$B(y_1, y_2) = y_1(1 - y_2)(1 - \kappa^2 y_2) + y_2(1 - y_1)(1 - \kappa^2 y_1) .$$

The polynomials A, B and S satisfy a relation:

$$B(y_1, y_2)^2 = 4S(y_1)S(y_2) + (y_1 - y_2)^2 A(y_1, y_2)^2 \tag{1}$$

which is verified in the following:

$$\begin{aligned}
&B(y_1, y_2)^2 - 4S(y_1)S(y_2)\\
&\quad = \{y_1 y_2^{-1}S(y_2) + y_2 y_1^{-1}S(y_1)\}^2 - 4S(y_1)S(y_2)\\
&\quad = \{y_1 y_2^{-1}S(y_2) - y_2 y_1^{-1}S(y_1)\}^2\\
&\quad = \{y_1(1 - y_2)(1 - \kappa^2 y_2) - y_2(1 - y_1)(1 - \kappa^2 y_1)\}^2\\
&\quad = \{y_1 - y_2 - \kappa^2(y_1^2 y_2 - y_1 y_2^2)\}^2\\
&\quad = (y_1 - y_2)^2 A(y_1, y_2)^2 .
\end{aligned}$$

By the definition of a

$$\{A(z, w)^2 a - B(z, w)\}^2 - 4\lambda^{-2}(w')^2(z')^2 = 0 .$$

Since w and z are solutions of $T = 0$ and (1), the left hand side is

$$\begin{aligned}
&\{A(z, w)^2 a - B(a, w)\}^2 - 4S(w)S(z)\\
&\quad = A(z, w)^4 a^2 - 2A(z, w)^2 B(z, w)a + B(z, w)^2 - 4S(w)S(z)\\
&\quad = A(z, w)^4 a^2 - 2A(z, w)^2 B(z, w)a + (z - w)^2 A(z, w)^2\\
&\quad = A(z, w)^2\{A(z, w)^2 a^2 - 2B(z, w)a + (z - w)^2\} .
\end{aligned}$$

Since $A(z, w) \neq 0$, we have an algebraic relation over k_1 between a and z:

$$A(z, w)^2 a^2 - 2B(z, w)a + (z - w)^2 = 0 . \tag{2}$$

The left hand side of (2) is

$$\begin{aligned}
&(1 - \kappa^2 zw)^2 a^2 - 2z(1 - w)(1 - \kappa^2 w) + w(1 - z)(1 - \kappa^2 z)\}a\\
&\qquad + (z - w)^2\\
&\quad = a^2(\kappa^4 z^2 w^2 - 2\kappa^2 zw + 1)\\
&\qquad - 2a[\kappa^2 wz^2 + \{\kappa^2 w^2 - 2(1 + \kappa^2)w + 1\}z + w]\\
&\qquad + z^2 - 2zw + w^2\\
&\quad = z^2(\kappa^4 a^2 w^2 - 2\kappa^2 aw + 1)\\
&\qquad - 2z[\kappa^2 wa^2 + \{\kappa^2 w^2 - 2(1 + \kappa^2)w + 1\}a + w]\\
&\qquad + a^2 - 2wa + w^2 .
\end{aligned}$$

Hence we have a relation equivalent to (2):

$$A(a, w)^2 z^2 - 2B(a, w)z + (a - w)^2 = 0 . \tag{3}$$

Since z is transcendental over k_1, a is transcendental over k_1 and satisfies $[k_1(a, z): k_1(z)] = 2$. For the discriminant of (2) is $16S(z)S(w)$ by (1). We have $k_1\langle z\rangle = k_1(a, z)$. We shall prove that a is a constant (cf. [2, p. 805]). Let us take an element α of $\bar{k}$ such that $\alpha^2 = 4/\lambda$ and define a new differentiation signed by the dot in $k_1\langle \alpha, z\rangle$ by $\dot{x} = \alpha x'$. Then,

$$(4) \qquad \begin{aligned} a &= \{B(z, w) - 2^{-1}\dot{w}\dot{z}\}/A(z, w)^2 , \\ (\dot{z})^2 &= 4S(z) , \qquad (\dot{w})^2 = 4S(w) . \end{aligned}$$

In what follows, we denote $A(z, w)$ and $B(z, w)$ by A and B respectively for simplicity. Differentiating both sides of $(\dot{w})^2 = 4S(w)$, we have $2\dot{w}\ddot{w} = 4S_w\dot{w}$ and $\ddot{w} = 2S_w$ since $\dot{w} \neq 0$. Hence,

$$\begin{aligned} B_z - \ddot{w}/2 &= B_z - S_w \\ &= (1 - w)(1 - \kappa^2 w) + w\{2\kappa^2 z - (1 + \kappa^2)\} \\ &\quad - \{3\kappa^2 w^2 - 2(1 + \kappa^2)w + 1\} \\ &= -2\kappa^2 w^2 + 2\kappa^2 wz \\ &= 2\kappa^2 w(z - w) . \end{aligned}$$

