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THE CHARACTER SPACE OF THE ALGEBRA OF REGULATED FUNCTIONS

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The character space of the algebra of regulated functions on a closed interval is computed and is identified with the character space of the algebra of functions of bounded variation. The dual space of the Banach space of regulated functions is analyzed in terms of the character space.

1. Introduction. The regulated complex functions on a compact interval form a commutative C^*-algebra with unity. The objective of this paper is to explore the character space of this algebra, and those of several related Banach algebras. E. Hewitt's computation of the character space of one of the related algebras [7] is given a new interpretation (§ 4). In pre-Banach algebra times, H. S. Kaltenborn gave an elegant description of the dual space of the algebra of regulated functions [10]; we re-examine his results from a Banach algebra point of view, recasting slightly his formula for the continuous linear forms (§ 6).

Let $I = [a, b]$ be a nondegenerate closed interval of the real line $\boldsymbol{R}$, fixed for the rest of the paper. A complex-valued function f on I is said to be *regulated* if it possesses (finite) one-sided limits at every point of I; that is, the limits

$$f(c+) = \lim_{x \to c, x > c} f(x)\,, \quad f(c-) = \lim_{x \to c, x < c} f(x)$$

exist in $\boldsymbol{C}$ for every $c \in [a, b)$ and every $c \in (a, b]$, respectively. Every regulated function on I is bounded. We write $\mathscr{R} = \mathscr{R}(I)$ for the set of all regulated functions $f: I \to \boldsymbol{C}$. With the pointwise operations and the norm $\|f\|_\infty = \sup_{x \in I} |f(x)|$, $\mathscr{R}$ is a commutative C^*-algebra with unity [16, p. 276, Def. 11.17]: completeness for the norm metric follows at once from the "iterated limits theorem" [15, p. 149, Th. 7.11]. To enlarge the perspective a bit, $\mathscr{R}$ is a closed $*$-subalgebra of the commutative C^*-algebra $\mathscr{B} = \mathscr{B}(I)$ of all bounded complex functions on I. In turn, the algebra $\mathscr{C} = \mathscr{C}(I)$ of all continuous complex functions on I is a closed subalgebra of $\mathscr{R}$; thus $\mathscr{C} \subset \mathscr{R} \subset \mathscr{B}$.

A complex function f on I is called a *step function* if there exists a partition of I into finitely many subintervals (possibly degenerate) on each of which f is constant. We write $\mathscr{S} = \mathscr{S}(I)$ for the algebra of all step functions on I, and $\mathscr{BV} = \mathscr{BV}(I)$ for the algebra of all complex functions on I of bounded variation. It is clear that $\mathscr{S} \subset \mathscr{BV}$ and that $\mathscr{S}$ is the linear span of the set of characteristic functions φ_J of subintervals J of I; and $\mathscr{BV} \subset \mathscr{R}$ because, by the Jordan theorem, $\mathscr{BV}$ is the linear span of the monotone functions (which are evidently regulated). Thus $\mathscr{S} \subset \mathscr{BV} \subset \mathscr{R}$; since $\mathscr{S}$ is dense in $\mathscr{R}$ for the sup-norm topology [2, Ch. II, §1, $n°$ 3], $\mathscr{R}$ is the completion of both $\mathscr{S}$ and $\mathscr{BV}$ for this norm. {In particular, $\mathscr{BV}$ is not complete for this norm. In §5 we study $\mathscr{BV}$ as a Banach algebra relative to another norm.}

2. Some decompositions of regulated functions. Regulated functions have a certain capacity for self-improvement:

PROPOSITION 1. *Let $f \in \mathscr{R}$. Define $f^*: I \to C$ by the formulas*

$$(1) \qquad f^*(x) = f(x+) \quad \text{for} \quad x \in [a, b)\,, \quad f^*(b) = f(b-)\,.$$

Then $f^* \in \mathscr{R}$. More precisely: (i) *f^* is right-continuous on* $[a, b)$; (ii) *f^* is left-continuous at b*; (iii) *$f^*(x-) = f(x-)$ for all $x \in (a, b]$*; (iv) $\|f^*\|_\infty \leqq \|f\|_\infty$; (v) *$f^*(x) = f(x)$ for all but denumerably many values of x.*

Proof. {The motivation for the definition of $f^*(b)$ is that, when considering possible discontinuities of f at the endpoints a, b, it is convenient to extend f to the interval $[a-1, b+1]$ by defining $f(x) = f(a+)$ on $[a-1, a)$ and $f(x) = f(b-)$ on $(b, b+1]$. The extended function will then have at most removable discontinuities at a, b, and f^* is continuous at these points.}

(iv) is obvious, and (v) is immediate from the fact that f is continuous at all but denumerably many points of I [2, Ch. II, §1, Th. 3].

(i) Let $c \in [a, b)$, $c < c_n < b$, $c_n \to c$. For each n, choose x_n, $c_n < x_n < b$, such that $|f(x_n) - f(c_n+)| < 1/n$ and $|x_n - c_n| < 1/n$; then $x_n \to c$, therefore $f(x_n) \to f(c+)$, therefore $|f(c+) - f(c_n+)| \to 0$.

(iii) is proved similarly, and implies (ii).

The mapping $P: \mathscr{R} \to \mathscr{R}$ defined by

$$(2) \qquad Pf = f^* \quad (f \in \mathscr{R})$$

is an algebra $*$-homomorphism, continuous for the sup-norm topology, idempotent ($P^2 = P$). We write

$$(3) \qquad \mathscr{R}^* = P(\mathscr{R})$$

for the range of P, and

$$(4) \qquad \mathscr{N} = \ker P$$

for the kernel of P; thus $\mathscr{N}$ is a closed ideal of $\mathscr{R}$, and $\mathscr{R}^* = \ker(1 - P)$ is a closed $*$-subalgebra of $\mathscr{R}$ that contains $\mathscr{C}$. {Remark: It is easy to see that $\|P\| = 1$ and $\|1 - P\| = 2$.} Every $f \in \mathscr{R}$ has a unique decomposition $f = g + h$ with $g \in \mathscr{R}^*$ (that is, $Pg = g$) and $h \in \mathscr{N}$ (that is, $Ph = 0$), namely $g = Pf$, $h = f - Pf$.

COROLLARY 1. (i) *$\mathscr{R}^*$ is the set of all $g \in \mathscr{R}$ such that g is right-continuous on $[a, b)$ and left-continuous at b;* (ii) *$\mathscr{N}$ is the set of all $h \in \mathscr{R}$ such that $h(x+) = 0$ for all $x \in [a, b)$ and $h(x-) = 0$ for all $x \in (a, b]$.*

Proof. This is clear from Proposition 1 and the definition of f^*.

From (ii) of Corollary 1, one sees that for $h \in \mathscr{N}$, the set of discontinuities of h is the set $\{x \in I: h(x) \neq 0\}$.

COROLLARY 2. (i) *$\mathscr{N}$ is the closed linear span in $\mathscr{R}$ of the characteristic functions $\varphi_{\{x\}}$, $x \in I$;* (ii) *$\mathscr{R}^*$ is the closed linear span in $\mathscr{R}$ of the characteristic functions φ_J, where J is a subinterval of I such that either $J = [c, d)$ with $d < b$, or $J = [c, b]$.*

Proof. (i) If $f = \varphi_J$, J a subinterval of I, then f^* can differ from f only at the endpoints of J; therefore if f is a step function, then $(1 - P)f = f - f^*$ vanishes at all but finitely many points (hence is a linear combination of functions $\varphi_{\{x\}}$). Suppose $h \in \mathscr{N}$. Choose a sequence of step functions f_n such that $f_n \to h$ uniformly in I; then $(1 - P)f_n \to (1 - P)h = h$ uniformly, whence (i).

(ii) Let $\mathscr{J}$ be the indicated set of subintervals of I; these are the subintervals J of I such that $\varphi_J \in \mathscr{R}^*$. Let $\mathscr{R}_0$ be the closed linear span of the φ_J, $J \in \mathscr{J}$; evidently $\mathscr{R}_0 \subset \mathscr{R}^*$. If f is the characteristic function of a subinterval of I with endpoints c, d, then f^* is the characteristic function of the interval in $\mathscr{J}$ with endpoints c, d; therefore $f^* \in \mathscr{R}_0$ whenever f is a step function. Suppose $g \in \mathscr{R}^*$. Choose a sequence of step functions f_n such that $f_n \to g$ uniformly in I; then $f_n^* \to g^* = g$ uniformly, whence $g \in \mathscr{R}_0$.

E. Hewitt studied the algebra of restrictions to $[a, b)$ of the functions in $\mathscr{R}^*$; identifying the two algebras in the obvious way, part (ii) of Corollary 2 corresponds to Theorem 4.5 of his paper [7, p. 87]. Here are some useful characterizations of $\mathscr{N}$:

COROLLARY 3. *The following conditions on a function $f: I \to C$ are equivalent:*

(a) $f \in \mathscr{N}$;
(b) $f \in \mathscr{R}$, $f(x+) = 0$ *for all* $x \in [a, b)$;
(b′) $f \in \mathscr{R}$, $f(x-) = 0$ *for all* $x \in (a, b]$;
(c) $f \in \mathscr{R}$ *and the set* $\{x \in I: f(x) \neq 0\}$ *is denumerable*;
(d) $f \in \mathscr{R}$ *and the set* $\{x \in I: f(x) = 0\}$ *is dense in* I;
(e) *there exist a sequence* $c_n \in I$ *and a sequence* $r_n \in C$ *with* $r_n \to 0$, *such that* $f(c_n) = r_n$ *for all* n *and* $f(x) = 0$ *for all other* x.