On the other hand

$$\begin{aligned} &2A_zB - (\dot{w})^2 A_w \\ &\quad = -2\kappa^2 wB + 4\kappa^2 zw(1 - w)(1 - \kappa^2 w) \\ &\quad = 2\kappa^2 w\{2z(1 - w)(1 - \kappa^2 w) - B\} \\ &\quad = 2\kappa^2 w\{z(1 - w)(1 - \kappa^2 w) - w(1 - z)(1 - \kappa^2 z)\} \\ &\quad = 2\kappa^2 w(z - w)A . \end{aligned}$$

Therefore

$$A(B_z - \ddot{w}/2) = 2A_zB - (\dot{w})^2 A_w = 2\kappa^2 w(z - w)A .$$

Similarly we have

$$A(B_w - \ddot{z}/2) = 2A_wB - (\dot{z})^2 A_z = 2\kappa^2 z(w - z)A .$$

From the above equalities and (4)

$$\begin{aligned} A^3\dot{a} &= \dot{z}\{A(B_z - \ddot{w}/2) - 2A_zB + (\dot{w})^2 A_w\} \\ &\quad + \dot{w}\{A(B_w - \ddot{z}/2) - 2A_wB + (\dot{z})^2 A_z\} \\ &= 0 . \end{aligned}$$

Hence, $\dot{a} = 0$, and $a' = 0$.

Let k_2 denote $k_1(\alpha)$ and b be an element of $k_2\langle z\rangle$ defined by

$$b = \{A(a, w)^2 z - B(a, w)\}/(\alpha w') .$$

Then, we have $b^2 = S(a)$. In fact from (1) and (3) we have

$$\begin{aligned} \{A(a, w)^2 z - B(a, w)\}^2 &= B(a, w)^2 - (a - w)^2 A(a, w)^2 \\ &= 4S(a)S(w) , \end{aligned}$$

and $(\alpha w')^2 = 4S(w)$ since w is a solution of $T = 0$. Hence, $k_2\langle z\rangle = k_2(a, b)$ because $[k_2\langle z\rangle : k_2(a)] = [k_2(a, b) : k_2(a)] = 2$ and $b \in k_2\langle z\rangle$.

By Lemma 1, there exists a nonsingular solution v of $(y')^2 = 4S(y)$ such that the field of constants of $k_2\langle v\rangle$ is k_0. Since a is a constant,

$$\text{trans.}\deg k^*(a)/k^* = \text{trans.}\deg k_0(a)/k_0 = 1\ ,$$

where $k^* = k_2\langle v\rangle$ (cf. [2, p. 767]). Hence, a is transcendental over k^*. Therefore, z is transcendental over k^* by (3).

Let us define an element ζ of $k^*\langle z\rangle$ by

$$\zeta = \{B(a, v) + bv'\}/A(a, v)^2\ .$$

Matsuda [3] proved that ζ is a solution of $(y')^2 = 4S(y)$ and $k^*(\zeta, \zeta') = k^*(a, b)$: We may take elements s_i, c_i, d_i $(1 \leqq i \leqq 3)$ of Ω such that

$$\begin{aligned}
&s_1^2 = v\ , \qquad c_1^2 = 1 - v\ , \qquad d_1^2 = 1 - \kappa^2 v\ , \qquad s_1' = c_1 d_1\ ;\\
&s_2^2 = a\ , \qquad c_2^2 = 1 - a\ , \qquad d_2^2 = 1 - \kappa^2 a\ , \qquad b = s_2 c_2 d_2\ ;\\
&s_3 = (s_1 c_2 d_2 + s_2 c_1 d_1)(1 - \kappa^2 s_1^2 s_2^2)^{-1}\ ;\\
&c_3 = (c_1 c_2 - s_1 s_2 d_1 d_2)(1 - \kappa^2 s_1^2 s_2^2)^{-1}\ ;\\
&d_3 = (d_1 d_2 - \kappa^2 s_1 s_2 c_1 c_2)(1 - \kappa^2 s_1^2 s_2^2)^{-1}\ .
\end{aligned}$$

We shall prove that

$$(5) \qquad c_3^2 = 1 - s_3^2\ , \qquad d_3^2 = 1 - \kappa^2 s_3^2\ , \qquad s_3' = c_3 d_3\ .$$

In fact by the definitions

$$c_1' = -s_1 d_1\ , \qquad d_1' = -\kappa^2 s_1 c_1\ , \qquad c_2' = d_2' = 0\ .$$

Since

$$1 - \kappa^2 s_1^2 s_2^2 = c_1^2 + s_1^2 d_2^2 = c_2^2 + s_2^2 d_1^2\ ,$$

we have

$$\begin{aligned}
&(1 - s_3^2)(1 - \kappa^2 s_1^2 s_2^2)^2\\
&\quad = (1 - \kappa^2 s_1^2 s_2^2)^2 - (s_1 c_2 d_2 + s_2 c_1 d_1)^2\\
&\quad = (c_1^2 + s_1^2 d_2^2)(c_2^2 + s_2^2 d_1^2) - (s_1 c_2 d_2 + s_2 c_1 d_1)^2\\
&\quad = c_1^2 c_2^2 + s_1^2 s_2^2 d_1^2 d_2^2 - 2 s_1 s_2 c_1 c_2 d_1 d_2\\
&\quad = (c_1 c_2 - s_1 s_2 d_1 d_2)^2\ .
\end{aligned}$$