Proof. (a) $\Rightarrow$ (b), (b′) by Corollary 1.

(b) $\Rightarrow$ (c), (b′) $\Rightarrow$ (c): Clear from the fact that every $f \in \mathscr{R}$ is continuous at all but denumerably many points of I.

(c) $\Rightarrow$ (d): Obvious.

(d) $\Rightarrow$ (e): From (d), it is clear that (b), (b′) hold, hence also (c). To prove (e), it will suffice to show that, given any $\varepsilon > 0$, the set $\{x \in I: |f(x)| \geqq \varepsilon\}$ is finite. Assume to the contrary that $|f(x_n)| \geqq \varepsilon$ for a (faithfully indexed) infinite sequence x_n. Passing to a subsequence, we can suppose that x_n is convergent, say $x_n \to x$, and that either $x_n < x$ for all n or $x_n > x$ for all n; in either case, $f(x_n) \to 0$ by (b) or (b′), contrary to $|f(x_n)| \geqq \varepsilon$.

(e) $\Rightarrow$ (a): In the notations of (e), let $f_n = \sum_{k=1}^{n} r_k \varphi_{\{c_k\}}$; then $\|f - f_n\|_\infty \to 0$, therefore $f \in \mathscr{N}$ by (i) of Corollary 2.

Dually, one can define, for each $f \in \mathscr{R}$, a function ${}^*f \in \mathscr{R}$ by the formulas

$$(5) \qquad {}^*f(x) = f(x-) \quad \text{for} \quad x \in (a, b], \quad {}^*f(a) = f(a+).$$

Defining $Q: \mathscr{R} \to \mathscr{R}$ by

$$(6) \qquad Qf = {}^*f \quad (f \in \mathscr{R}),$$

one sees that Q has properties analogous to those of P; in view of the symmetry in Corollary 3 of Proposition 1, the kernel of Q is also $\mathscr{N}$; we write ${}^*\mathscr{R} = Q(\mathscr{R})$, which is the set of all $f \in \mathscr{R}$ that are left-continuous on $(a, b]$ and right-continuous at a. In view of Proposition 1 and its dual, inspection of the definitions yields the formulas

$$(7) \qquad PQ = P, \quad QP = Q.$$

{We remark that if $f \in \mathscr{R}$ is real-valued, then the upper and lower semicontinuous regularizations of f [1, Ch. IV, §6, n° 2], namely the functions $\bar{f} = \limsup f$, $\underline{f} = \liminf f$, are given by the formulas $\bar{f} = \max\{f, Pf, Qf\}$, $\underline{f} = \min\{f, Pf, Qf\}$, hence are obviously regulated. Writing $g = \bar{f}$, $h = f - \bar{f}$, one obtains a decomposition $f = g + h$ with $g \in \mathscr{R}$ upper semicontinuous and $h \in \mathscr{N}$ (by criterion (c) of Corollary 3 of Proposition 1). However, such decompositions are not unique, nor do they possess the algebraic properties enjoyed by the decompositions described earlier; they will play no role in the rest of the paper.}

3. **The character space of $\mathscr{R}$.** A *character* of a commutative Banach algebra $\mathscr{A}$ is an algebra epimorphism $\mathscr{A} \to \boldsymbol{C}$. The set $X(\mathscr{A})$ of all characters of $\mathscr{A}$, equipped with the topology of simple convergence in $\mathscr{A}$, is a locally compact space, called the *character space* of $\mathscr{A}$ (this topology on $X(\mathscr{A})$ is called the *Gelfand topology*); when $\mathscr{A}$ has a unity element, $X(\mathscr{A})$ is compact [4, Ch. I, §3]. Since $\mathscr{R}$ is a commutative C^*-algebra with unity, it is isomorphic to the algebra $\mathscr{C}(X(\mathscr{R}))$ of all continuous complex functions on the compact space $X(\mathscr{R})$ [4, Ch. I, §6, Th. 1]; our central objective is to explore the space $X(\mathscr{R})$.

THEOREM 1. *$X(\mathscr{R})$ is totally disconnected.*

Proof. Let $\mathscr{J}$ be the family of all subintervals J of I. The mapping $\beta \mapsto (\beta(\varphi_J))_{J \in \mathscr{J}}$ is a continuous mapping of $X(\mathscr{R})$ into a cartesian product $\mathscr{X}$ of copies of the discrete space $\{0, 1\}$; it is injective because $\mathscr{S}$ is dense in $\mathscr{R}$; since $X(\mathscr{R})$ is compact and $\mathscr{X}$ is separated, the mapping is a homeomorphism of $X(\mathscr{R})$ onto a subspace of the totally disconnected space $\mathscr{X}$.

THEOREM 2. *$X(\mathscr{R})$ does not have a denumerable base for open sets (hence is not metrizable).*

Proof. Since $\mathscr{C}(X(\mathscr{R}))$ is isomorphic to $\mathscr{R}$, it suffices to prove that the Banach algebra $\mathscr{R}$ is not separable [1, Ch. X, §3, Th. 1]. This is shown by the family of functions $f_x = \varphi_{\{x\}}$, $x \in I$, which satisfies $\|f_x - f_y\|_\infty = 1$ for $x \neq y$.

The closed open sets of $X(\mathscr{R})$ (there are lots of them, by Theorem 1) correspond, via the Gelfand isomorphism, to the idempotents in the algebra $\mathscr{R}$. The idempotents of $\mathscr{R}$ are all in $\mathscr{S}$:

THEOREM 3. *The idempotents of $\mathscr{R}$ are the characteristic functions φ_A, where A is the union of finitely many subintervals of I.*

Proof. Let $f \in \mathscr{R}$ be idempotent, say $f = \varphi_A$, $A \subset I$. Let $c \in [a, b)$ and consider open intervals $J \subset I$ with left endpoint c; if one had both $J \cap A \neq \varnothing$ and $J \cap (I - A) \neq \varnothing$ for every such J, then $f(c+)$ would fail to exist, a contradiction. Thus, for every $c \in [a, b)$, there exists an open interval $J_c \subset I$ with left endpoint c, such that either $J_c \subset A$ or $J_c \subset I - A$. Similarly, for each $c \in (a, b]$, there exists an open interval $K_c \subset I$ with right endpoint c, such that either $K_c \subset A$ or $K_c \subset I - A$. Defining

$$U_a = \{a\} \cup J_a,\ U_b = K_b \cup \{b\},\ U_c = K_c \cup \{c\} \cup J_c \quad \text{for} \quad c \in (a, b)\ ,$$

one has an open covering of the compact space I; say $I = U_{x_1} \cup U_{x_2} \cup \cdots \cup U_{x_n}$; then $A = (A \cap U_{x_1}) \cup \cdots \cup (A \cap U_{x_n})$, and each $A \cap U_{x_i}$ is either an interval or the union of two intervals.

COROLLARY. *In the lattice of closed open subsets of $X(\mathscr{R})$, there exists a denumerable family that does not have a supremum.*

Proof. Let c_n be a sequence in I such that $c_1 < c_2 < c_3 < \cdots$ and $c_n \to b$. Let $J_n = [c_n, c_{n+1})$ and $f_n = \varphi_{J_n}$. Assuming that the corollary is false, let f (resp. g) be the supremum of the f_n for n odd (resp. even). {Caution: We mean here supremum in the lattice sense, which must be distinguished from the pointwise supremum.} Say $f = \varphi_A$, $g = \varphi_B$. For $m \neq n$ one has $f_m f_n = 0$, thus $f_m \leqq 1 - f_n$; it follows that $fg = 0$, therefore $A \cap B = \varnothing$. By Theorem 3, A is the disjoint union of a finite number of intervals; since $A \supset J_n$ for all odd n, the rightmost nondegenerate interval in this representation of A has right endpoint b (which it need not contain). Similarly, B contains a nondegenerate interval with right endpoint b. Then $A \cap B$ contains a nondegenerate interval, contrary to $A \cap B = \varnothing$.

We write β, γ, δ for generic elements of $X(\mathscr{R})$; there are the obvious characters

$$\beta_x(f) = f(x) \quad \text{for} \quad f \in \mathscr{R}\ , \tag{8}$$

$$\delta_x(f) = f(x+) \quad \text{for} \quad f \in \mathscr{R}\ , \tag{9}$$

$$\gamma_x(f) = f(x-) \quad \text{for} \quad f \in \mathscr{R}\ , \tag{10}$$

defined for $x \in I$, $x \in [a, b)$ and $x \in (a, b]$, respectively; we shall see in Theorem 5 that there are no others.

THEOREM 4. *Define $\Psi: I \to X(\mathscr{R})$ by $\Psi(x) = \delta_x$ for $x \in [a, b)$ and $\Psi(b) = \gamma_b$. Then Ψ is right-continuous on $[a, b)$ and left-continuous at b.*

Proof. Let $f \in \mathscr{R}$. For all $x \in I$ one has $\Psi(x)(f) = f^*(x)$, thus the theorem is immediate from §2, Proposition 1.

THEOREM 5.

$$X(\mathscr{R}) = \{\beta_x : x \in [a, b]\} \cup \{\delta_x : x \in [a, b)\} \cup \{\gamma_x : x \in (a, b]\}\ .$$

Proof. Let $\beta \in X(\mathscr{R})$ and assume to the contrary that β does not have one of the three indicated forms, equivalently, that $\ker\beta$ is not equal to (hence is not contained in) $\ker\beta_x$, $\ker\delta_x$ or $\ker\gamma_x$, for any x. For each $x\in I$, choose $f_x\in\ker\beta$ with $f_x(x)\neq 0$; replacing f_x by $\bar{f}_x f_x$, we can suppose that $f_x\geqq 0$, $f_x(x)>0$. For each $x\in[a,b)$, choose $g_x\in\ker\beta$ with $g_x\geqq 0$ and $g_x(x+)>0$; let $J_x\subset I$ be an open interval with left endpoint x, such that g_x is bounded away from zero on J_x. Similarly, for each $x\in(a,b]$, there exist $h_x\in\ker\beta$, $h_x\geqq 0$, and an open interval $K_x\subset I$ with right endpoint x, such that h_x is bounded away from zero on K_x. For $x\in(a,b)$ define

$$V_x = K_x\cup\{x\}\cup J_x\,,\quad k_x = h_x + f_x + g_x\,,$$

and define $V_a=\{a\}\cup J_a$, $k_a=f_a+g_a$, $V_b=K_b\cup\{b\}$, $k_b=h_b+f_b$. For every $x\in I$, V_x is a neighborhood of x in I, and k_x is a positive function, belonging to $\ker\beta$, such that k_x is bounded away from zero on V_x. Say $I=V_{x_1}\cup\cdots\cup V_{x_n}$. Then the function $k=k_{x_1}+\cdots+k_{x_n}$ belongs to $\ker\beta$ and is bounded away from zero on I; it follows that $1/k$ is regulated, whence $1=(1/k)k\in\ker\beta$, which is absurd.

COROLLARY 1. *$X(\mathscr{R})$ has the cardinality of the continuum.*

We write α for generic elements of $X(\mathscr{C})$, specifically $\alpha_x(f)=f(x)$ for $x\in I$, $f\in\mathscr{C}$; the mapping $I\to X(\mathscr{C})$ defined by $x\mapsto\alpha_x$ is a homeomorphism [4, Ch. I, §3, Cor. 2 of Prop. 1]. If $\beta\in X(\mathscr{R})$, then $\beta|\mathscr{C}\in X(\mathscr{C})$, thus $\beta|\mathscr{C}=\alpha_x$ for a unique $x\in I$; defining $\Phi(\beta)=x$, we have a continuous surjection

$$\Phi\colon X(\mathscr{R})\longrightarrow I\,,\quad \alpha_{\Phi(\beta)}=\beta|\mathscr{C}\ \text{ for } \beta\in X(\mathscr{R})\,. \tag{11}$$

For the mapping Ψ of Theorem 4, one has $\Phi\circ\Psi=\mathrm{id}_I$; so to speak, Ψ is a right-continuous cross-section for Φ. {No continuous cross-section exists, since $X(\mathscr{R})$ contains no connected subset with more than one element.}

COROLLARY 2. *$\Phi^{-1}(\{a\})=\{\beta_a,\delta_a\}$, $\Phi^{-1}(\{b\})=\{\beta_b,\gamma_b\}$, and $\Phi^{-1}(\{x\})=\{\beta_x,\gamma_x,\delta_x\}$ for $x\in(a,b)$.*

COROLLARY 3. *If $\xi\in X(\mathscr{B})$, there exists $c\in I$ satisfying one of the following three conditions*: (i) *$\xi(f)=f(c)$ for all $f\in\mathscr{R}$*; (ii) *$\xi(f)=f(c+)$ for all $f\in\mathscr{R}$*; (iii) *$\xi(f)=f(c-)$ for all $f\in\mathscr{R}$.*

Proof. Let $\beta=\xi|\mathscr{R}$ and cite Theorem 5.

Here is another perspective on Corollary 3. Regard I_d (I with the discrete topology) as a dense subspace of its Stone-Čech compactification $X(\mathscr{B})$. Every bounded function $f: I \to C$ may be uniquely extended to a continuous function $\hat{f}: X(\mathscr{B}) \to C$. The message of Corollary 3: for each $\xi \in X(\mathscr{B})$, there exists $c \in I$ such that one of the following three conditions holds: (i) $\hat{f}(\xi) = f(c)$ for all $f \in \mathscr{R}$; (ii) $\hat{f}(\xi) = f(c+)$ for all $f \in \mathscr{R}$; (iii) $\hat{f}(\xi) = f(c-)$ for all $f \in \mathscr{R}$. {We remark that all three cases occur, since the mapping $X(\mathscr{B}) \to X(\mathscr{R})$ defined by restriction of characters is surjective [13, p. 126, (3.2.16)].}

COROLLARY 4. *For every $x \in I$, $\{\beta_x\}$ is an isolated point of $X(\mathscr{R})$.*

Proof. It is clear from Theorem 5 that

$$\{\beta_x\} = \{\beta \in X(\mathscr{R}): |\beta(\varphi_{\{x\}}) - \beta_x(\varphi_{\{x\}})| < 1\} .$$

Theorem 2 is an obvious consequence of Corollary 4. The proof of the next corollary employs the 'hull-kernel' characterization of the topology of $X(\mathscr{R})$. If $\mathscr{A}$ is any commutative Banach algebra, the Gelfand topology on $X(\mathscr{A})$ is finer than the hull-kernel topology; when the two topologies coincide, $\mathscr{A}$ is said to be completely regular. Every commutative C^*-algebra is completely regular [13, p. 174, (3.7.2)]. In particular, $\mathscr{R}$ is completely regular; thus, if S is any subset of $X(\mathscr{R})$, the closure of S is given by the formula

$$\bar{S} = hk(S) = \left\{\beta' \in X(\mathscr{R}): \ker \beta' \supset \bigcap_{\beta \in S} \ker \beta\right\} . \tag{12}$$

COROLLARY 5. *Let $X(\mathscr{R}) = E \cup F \cup G$ be the partition of $X(\mathscr{R})$ defined by*

$$E = \{\beta_x : x \in [a, b]\} , \quad F = \{\delta_x : x \in [a, b)\}, G = \{\gamma_x : x \in (a, b]\} . \tag{13}$$

Then E is a dense, open, discrete subspace of $X(\mathscr{R})$, and

$$\bar{F} = \bar{G} = F \cup G = X(\mathscr{R}) - E \tag{14}$$

is a perfect subset of $X(\mathscr{R})$ with empty interior.

Proof. That E is dense results from $\bigcap_{x \in I} \ker \beta_x = \{0\}$, thus it is clear from Corollary 4 that E has the properties claimed for it. From §2, Corollary 3 of Proposition 1, one sees that $k(F) = \mathscr{N} = k(G)$, therefore $\bar{F} = hk(F) = hk(G) = \bar{G}$; since $F \cup G = \complement E$ is closed, it follows that $F \cup G = \overline{F \cup G} = \bar{F} \cup \bar{G} = \bar{F} = \bar{G}$. Evidently $F \subset \bar{G}$ and $G \subset \bar{F}$, therefore $F \cup G$ is dense in itself. Since $F \cup G$ is disjoint from the dense set E, its interior is empty.

From Corollaries 4 and 5, one sees that a subset of $X(\mathscr{R})$ is dense if and only if it contains E, and that $X(\mathscr{R})$ has 2^c open subsets. The second assertion of Corollary 5 is an instance of a theorem of A. Pelczyinski and Z. Semadeni: a compact space T contains a non-empty perfect subset if and only if there exists a continuous surjection $T \to I$ ([12], [11, p. 29, Th. 2]). Incidentally, $X(\mathscr{R})$ is not the Stone-Čech compactification of its dense open subspace E; for, the Stone-Čech compactification of a discrete space is extremally disconnected [19, p. 300], and $X(\mathscr{R})$ is not extremally disconnected (by the corollary of Theorem 3 [cf. 17, p. 185, Th. 14]). {Alternative proof: $\mathscr{R}$ has c idempotents (Theorem 3), whereas $\mathscr{B}$ has 2^c idempotents, thus $\mathscr{R}$ and $\mathscr{B}$ are not isomorphic.} The fact that $X(\mathscr{R})$ contains a dense subspace E homeomorphic to I_d (I with the discrete topology) was predictable from [5, p. 225, Th. 1], since $\mathscr{R}$ contains the functions $\varphi_{\{x\}}$.

An explicit formula for the closure operation in $X(\mathscr{R})$ is given in Corollary 7; the following notations prepare the way. For a subset A of I, we write $\mathscr{I}_A$ for the kernel of the set $\{\beta_x : x \in A\}$, that is,

$$\mathscr{I}_A = \bigcap_{x \in A} \ker \beta_x = \{f \in \mathscr{R} :\ f \mid A = 0\}\ . \tag{15}$$

We write A'_+ for the set of $x \in I$ for which there exists a sequence $x_n \in A$ such that $x_n > x$ and $x_n \to x$, thus

$$A'_+ = \{x \in I:\ x \in \overline{(x, +\infty) \cap A}\}\ ; \tag{16}$$

this is, so to speak, the 'right derived set' of A. Similarly, we write

$$A'_- = \{x \in I:\ x \in \overline{(-\infty, x) \cap A}\}\ . \tag{17}$$

Then $A'_+ \cup A'_- = A'$, the usual derived set of A. One has $(A \cup B)'_+ = A'_+ \cup B'_+$, $(A \cup B)'_- = A'_- \cup B'_-$. If J is a subinterval of I with endpoints c, d ($c \leqq d$), then $J'_+ = [c, d)$ and $J'_- = (c, d]$.