Hence, $c_3^2 = 1 - s_3^2$. Similarly, we have $d_3^2 = 1 - \kappa^2 s_2^3$, since

$$1 - \kappa^2 s_1^2 s_2^2 = d_1^2 + \kappa^2 s_1^2 c_2^2 = d_2^2 + \kappa^2 s_2^2 c_1^2\ .$$

We have $s_3' = c_3 d_3$ according to the following:

$$\begin{aligned}
&(1 - \kappa^2 s_1^2 s_2^2)^2 s_3'\\
&\quad = (1 - \kappa^2 s_1^2 s_2^2)(s_1 c_2 d_2 + s_2 c_1 d_1)'
\end{aligned}$$

$$
\begin{aligned}
&- (1 - \kappa^2 s_1^2 s_2^2)'(s_1 c_2 d_2 + s_2 c_1 d_1) \\
&= (1 - \kappa^2 s_1^2 s_2^2)(s_1' c_2 d_2 + s_2 c_1' d_1 + s_2 c_1 d_1') \\
&\quad + 2\kappa^2 s_1 s_1' s_2^2 (s_1 c_2 d_2 + s_2 c_1 d_1) \\
&= (1 - \kappa^2 s_1^2 s_2^2)(c_1 c_2 d_1 d_2 - s_1 s_2 d_1^2 - \kappa^2 s_1 s_2 c_1^2) \\
&\quad + 2\kappa^2 s_1 s_2^2 c_1 d_1 (s_1 c_2 d_2 + s_2 c_1 d_1) \\
&= c_1 c_2 d_1 d_2 - s_1 s_2 d_1^2 - \kappa^2 s_1 s_2 c_1^2 - \kappa^2 s_1^2 s_2^2 c_1 c_2 d_1 d_2 \\
&\quad + \kappa^2 s_1^3 s_2^3 d_1^2 + \kappa^4 s_1^3 s_2^3 c_1^2 + 2\kappa^2 s_1^2 s_2^2 c_1 c_2 d_1 d_2 + 2\kappa^2 s_1 s_2^3 c_1^2 d_1^2 \\
&= c_1 c_2 d_1 d_2 + \kappa^2 s_1^2 s_2^2 c_1 c_2 d_1 d_2 \\
&\quad - s_1 s_2 (d_1^2 + \kappa^2 c_1^2 - \kappa^2 s_1^2 s_2^2 d_1^2 - \kappa^4 s_1^2 s_2^2 c_1^2 - 2\kappa^2 s_2^2 c_1^2 d_1^2) \ ;
\end{aligned}
$$

here

$$
\begin{aligned}
&d_1^2 + \kappa^2 c_1^2 - \kappa^2 s_1^2 s_2^2 d_1^2 - \kappa^4 s_1^2 s_2^2 c_1^2 - 2\kappa^2 s_2^2 c_1^2 d_1^2 \\
&\quad = d_1^2 (1 - \kappa^2 s_1^2 s_2^2 - \kappa^2 s_2^2 c_1^2) \\
&\qquad + \kappa^2 c_1^2 (1 - \kappa^2 s_1^2 s_2^2 - s_2^2 d_1^2) \\
&\quad = d_1^2 \{1 - \kappa^2 s_2^2 (s_1^2 + c_1^2)\} + \kappa^2 c_1^2 \{1 - s_2^2 (\kappa^2 s_1^2 + d_1^2)\} \\
&\quad = d_1^2 d_2^2 + \kappa^2 c_1^2 c_2^2 \ .
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&(1 - \kappa^2 s_1^2 s_2^2)^2 s_3' \\
&\quad = c_1 c_2 d_1 d_2 + \kappa^2 s_1^2 s_2^2 c_1 c_2 d_1 d_2 - s_1 s_2 (d_1^2 d_2^2 + \kappa^2 c_1^2 c_2^2) \\
&\quad = (c_1 c_2 - s_1 s_2 d_1 d_2)(d_1 d_2 - \kappa^2 s_1 s_2 c_1 c_2) \ ,
\end{aligned}
$$

and we have $s_3' = c_3 d_3$.

By the definition of ζ we have irreducible equations over k^*:

$$
\begin{aligned}
&A(a, v)^2 \zeta^2 - 2B(a, v) + (a - v)^2 = 0 \ , \\
&A(\zeta, v)^2 a^2 - 2B(\zeta, v) a + (\zeta - v)^2 = 0 \ ,
\end{aligned}
$$

as we get (2) and (3). Hence, $k^*(\zeta, \zeta') = k^*(a, b) = k^*(z, z')$. For we have $[k^*(\zeta, \zeta') : k^*(\zeta)] = [k^*(a, \zeta) : k^*(\zeta)] = [k^*(a, \zeta) : k^*(a)] = [k^*(a, b) : k^*(a)] = 2$ by above equalities.

We remark that the adopting of the s, c and d gives an expository verification of the identity $(\zeta')^2 = 4S(\zeta)$ proved by Matsuda [3].

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