LEMMA 1. *Let* $A \subset I$, $S = \{\beta_x : x \in A\}$, $c \in I$. *Then*: (i) $\beta_c \in \bar{S}$ *if and only if* $c \in A$; (ii) $\delta_c \in \bar{S}$ *if and only if* $c \in A'_+$; (iii) $\gamma_c \in \bar{S}$ *if and only if* $c \in A'_-$.

Proof. (ii) Suppose $\delta_c \in \bar{S}$ (in particular, $c < b$), that is, $\ker \delta_c \supset k(S) = \mathscr{I}_A$. Assuming to the contrary that $c \notin A'_+$, choose an open interval $J \subset I$ with left endpoint c, such that $A \cap J = \varnothing$; then $\varphi_J \mid A = 0$ but $\varphi_J(c+) = 1$, contrary to the supposition that $\mathscr{I}_A \subset \ker \delta_c$. Conversely, suppose $c \in A'_+$ and choose a sequence $c_n \in A$ such that $c_n > c$ and $c_n \to c$. For every $f \in \mathscr{R}$, $f(c_n) \to f(c+)$; if, moreover, $f \in \mathscr{I}_A$, then $f(c_n) = 0$ for all n, whence $f(c+) = 0$, $f \in \ker \delta_c$.

(iii) is proved similarly, and (i) is obvious.

LEMMA 2. *Let* $A \subset I$, $S = \{\beta_x : x \in A\}$. *Then* $\bar{S} = S \cup \{\delta_x : x \in A'_+\} \cup \{\gamma_x : x \in A'_-\}$.

Proof. Immediate from Lemma 1 and Theorem 5.

COROLLARY 6. *The closed open subsets of* $X(\mathscr{R})$ *are the sets of the form* $\bar{S}_A$, *where* A *is the union of finitely many subintervals of* I *and* $S_A = \{\beta_x : x \in A\}$.

Proof. The closed open sets in $X(\mathscr{R})$ correspond to the idempotents of $\mathscr{R}$ via the Gelfand isomorphism. By Theorem 3, the idempotents of $\mathscr{R}$ are the functions φ_A, A the union of finitely many subintervals of I. For such an A, the closed open subset of $X(\mathscr{R})$ corresponding to φ_A is the set $U(A) = \{\beta \in X(\mathscr{R}) : \beta(\varphi_A) = 1\}$; evidently $\beta_x(\varphi_A) = 1$ if and only if $x \in A$; $\delta_x(\varphi_A) = 1$ if and only if $x \in A'_+$; and $\gamma_x(\varphi_A) = 1$ if and only if $x \in A'_-$; therefore

$$U(A) = \{\beta_x : x \in A\} \cup \{\delta_x : x \in A'_+\} \cup \{\gamma_x : x \in A'_-\} ,$$

thus $U(A) = \bar{S}_A$ by Lemma 2.

From Corollary 6 (or Theorem 3) one sees that $X(\mathscr{R})$ has c closed open sets; a closed open subset of $X(\mathscr{R})$ either is a finite subset of E or has cardinality c.

LEMMA 3. *Let* $B \subset [a, b)$, $S = \{\delta_x : x \in B\}$. *Then* $\bar{S} \cap F = S \cup \{\delta_x : x \in B'_+\}$.

Proof. Let $x \in [a, b)$; the problem is to show that $\delta_x \in \bar{S} \Leftrightarrow x \in B \cup B'_+$.

Proof of $\Rightarrow$: Suppose $x \notin B \cup B'_+$. Since $x < b$ and $x \notin B'_+$ there exists y, $x < y < b$, such that $B \cap (x, y) = \emptyset$. Since $x \notin B$, also $B \cap [x, y) = \emptyset$. Let $J = [x, y)$, $f = \varphi_J$. If $t \in B$ then $t < x$ or $t \geqq y$, and in either case $f(t+) = 0$; thus $f \in \bigcap_{t \in B} \ker \delta_t = k(S)$. But $f(x+) = 1$, thus $\ker \delta_x \not\supset k(S)$, that is, $\delta_x \notin \bar{S}$.

Proof of $\Leftarrow$: Suppose $x \in B'_+$. Choose a sequence $x_n \in B$ such that $x_n > x$ and $x_n \to x$. Then $\delta_{x_n} \to \delta_x$ by Theorem 4, thus $\delta_x \in \bar{S}$.

LEMMA 4. *Let* $B \subset [a, b)$, $S = \{\delta_x : x \in B\}$. *Then* $\bar{S} \cap G = \{\gamma_x : x \in B'_-\}$.

Proof. Let $x \in (a, b]$; the problem is to show that $\gamma_x \in \bar{S} \Leftrightarrow x \in B'_-$.

Proof of $\Rightarrow$: Suppose $x \notin B'_-$. Then, since $a < x$, there exists

y, $a < y < x$, such that the interval $J = [y, x)$ is disjoint from B. Let $f = \varphi_J$. If $t \in B$ then $t < y$ or $t \geqq x$, and in either case $f(t+) = 0$; thus $f \in k(S)$. But $f(x-) = 1$, thus $\ker \gamma_x \not\supset k(S)$, that is, $\gamma_x \notin \bar{S}$.

Proof of $\Leftarrow$: Suppose $x \in B'_-$. Choose a sequence $x_n \in B$ such that $x_n < x$, $x_n \rightarrow x$. If $f \in k(S)$ then $f(x_n+) = 0$ for all n; thus, in the notation of §2, Proposition 1, one has $f^*(x_n) = 0$ and $\gamma_x(f) = f(x-) = f^*(x-) = \lim f^*(x_n) = 0$. Thus $\ker \gamma_x \supset k(S)$, that is, $\gamma_x \in \bar{S}$.

LEMMA 5. *Let* $B \subset [a, b)$, $S = \{\delta_x : x \in B\}$. *Then*

$$\bar{S} = S \cup \{\delta_x : x \in B'_+\} \cup \{\gamma_x : x \in B'_-\} .$$

Proof. Since $S \subset F$, one has $\bar{S} \subset \bar{F} = F \cup G$, therefore $\bar{S} = (\bar{S} \cap F) \cup (\bar{S} \cap G)$; cite Lemmas 3 and 4.

By dual arguments one shows:

LEMMA 6. *Let* $C \subset (a, b]$, $S = \{\gamma_x : x \in C\}$. *Then*

$$\bar{S} = S \cup \{\gamma_x : x \in C'_-\} \cup \{\delta_x : x \in C'_+\} .$$

COROLLARY 7. *Given any set* $S \subset X(\mathscr{R})$, *say*

$$S = \{\beta_x : x \in A\} \cup \{\delta_x : x \in B\} \cup \{\gamma_x : x \in C\} ,$$

where $A \subset [a, b]$, $B \subset [a, b)$, $C \subset (a, b]$. *Then*

$$\bar{S} = S \cup \{\delta_x : x \in (A \cup B \cup C)'_+\} \cup \{\gamma_x : x \in (A \cup B \cup C)'_-\} .$$

In particular, S *is closed if and only if* $(A \cup B \cup C)'_+ \subset B$ *and* $(A \cup B \cup C)'_- \subset C$.

Proof. This is immediate from Lemmas 2, 5, and 6, and the 'additivity' of the derived set operations.

Here is an amusing consequence of Corollaries 6 and 7. Let $A \subset I$. In order that A be the union of finitely many intervals, it is necessary and sufficient that there exist partitions $I = A_1 \cup A_2$, $[a, b) = B_1 \cup B_2$, $(a, b] = C_1 \cup C_2$, with $A = A_1$, such that $(A_i \cup B_i \cup C_i)'_+ \subset B_i$ and $(A_i \cup B_i \cup C_i)'_- \subset C_i$ for $i = 1, 2$.

The next proposition is for application in §6. Let $\Phi : X(\mathscr{R}) \rightarrow I$ be the mapping (11); let $\beta \sim \beta'$ denote the equivalence relation defined in $X(\mathscr{R})$ by $\Phi(\beta) = \Phi(\beta')$ (that is, by $\beta | \mathscr{C} = \beta' | \mathscr{C}$), write $\Lambda : X(\mathscr{R}) \rightarrow X(\mathscr{R})/\sim$ for the quotient mapping, and equip $X(\mathscr{R})/\sim$ with the quotient topology. The equivalence classes for $\sim$ are the sets $\Phi^{-1}(\{x\})$,

$x \in I$ (cf. Cor. 2 of Th. 5). Let $\Phi' \colon X(\mathscr{R})/\!\sim\, \to I$ be the continuous bijection derived from Φ by passage to quotients (thus $\Phi = \Phi' \circ \Lambda$); it follows from compactness that Φ' is a homeomorphism [1, Ch. I, §9, Cor. 2 of Th. 2]. Finally, if $\Gamma \colon \mathscr{R} \to \mathscr{C}(X(\mathscr{R}))$ is the Gelfand isomorphism, then for every $f \in \mathscr{C} = \mathscr{C}(I)$, one has $\Gamma f = f \circ \Phi$. {Proof: For all $\beta \in X(\mathscr{R})$, $(\Gamma f)(\beta) = \beta(f) = (\beta \mid \mathscr{C})(f) = \alpha_{\Phi(\beta)}(f) = f(\Phi(\beta)) = (f \circ \Phi)(\beta)$.}

PROPOSITION 2. *For a function $u \colon X(\mathscr{R}) \to \boldsymbol{C}$, the following conditions are equivalent*:

(a) *$u = f \circ \Phi$ for some continuous $f \colon I \to \boldsymbol{C}$*;

(b) *$u = \Gamma g$ for some continuous $g \colon I \to \boldsymbol{C}$*;

(c) *u is continuous, and is constant on each of the sets $\Phi^{-1}(\{x\})$, $x \in I$.*

In the notations of (a) *and* (b), *necessarily $f = g$.*

Proof. The remark immediately preceding the proposition proves the equivalence of (a) and (b), as well as the final assertion of the proposition. It is obvious that (a) implies (c).

(c) $\Rightarrow$ (*a*): Since u is constant on each set $\Phi^{-1}(\{x\})$, there exists a factorization $u = f \circ \Phi$ with $f \colon I \to \boldsymbol{C}$; from the continuity of u, we are to infer the continuity of f. Since $(f \circ \Phi') \circ \Lambda = f \circ (\Phi' \circ \Lambda) = f \circ \Phi = u$ is continuous, and since $X(\mathscr{R})/\!\sim$ bears the quotient topology, it follows that $f \circ \Phi'$ is continuous; but Φ' is a homeomorphism, therefore f is continuous.

4. The character spaces of $\mathscr{R}^*$ and $\mathscr{N}$. The notations P, Q, $\mathscr{N}$, $\mathscr{R}^*$, ${}^*\mathscr{R}$ have the same meanings as in §2, and $X(\mathscr{R}) = E \cup F \cup G$ is the partition described in §3, Corollary 5 of Theorem 5. The commutative C^*-algebras $\mathscr{R}$, $\mathscr{N}$, $\mathscr{R}^*$ are related by the isomorphism $\mathscr{R}^* \cong \mathscr{R}/\mathscr{N}$; the relation between their character spaces is given by the following general lemma [cf. 13, p. 193, Th. 4.2.4]:

LEMMA. *If T is a compact space, S is a closed subset of T, and $\mathscr{J}$ is the ideal of $\mathscr{C}(T)$ annihilating S, then $\mathscr{C}(T)/\mathscr{J} \cong \mathscr{C}(S)$ and $\mathscr{J} \cong \mathscr{C}_0(T - S)$, therefore $X(\mathscr{C}(T)/\mathscr{J}) = S$ and $X(\mathscr{J}) = T - S$.*

Proof. Consider the mappings $\mathscr{C}(T) \to \mathscr{C}(S)$ and $\mathscr{J} \to \mathscr{C}_0(T - S)$ defined by $f \mapsto f \mid S$ and $f \mapsto f \mid T - S$ and cite, respectively, the Tietze extension theorem and the Stone-Weierstrass theorem. {Here $\mathscr{C}_0(T - S)$ denotes the algebra of continuous functions vanishing at infinity on the locally compact space $T - S$.}

The first assertion of the following theorem was proved by E. Hewitt [7, p. 91, Th. 5.1]:

THEOREM 6. *$X(\mathscr{R}^*)$ is homeomorphic to $F \cup G$; $X(\mathscr{N})$ is homeomorphic to E, that is, to I_d (I equipped with the discrete topology).*

Proof. Identifying $\mathscr{R}$ (via the Gelfand isomorphism) with $\mathscr{C}(X(\mathscr{R}))$, $\mathscr{N}$ is the ideal of $\mathscr{R}$ annihilating the closed set $F \cup G$ (§2, Cor. 1 of Prop. 1). Cite the lemma. {It is easy to see that the homeomorphisms $E \to X(\mathscr{N})$, $F \cup G \to X(\mathscr{R}^*)$ are effected by restriction of characters. Note, incidentally, that $\delta_x | \mathscr{R}^* = \beta_x | \mathscr{R}^*$ for $x \in [a, b)$, and $\gamma_b | \mathscr{R}^* = \beta_b | \mathscr{R}^*$.}

The same reasoning shows that $X({}^*\mathscr{R})$ is homeomorphic to $F \cup G$. This reminds us that ${}^*\mathscr{R}$ and $\mathscr{R}^*$ are isomorphic; indeed, it follows from the relations $\mathscr{R}^* = P(\mathscr{R}) = P({}^*\mathscr{R} + \mathscr{N}) = P({}^*\mathscr{R})$ that P effects an isomorphism of ${}^*\mathscr{R}$ onto $\mathscr{R}^*$. In view of Theorem 6, the message of §2, Corollary 3 of Proposition 1 is that $\mathscr{N} = \mathscr{C}_0(I_d)$.

5. The character space of $\mathscr{B}\mathscr{V}$. It will be shown that $\mathscr{B}\mathscr{V}$ has the same character space as $\mathscr{R}$; first, we see that $\mathscr{B}\mathscr{V}$ can be normed to be a Banach algebra [13, p. 302]:

LEMMA. *Equipped with pointwise operations and the norm*

$$\|f\| = V_a^b f + \|f\|_\infty , \tag{18}$$

$\mathscr{B}\mathscr{V}$ is a commutative, involutive Banach algebra with unity.

Proof. Here $V_a^b f$ denotes the total variation of f in I [8, p. 266, (17.14)]. Let $f, g \in \mathscr{B}\mathscr{V}$; from the identity

$$(fg)(x) - (fg)(y) = f(x)[g(x) - g(y)] + [f(x) - f(y)]g(y)$$

one sees that $V_a^b(fg) \leqq \|f\|_\infty V_a^b g + \|g\|_\infty V_a^b f$ (this shows, in particular, that $fg \in \mathscr{B}\mathscr{V}$); it follows that the norm (18) satisfies $\|fg\| \leqq \|f\| \, \|g\|$. Completeness for this norm is easily deduced from [9, p. 43, Th. 8.6].

THEOREM 7. *The mapping $X(\mathscr{R}) \to X(\mathscr{B}\mathscr{V})$ defined by $\beta \mapsto \beta | \mathscr{B}\mathscr{V}$ is a homeomorphism.*

Proof. The mapping is continuous (for the Gelfand topologies) and injective (e.g., because $\mathscr{B}\mathscr{V}$ is dense in $\mathscr{R}$ for the sup-norm topology). It will suffice (by compactness) to show that it is surjective. Let $\varepsilon \in X(\mathscr{B}\mathscr{V})$ and suppose to the contrary that $\varepsilon \neq \beta | \mathscr{B}\mathscr{V}$, equivalently $\ker \varepsilon \not\subset \ker \beta$, for all $\beta \in X(\mathscr{R})$. Arguing exactly as in the proof of Theorem 5, one constructs $k \in \ker \varepsilon$ such that k is bounded away from zero in I. It is then immediate from the identity

$$(1/k)(x) - (1/k)(y) = [k(y) - k(x)]/k(x)k(y)$$

that $1/k \in \mathscr{BV}$, whence $1 = (1/k)k \in \ker \varepsilon$, which is absurd.

It follows that the closure operation in $X(\mathscr{BV})$ is also given by the formula in Corollary 7 of Theorem 5.

THEOREM 8. *$\mathscr{BV}$ is completely regular.*

Proof. As is true for every commutative Banach algebra with unity, $X(\mathscr{BV})$ is compact for the Gelfand topology and quasicompact for the hull-kernel topology [4, Ch. I, §1, Prop. 4], and the Gelfand topology is finer than the hull-kernel topology; to prove that the two topologies coincide, it will suffice to show that the hull-kernel topology is separated. To this end, it suffices [6, p. 111, 7M] to show that if $\varepsilon_1, \varepsilon_2$ are distinct characters of $\mathscr{BV}$, then there exist functions f_1, f_2 in $\mathscr{BV}$ such that $\varepsilon_1(f_1) = \varepsilon_2(f_2) = 1$ and $f_1 f_2 = 0$. In view of Theorem 7, one is reduced to the consideration of a small number of cases. For example, suppose $\varepsilon_1 = \beta_x | \mathscr{BV}$ and $\varepsilon_2 = \delta_y | \mathscr{BV}$; if $x \leqq y$, take $f_1 = \varphi_{\{x\}}$ and $f_2 = \varphi_{(y,b)}$; if $x > y$, take $f_1 = \varphi_{\{x\}}$ and $f_2 = \varphi_{(y,x)}$. The remaining cases are equally transparent. {We remark that $\mathscr{BV}$ is not isomorphic (as an algebra) to a C^*-algebra; for, the spectral radius of $f \in \mathscr{BV}$ is $\|f\|_\infty$, and $\mathscr{BV}$ is not complete for this norm.}

An example due to G. Šilov [cf. 13, p. 302, A.2.5] fits into the present circle of ideas. Let us write $\mathscr{BV}_c$ for the subalgebra $\mathscr{C} \cap \mathscr{BV}$ of $\mathscr{BV}$. It is elementary that $\mathscr{BV}_c$ is closed in $\mathscr{BV}$ for the norm (18), hence is a Banach algebra for this norm.

THEOREM 9 (Šilov). *$X(\mathscr{BV}_c)$ is homeomorphic to $X(\mathscr{C})$ (that is, to I).*

Proof. Recall that $X(\mathscr{C}) = \{\alpha_x : x \in I\}$, where $\alpha_x(f) = f(x)$ for $f \in \mathscr{C}$. The mapping $X(\mathscr{C}) \to X(\mathscr{BV}_c)$ defined by $\alpha \mapsto \alpha | \mathscr{BV}_c$ is continuous and injective; to prove that it is a homeomorphism, it suffices to show that it is surjective. Let $\varepsilon \in X(\mathscr{BV}_c)$ and assume to the contrary that $\varepsilon \neq \alpha_x | \mathscr{BV}_c$ for all $x \in I$, that is, $\ker \varepsilon \not\subset \ker \alpha_x$ for all $x \in I$. Repeating a classical argument (simpler than that in Theorem 5), one constructs a function $k \in \ker \varepsilon$ such that k vanishes at no point of I; since k is continuous, it is bounded away from zero; therefore $1/k \in \mathscr{BV}_c$, whence $1 = (1/k)k \in \ker \varepsilon$, which is absurd. {Incidentally, by a simplification of the argument in Theorem 8, one sees that $\mathscr{BV}_c$ is completely regular [13, p. 302]. (There is only one case to consider: $\varepsilon_1 = \alpha_x | \mathscr{BV}_c$, $\varepsilon_2 = \alpha_y | \mathscr{BV}_c$ with $x \neq y$.)}

Next, we study the decomposition $f = Pf + (1 - P)f$ of §2 for functions of bounded variation.

LEMMA 1. *If* $f \in \mathscr{BV}$ *then* $Pf \in \mathscr{BV}$ *and* $V_a^b(Pf) \leqq V_a^b f$.

Proof. Let $f \in \mathscr{BV}$ and let $a = x_0 < x_1 < \cdots < x_n = b$ be any subdivision of I; we are to show that $\sum_{i=1}^n |f^*(x_i) - f^*(x_{i-1})| \leqq V_a^b f$. Choose points y_i, $1 \leqq i \leqq n + 1$, such that $x_{i-1} < y_i < x_i$ for $1 \leqq i \leqq n - 1$ and $x_{n-1} < y_n < y_{n+1} < x_n = b$. Then $V_a^b f \geqq \sum_{i=2}^{n+1} |f(y_i) - f(y_{i-1})|$; the desired inequality results on letting $y_i \to x_{i-1}$ for $1 \leqq i \leqq n$ and $y_{n+1} \to b$.

It follows from Lemma 1 that $P(\mathscr{BV}) = \mathscr{R}^* \cap \mathscr{BV}$, and that $\|Pf\| \leqq \|f\|$ for all $f \in \mathscr{BV}$; the restriction of P to $\mathscr{BV}$ is an algebra $*$-homomorphism $P_0: \mathscr{BV} \to \mathscr{BV}$, continuous for the norm $\| \ \|$, idempotent; therefore, in the Banach algebra $(\mathscr{BV}, \| \ \|)$, $\ker P_0 = \mathscr{N} \cap \mathscr{BV}$ is a closed ideal and $P_0(\mathscr{BV}) = \mathscr{R}^* \cap \mathscr{BV}$ is a closed $*$-subalgebra. Passing to quotients, one has an isomorphism

$$\mathscr{BV}/(\mathscr{N} \cap \mathscr{BV}) \cong \mathscr{R}^* \cap \mathscr{BV} ;$$

it follows from Theorem 8 that the algebras $\mathscr{N} \cap \mathscr{BV}$ and $\mathscr{R}^* \cap \mathscr{BV}$ are also completely regular [13, p. 84, (2.7.2)]. {We remark that the upper and lower semicontinuous regularizations of a real-valued function in $\mathscr{BV}$ are also in $\mathscr{BV}$; this is immediate from the formulas mentioned at the end of §2.} Here are some useful characterizations of $\mathscr{N} \cap \mathscr{BV}$:

LEMMA 2. *The following conditions on a function* $f: I \to \boldsymbol{C}$ *are equivalent*:

(a) $f \in l^1(I)$, *that is*, $\|f\|_1 = \sum_{x \in I} |f(x)| < +\infty$;

(b) $f \in \mathscr{N} \cap \mathscr{BV}$;

(c) $f \in \mathscr{BV}$ *and the set* $\{x \in I: f(x) \neq 0\}$ *is denumerable*;

(d) $f \in \mathscr{BV}$ *and the set* $\{x \in I: f(x) = 0\}$ *is dense in* I.

For such a function f *one has*

$$\|f\|_1 \leqq V_a^b f \leqq 2\|f\|_1 \ . \tag{19}$$

Proof. (a) $\Rightarrow$ (b): If $f \in l^1(I)$, it is obvious that $V_a^b f \leqq 2\|f\|_1 < +\infty$, therefore $f \in \mathscr{BV} \subset \mathscr{R}$, and the set $\{x \in I: f(x) \neq 0\}$ is denumerable, thus $f \in \mathscr{N}$ by §2, Corollary 3 of Proposition 1.

(b) $\Rightarrow$ (c) is immediate from §2, Corollary 3 of Proposition 1, and (c) $\Rightarrow$ (d) is obvious.

(d) $\Rightarrow$ (a): Let $f \in \mathscr{BV}$ and suppose that the set $D = \{x \in I: f(x) = 0\}$ is dense in I. Consider any finite subset of I, say $x_1 < x_2 < \cdots < x_n$.

Choose $y_1, \cdots, y_n \in D$ such that $y_i \in (x_i, x_{i+1})$ for $1 \leqq i \leqq n-2$ and $x_{n-1} < y_{n-1} < y_n < x_n$; then

$$V_a^b f \geqq \sum_{i=1}^{n-1} |f(x_i) - f(y_i)| + |f(y_n) - f(x_n)| = \sum_{i=1}^{n} |f(x_i)| ,$$

thus $f \in l^1(I)$ and $\|f\|_1 \leqq V_a^b f$.

LEMMA 3. *$\mathscr{N} \cap \mathscr{BV}$ is the closed linear span in $\mathscr{BV}$ of the functions $\varphi_{\{x\}}$, $x \in I$, for the norm* (18).

Proof. Let $\mathscr{F}$ be the linear span of the functions $\varphi_{\{x\}}$, $x \in I$. Since $\mathscr{N} \cap \mathscr{BV}$ is closed for the norm $\|\ \|$ (remarks following Lemma 1), it contains the closure of $\mathscr{F}$. Conversely, suppose $f \in \mathscr{N} \cap \mathscr{BV}$. By Lemma 2, there exists a sequence $f_n \in \mathscr{F}$ such that $\|f_n - f\|_1 \to 0$, and $V_a^b(f_n - f) \to 0$ by (19); since $\|f_n - f\|_\infty \leqq \|f_n - f\|_1$, it follows that $\|f_n - f\| \to 0$.

THEOREM 10. (i) *$X(\mathscr{N} \cap \mathscr{BV})$ is homeomorphic to E, that is, to I_d (I with the discrete topology)*; (ii) *$X(\mathscr{R}^* \cap \mathscr{BV})$ is homeomorphic to $F \cup G$.*

Proof. (i) {To put it less obliquely, $X(\mathscr{N} \cap \mathscr{BV})$ is discrete and consists of the point evaluations.} Suppose $\varepsilon \in X(\mathscr{N} \cap \mathscr{BV})$; by Lemma 3, there exists $x \in I$ such that $\varepsilon(\varphi_{\{x\}}) = 1$; for $y \neq x$, $1 + \varepsilon(\varphi_{\{y\}}) = \varepsilon(\varphi_{\{x\}} + \varphi_{\{y\}}) = \varepsilon(\varphi_{\{x,y\}})$ is 0 or 1, therefore $\varepsilon(\varphi_{\{y\}}) = 0$; it follows from Lemma 3 that $\varepsilon = \beta_x | \mathscr{N} \cap \mathscr{BV}$. Thus, the mapping $E \to X(\mathscr{N} \cap \mathscr{BV})$ defined by $\beta \mapsto \beta | \mathscr{N} \cap \mathscr{BV}$ is bijective, and $X(\mathscr{N} \cap \mathscr{BV})$ is discrete by the proof of §3, Corollary 4 of Theorem 5.

(ii) If $\varepsilon \in X(\mathscr{R}^* \cap \mathscr{BV})$ then $\varepsilon \circ P_0 \in X(\mathscr{BV})$, therefore (Theorem 7) there exists $\beta \in F \cup G$ such that $\beta | \mathscr{BV} = \varepsilon \circ P_0$, whence $\beta | \mathscr{R}^* \cap \mathscr{BV} = \varepsilon$. Thus, the continuous mapping $F \cup G \to X(\mathscr{R}^* \cap \mathscr{BV})$ defined by restriction of characters is surjective; to show that it is a homeomorphism, it suffices to show that it is injective. Let $\beta_1, \beta_2 \in F \cup G$, $\beta_1 \neq \beta_2$; we seek $f \in \mathscr{R}^* \cap \mathscr{BV}$ such that $\beta_1(f) \neq \beta_2(f)$. If β_1, β_2 are induced by the same point x of I, say $\beta_1 = \gamma_x$ and $\beta_2 = \delta_x$, then $a < x < b$ and we may take $f = \varphi_{[a,x)}$. Suppose β_1, β_2 are induced by distinct points x, y of I, $x < y$; if $\beta_1 = \gamma_x$ (and $\beta_2 = \gamma_y$ or δ_y), take $f = \varphi_{[a,x)}$; if $\beta_1 = \delta_x$ (and $\beta_2 = \gamma_y$ or δ_y), take $f = \varphi_{[x,r)}$ for any r such that $x < r < y$.

6. The dual spaces of $\mathscr{R}$, $\mathscr{R}^*$, $\mathscr{N}$. The dual space $\mathscr{R}'$ of $\mathscr{R}$ has been calculated by H. S. Kaltenborn [10]; we review here his formula for the continuous linear forms on $\mathscr{R}$. To conform to his notations, we write T for elements of $\mathscr{R}'$, χ for elements of $\mathscr{BV}$, and ϕ for elements of $l^1(I)$ (by §5, Lemma 2, these are the functions in $\mathscr{N} \cap \mathscr{BV}$). Each $\chi \in \mathscr{BV}$ defines $T_\chi \in \mathscr{R}'$ by the formula

$$T_\chi f = \int_a^b f d\chi \quad \text{for} \quad f \in \mathscr{R}, \tag{20}$$

where the right side is the 'modified Stieltjes integral' whose existence was proved by B. Dushnik (see [10]): it is the limit, in the sense of refinement of subdivisions $a = x_0 < x_1 < \cdots < x_n = b$ of I, of the sums

$$\sum_{i=1}^{n} f(\xi_i)[\chi(x_i) - \chi(x_{i-1})] ,$$

where ξ_i is required to be a point of the open interval (x_{i-1}, x_i). On the other hand, if $\phi \in l^1(I)$ then the formula

$$T_\phi f = \sum_{x \in I} [f(x) - f(x+)]\phi(x) \quad \text{for} \quad f \in \mathscr{R} \tag{21}$$

defines $T_\phi \in \mathscr{R}'$. {Note that $f(b+) = f(b-)$ by the convention pertaining to the definition of f on $(b, b+1]$: see the remarks in the proof of §2, Proposition 1.} Caution: The subscript on T determines the formula of definition. The following lemma is readily verified:

LEMMA. (i) *If* $\chi \in \mathscr{BV}$ *then* $T_\chi | \mathscr{N} = 0$ *and* $\| T_\chi \| = V_a^b \chi = \| T_\chi | \mathscr{R}^* \|$. (ii) *If* $\phi \in l^1(I)$ *then* $T_\phi | \mathscr{R}^* = 0$, $\| T_\phi \| = 2 \| \phi \|_1$ *and* $\| T_\phi | \mathscr{N} \| = \| \phi \|_1$.

THEOREM 11 (Kaltenborn). *Every* $T \in \mathscr{R}'$ *has a representation* $T = T_\chi + T_\phi$, *where* $\chi \in \mathscr{BV}$ *and* $\phi \in l^1(I)$; ϕ *is unique and* χ *is unique up to an additive constant.*

If, in Theorem 11, one requires that $\chi \in \mathscr{BV}_0$ (the functions in $\mathscr{BV}$ that vanish at a), then χ is uniquely determined by T. Note that $V_a^b \chi$ is a norm on $\mathscr{BV}_0$.

COROLLARY 1. $(\mathscr{R}^*)' = \mathscr{BV}_0$.

Proof. If $\chi \in \mathscr{BV}_0$ then $T_\chi | \mathscr{R}^* \in (\mathscr{R}^*)'$ and $\| T_\chi | \mathscr{R}^* \| = V_a^b \chi$ by the lemma. Conversely, if $\lambda \in (\mathscr{R}^*)'$ then $T = \lambda \circ P \in \mathscr{R}'$ and $T | \mathscr{N} = 0$; therefore by Kaltenborn's theorem there exists a unique $\chi \in \mathscr{BV}_0$ such that $T = T_\chi$, whence $T_\chi | \mathscr{R}^* = \lambda$. {In particular, $\mathscr{BV}_0$ is a Banach space for the norm $V_a^b \chi$ [8, p. 271, (17.35)].}

E. Hewitt has shown that $(\mathscr{R}^*)'$ may be represented as the space of all bounded, finitely additive measures defined on the ring of subsets of I generated by the intervals of the form $[c, d)$, $a \leqq c < d \leqq b$ [7, p. 90, Th. 4.10].

COROLLARY 2. $\mathscr{N}' = l^1(I)$.

Proof. If $\phi \in l^1(I)$ then $T_\phi | \mathscr{N} \in \mathscr{N}'$ and $\| T_\phi | \mathscr{N} \| = \|\phi\|_1$. Conversely, if $\mu \in \mathscr{N}'$ then $T = \mu \circ (1 - P) \in \mathscr{R}'$ and $T | \mathscr{R}^* = 0$; therefore by Kaltenborn's theorem there exists a unique $\phi \in l^1(I)$ such that $T = T_\phi$, whence $T_\phi | \mathscr{N} = \mu$. {Alternate proof: Since $\mathscr{N} = \mathscr{C}_0(I_d)$ (§2, Cor. 3 of Prop. 1), it is elementary that $\mathscr{N}' = l^1(I)$.}

Thus, the structure of $\mathscr{R}'$ is as follows: given $\lambda \in (\mathscr{R}^*)'$ and $\mu \in \mathscr{N}'$, one forms $T = \lambda \circ P + \mu \circ (1 - P)$; this is the general element of $\mathscr{R}'$. One can also deduce Theorem 5 from Kaltenborn's formula; the computations are tedious but thoroughly elementary (nothing so fancy as the Gelfand theory is needed).

Since $\mathscr{R}$ is isomorphic to $\mathscr{C}(X(\mathscr{R}))$, $\mathscr{R}'$ may be identified with the space $\mathscr{M}(X(\mathscr{R}))$ of measures on $X(\mathscr{R})$. The rest of the paper is devoted to the measure-theoretic description of $\mathscr{R}'$.

By *measure* we mean complex (Radon) measure [3, Ch. III, §1, Def. 2] (equivalently, complex regular Borel measure). The measures on I are the Lebesgue-Stieltjes measures induced by functions of bounded variation ([8, p. 331, (19.48)], [14, p. 263]). The mapping $\Phi: X(\mathscr{R}) \to I$ described in formula (11) will figure in the following results. If μ is a measure on $X(\mathscr{R})$, the image of μ under Φ is the measure $\nu = \Phi(\mu)$ on I defined by $\nu(f) = \mu(f \circ \Phi)$ for $f \in \mathscr{C}$. For every measure ν on I, there exist measures μ on $X(\mathscr{R})$ such that $\Phi(\mu) = \nu$, and if ν is positive then μ can be chosen to be positive [3, Ch. IX, p. 33, Lemma 1]. Thus, the mapping $\mathscr{M}(X(\mathscr{R})) \to \mathscr{M}(I)$ defined by $\mu \mapsto \Phi(\mu)$ is surjective; it is linear, positive and contractive; in view of §3, Proposition 2, its kernel is the subspace $\Gamma(\mathscr{C})^\perp$, consisting of all measures μ on $X(\mathscr{R})$ that are zero on the image of $\mathscr{C} = \mathscr{C}(I)$ under the Gelfand isomorphism $\Gamma: \mathscr{R} \to \mathscr{C}(X(\mathscr{R}))$. We remark that the quotient mapping $\mathscr{M}(X(\mathscr{R}))/\Gamma(\mathscr{C})^\perp \to \mathscr{M}(I)$ is isometric. {Proof: Identify $\mathscr{R}$ with $\mathscr{C}(X(\mathscr{R}))$ via Γ; the mapping $\mu \mapsto \Phi(\mu)$ then becomes the mapping $\mathscr{R}' \to \mathscr{C}'$ defined by $\mu \mapsto \mu | \mathscr{C}$, with kernel $\mathscr{C}^\perp$. The quotient mapping $\mathscr{R}'/\mathscr{C}^\perp \to \mathscr{C}'$ is an isometric vector space isomorphism [16, p. 91, Th. 4.9].}

THEOREM 12. (i) *$X(\mathscr{R})$ admits nonzero diffuse measures.* (ii) *The support of every diffuse measure on $X(\mathscr{R})$ is contained in $F \cup G$.* (iii) *A positive measure μ on $X(\mathscr{R})$ is diffuse if and only if $\Phi(\mu)$ is diffuse.*

Proof. A measure on a space is said to be diffuse if every one-point subset of the space is negligible. The support of a measure is the complement of the largest negligible open set in the space.

(iii) Let μ be a positive measure on $X(\mathscr{R})$ and let $\nu = \Phi(\mu)$. For every $x \in I$, one has $\mu(\Phi^{-1}(\{x\})) = \nu(\{x\})$ [3, Ch. V, §6, Cor. 1 of Proof. 2]. Since the $\Phi^{-1}(\{x\})$ form a covering of $X(\mathscr{R})$, it is clear that the diffuseness of ν implies that of μ; since the sets $\Phi^{-1}(\{x\})$ are finite (§3, Cor. 2 of Th. 5), the converse is also true. {The diffuse measures on I are the Lebesgue-Stieltjes measures induced by continuous functions of bounded variation [8, p. 332, (19.52)].}

(i) Let ν be a nonzero, diffuse positive measure on I (for instance, Lebesgue measure), let μ be a positive measure on $X(\mathscr{R})$ such that $\Phi(\mu) = \nu$, and cite (iii). {In view of §3, Corollary 5 of Theorem 5, this is a manifestation of a general theorem of A. Pelczynski and Z. Semadeni: a compact space admits a nonzero diffuse measure if and only if it has a nonempty perfect subset ([12, p. 214], [11, p. 52, Th. 10]).}

(ii) Let μ be a diffuse measure on $X(\mathscr{R})$. Replacing μ by $|\mu|$, we can suppose that μ is positive. Since the set E in Corollary 5 of Theorem 5 is open, one has $\mu(E) = \sup\{\mu(K): K \subset E, K \text{ compact}\}$ [3, Ch. IV, §4, Cor. 4 of Th. 4]; every compact subset K of the discrete space E is finite, hence is negligible by the hypothesis on μ; therefore $\mu(E) = 0$, thus $E \subset \complement \operatorname{Supp} \mu$.

To each $T \in \mathscr{R}'$ there corresponds a unique measure μ_T on $X(\mathscr{R})$ such that $Tf = \mu_T(\Gamma f)$ for all $f \in \mathscr{R}(\Gamma$ the Gelfand isomorphism); $T \mapsto \mu_T$ is an isometric, positivity-preserving vector space isomorphism $\mathscr{R}' \to \mathscr{M}(X(\mathscr{R})) = \mathscr{C}(X(\mathscr{R}))'$. If, in particular, $T = T_\chi$ for $\chi \in \mathscr{BV}$ (resp. $T = T_\phi$ for $\phi \in l^1(I)$), we write μ_χ (resp. μ_ϕ) for μ_T.

LEMMA 1. *$\Phi(\mu_T) = T|\mathscr{C}$ for all $T \in \mathscr{R}'$.*

Proof. Let $T \in \mathscr{R}'$, $\nu = \Phi(\mu_T)$. For all $f \in \mathscr{C}(I)$ one has $\nu(f) = \mu_T(f \circ \Phi) = \mu_T(\Gamma f) = Tf$.

LEMMA 2. (i) *For every $\phi \in l^1(I)$, $\Phi(\mu_\phi) = 0$;* (ii) *for every $\chi \in \mathscr{BV}$, $\Phi(\mu_\chi)$ is the Stieltjes integral ('unmodified') induced by χ.*

Proof. (i) $T_\phi|\mathscr{C} = 0$ (lemma to Theorem 11).

(ii) For $f \in \mathscr{C}$, $T_\chi f = \int_a^b f d\chi$ is the limit, in the sense of refinement of subdivisions, of the Riemann-Stieltjes sums $\sum_i f(\xi_i)[\chi(x_i) - \chi(x_{i-1})]$, where, since f is continuous, ξ_i can be any point of the closed interval $[x_{i-1}, x_i]$.

LEMMA 3. *For $T \in \mathscr{R}'$, the following conditions are equivalent:* (a) Supp $\mu_T \subset F \cup G$; (b) *$T = T_\chi$ with $\chi \in \mathscr{BV}$.*

Proof. (a) $\Rightarrow$ (b): By hypothesis, the open set E is contained in $\complement$ Supp μ_T, therefore $\mu_T|E = 0$. Thus, if $g \in \mathscr{C}(X(\mathscr{R}))$ with Supp $g \subset E$ (in other words, if g is a finite linear combination of characteristic functions $\varphi_{\{\beta_x\}}$), then $\mu_T(g) = 0$. In particular, for all $x \in I$ one has $0 = \mu_T(\varphi_{\{\beta_x\}}) = \mu_T(\Gamma\varphi_{\{x\}}) = T\varphi_{\{x\}}$, therefore $T = 0$ on $\mathscr{N}$ (§2, Cor. 2 of Prop. 1). Thus, writing $T = T_\chi + T_\phi$ as in Theorem 11, one has $0 = T|\mathscr{N} = T_\chi|\mathscr{N} + T_\phi|\mathscr{N} = T_\phi|\mathscr{N}$, whence $\phi = 0$, $T = T_\chi$.

(b) $\Rightarrow$ (a): If $T = T_\chi$, $\chi \in \mathscr{BV}$, then for all $x \in I$ one has $0 = T\varphi_{\{x\}} = \mu_T(\varphi_{\{\beta_x\}})$; reversing the preceding argument, we conclude that $\mu_T|E = 0$, thus Supp $\mu_T \subset F \cup G$.

LEMMA 4. *For $T \in \mathscr{R}'$, the following conditions are equivalent*: (a) μ_T *is diffuse*; (b) $T = T_\chi$ *with* $\chi \in \mathscr{BV}_c$.

Proof. (a) $\Rightarrow$ (b): Write $\mu = \mu_T$ and suppose μ is diffuse. Let $\mu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4$ be the canonical decomposition of μ, where the μ_i are positive measures [3, Ch. III, §1, Th. 3]. The μ_i are also diffuse, therefore Supp $\mu_i \subset F \cup G$ by Theorem 12. Say $\mu_i = \mu_{T_i}$, $T_i \in \mathscr{R}'$. By Lemma 3, one has $T_i = T_{\chi_i}$ with $\chi_i \in \mathscr{BV}$, thus $\mu_i = \mu_{\chi_i}$. We can suppose $\chi_i(a) = 0$; then, since $\mu_i \geqq 0$, equivalently $T_i \geqq 0$, it follows that χ_i is an increasing real-valued function $\geqq 0$. By Lemma 1, $T_{\chi_i}|\mathscr{C} = T_i|\mathscr{C} = \Phi(\mu_{T_i}) = \Phi(\mu_i)$, which is diffuse (Theorem 12), therefore χ_i is continuous. Finally, $T = T_\chi$, where $\chi = \chi_1 - \chi_2 + i\chi_3 - i\chi_4$ is continuous and of bounded variation.

(b) $\Rightarrow$ (a): Suppose $T = T_\chi$ with $\chi \in \mathscr{BV}_c$; one can suppose, by linearity, that χ is real-valued, increasing and continuous. Then $\Phi(\mu_\chi) = T_\chi|\mathscr{C}$ is a diffuse measure on I [8, p. 332, (19.52)], therefore μ_χ is diffuse (Theorem 12).

If $\phi \in l^1(I)$, a suggestive notation for T_ϕ is $\sum_{x \in I} \phi(x)(\beta_x - \delta_x)$ (with the convention that $\delta_b = \gamma_b$); since $\|\beta_x - \delta_x\| \leqq 2$, the sum is convergent for the norm of $\mathscr{R}'$. In the same way, one can form 'weighted sums' of arbitrary families in $X(\mathscr{R})$, indexed by I, with 'weight function' $\phi \in l^1(I)$.

THEOREM 13. *Every $T \in \mathscr{R}'$ has a representation*

$$T = T_\psi + \sum_{x \in [a,b]} \phi_1(x)\beta_x + \sum_{x \in [a,b)} \phi_2(x)\delta_x + \sum_{x \in (a,b]} \phi_3(x)\gamma_x ,$$

where $\psi \in \mathscr{BV}_c$ *and* $\phi_1, \phi_2, \phi_3 \in l^1(I)$.

Proof. In view of Theorem 11, it suffices to consider the case that $T = T_\chi$ with $\chi \in \mathscr{BV}$, and by linearity one can suppose that χ is real-valued and increasing. Decomposing χ as the sum of an increasing continuous function ψ and an increasing 'saltus' function

[18, p. 92, 2.4.2], we are reduced to the case that χ is a saltus function (positive and increasing). Then $\chi = \sum_{n=1}^{\infty} \chi_n$, a uniformly convergent series, where χ_n is an increasing step function with a single discontinuity c_n, say $\chi_n = u'_n \varphi_{\{c_n\}} + (u'_n + u''_n)\varphi_{(c_n, b]}$, and where $\sum u'_n$, $\sum u''_n$ are convergent positive term series. Since, for each n, $\chi - \sum_{k=1}^{n} \chi_k = \sum_{k>n} \chi_k$ is an increasing function ψ_n, it follows that $V_a^b(\chi - \sum_{k=1}^{n} \chi_k) = \psi_n(b) - \psi_n(a) \to 0 - 0$, therefore $T_\chi f = \sum_{n=1}^{\infty} T_{\chi_n} f$ for all $f \in \mathscr{R}$. Since $T_{\chi_n} f = u'_n f(c_n -) + u''_n f(c_n +)$ for all $f \in \mathscr{R}$, one has $T_{\chi_n} = u'_n \gamma_{c_n} + u''_n \delta_{c_n}$, therefore $T_\chi = \sum u'_n \gamma_{c_n} + \sum u''_n \delta_{c_n}$.

With notation as in Theorem 13, let $S = T - T_\psi$; then $\mu_T = \mu_\psi + \mu_S$, where μ_ψ is diffuse (Lemma 4) and μ_S is atomic; such a decomposition of a measure is unique [3, Ch. V, §5, Prop. 15], thus the representation of Theorem 13 is unique: the ϕ_i are unique and ψ is unique up to an additive constant. One can also write S in the form $S = \sum_\beta \phi(\beta) \cdot \beta$, where ϕ: $X(\mathscr{R}) \to \boldsymbol{C}$ and $\sum_\beta |\phi(\beta)| < +\infty$.

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