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QUADRATIC FORMS WITH PRESCRIBED STIEFEL–WHITNEY INVARIANTS

RON BROWN

Milnor's construction of Stiefel–Whitney invariants for quadratic forms gives a map $\hat{w}$ from the Witt–Grothendieck ring of a field to a group arising in the K-theory of the field. Analogous maps are introduced here on the Witt ring and reduced Witt ring of the field. The images of these maps are studied. A central role is played by the degree of stability, in the sense of Elman and Lam, present in the Witt ring of the field.

In §1, we review Milnor's construction of $\hat{w}$ [**13**; also see 6] and show how it can be modified so as to give a well-defined map w on the Witt ring of a field. This construction systematizes and generalizes the way in which the determinant and Hasse symbol are modified to give the discriminant and Witt symbol. The problems of computing the images of w and $\hat{w}$ are equivalent. In §2, we show that $\hat{w}$ maps into an easily described subgroup k_{reg} of the target group of $\hat{w}$. Those fields with $\mathrm{Im}\,\hat{w} = k_{\mathrm{reg}}$ are shown in §3 to be precisely those with 3-stable Witt ring [**8**]. This is a special case of a fact about m-stability in the Witt ring for arbitrary m. Similar facts are established for w and for a map, w_{red}, which w induces on the reduced Witt ring. The exponent of the "cokernel" $k_{\mathrm{reg}}/\mathrm{Im}\,\hat{w}$ is studied in §4. If the Witt ring is n-stable, then the exponent is shown to be at most 2^f where $f = n - 1 + [-\log_2 n]$. (2^f equals the exponent for formally real algebraic function fields in n variables over the real numbers.) A similar estimate is given for fields of finite level. The exponent of the cokernel of w_{red} is computed explicitly. In §5 we provide examples of stability in Witt rings and reduced Witt rings. Particular attention is paid to certain familiar classes of algebraic function fields and Henselian valued fields. Finally, §6 is devoted to computing $\mathrm{Im}\,\hat{w}$ for superpythagorean fields. We hope this computation will be relevant to the computation of $\mathrm{Im}\,w_{\mathrm{red}}$ for all fields [**4**].

Throughout this paper F will denote a field not of characteristic two. Our notation closely follows that of Lam and Milnor [**12**; **14**]. (It will, however, be convenient for us to write "$\hat{w}$" in place of Milnor's "w".) Thus we denote the semigroup of equivalence (i.e., isometry) classes of nonsingular quadratic forms by $M(F)$, the Witt–Grothendieck ring by $\hat{W}(F)$, the Witt ring by $W(F)$, the torsion subgroup of $W(F)$ by $W_t(F)$, the reduced Witt ring by $W_{\mathrm{red}}(F)$, and the augmentation ideals of

$\hat{W}(F)$ and $W(F)$ by $\hat{I} = \hat{I}(F)$ and $I = I(F)$, respectively. Elements of $W(F)$ will often be denoted by their representatives in $\hat{W}(F)$. Thus the Pfister form $\langle\langle a_1, \cdots, a_n\rangle\rangle = \Pi_{i=1}^{n}\langle a_i, 1\rangle$ will be interpreted as a member of $\hat{W}(F)$ or $W(F)$ depending on the context.

We let Z, Q, R and C denote the sets of integers, rationals, reals, and complexes, respectively. $S^{\cdot}$ denotes the group of multiplicative units of a unitary ring S (so $Z^{\cdot} = \{1, -1\}$). $|A|$ denotes the number of elements in the set A if A is finite, and ∞ otherwise. $\operatorname{Im} f$ denotes the image of the function f.

1. Stiefel–Whitney invariants. We recall Milnor's construction of Stiefel-Whitney invariants for quadratic forms. Let $k_* = k_*(F)$ denote the commutative unitary ring generated by the symbols $l(a), a \in F^{\cdot}$, subject to the relations $1 + 1 = 0$, $l(ab) = l(a) + l(b)$ and $l(c)l(1-c) = 0$ for all $a, b, c \in F^{\cdot}$ with $c \neq 1$. For each $n \geqq 0$ let $k_n = k_n(F)$ be the additive subgroup of k_* generated by the n-fold products $l(a_1)\cdots l(a_n)$, $a_i \in F^{\cdot}$. Let $k_\pi = k_\pi(F)$ be the associated ring of formal series $\alpha = \alpha_0 + \alpha_1 + \alpha_2 + \cdots (\alpha_i \in k_i$ for all $i \geqq 0)$. Then there is a unique homomorphism $\hat{w}\colon \hat{W}(F) \to k_\pi^{\cdot}$ with $\hat{w}(\langle a\rangle) = 1 + l(a)$ for all $a \in F^{\cdot}$. For $q \in \hat{W}(F)$, Milnor calls $\hat{w}(q)$ the Stiefel–Whitney invariant of q. (See [**13**, especially §3] for details.)

The Stiefel–Whitney invariant is an invariant of the isometry class of a quadratic form, but not of its Witt (i.e., similarity) class. We now introduce Stiefel–Whitney invariants for Witt classes. Give $Z^{\cdot} \times k_\pi^{\cdot}$ the multiplication

$$(i, \alpha)(j, \beta) = (ij, \alpha\beta(1 + l(-1))^k)$$

where $k = -1$ if i and j both equal -1 and $k = 0$ otherwise. (Thus, $k = (1-i)(j-1)/4$.) Then $Z^{\cdot} \times k_\pi^{\cdot}$ is simply the group extension of $k_\pi^{\cdot}$ by $Z^{\cdot}$ associated with the factor set

$$(i, j) \mapsto (1 + l(-1))^k, \quad k = (1-i)(j-1)/4.$$

Proposition and Definition 1.1. There is a unique homomorphism $w\colon W(F) \to Z^{\cdot} \times k_\pi^{\cdot}$ with $w(\langle a\rangle) = (-1, 1 + l(a))$ for all $a \in F^{\cdot}$. For any $q \in M(F)$, say of dimension n, we have

$$w(q) = ((-1)^n, \hat{w}(q)(1 + l(-1))^{-[n/2]}). \tag{1}$$

Here, "[ ]" denotes the greatest integer function.

Proof. The second sentence follows from the first (and the definition of multiplication in $Z^{\cdot} \times k_\pi^{\cdot}$), which we now prove. For $a, b, c \in F^{\cdot}$

with $a+b\neq 0$ straightforward computation shows that

$$(-1,1+l(a))(-1,1+l(-a))=(1,1),$$
$$(-1,1+l(ac^2))=(-1,1+l(a)),$$

and

$$(-1,1+l(a))(-1,1+l(b))$$
$$=(-1,1+l(a+b))(-1,1+l(ab(a+b))).$$

(The last formula also follows from [**12**, p. 46] and [**13**, Lemma 3.1].) Thus the elements $(-1,1+l(a))$ of $Z^{\cdot}\times k^{\cdot}_{\pi}$ satisfy all the relations that the corresponding generators $\langle a\rangle$ of $W(F)$ satisfy [**14**, p. 85], so there is an additive homomorphism carrying $\langle a\rangle$ to $(-1,1+l(a))$ for all $a\in F^{\cdot}$.

REMARK 1.2. Suppose $q\in W(F)$ and $w(q)=(i,1+\alpha_1+\alpha_2+\cdots)$ where $\alpha_n\in k_n$ for $n=1,2,\cdots$. We now relate i,α_1, and α_2 to the "classical invariants" of q. First, i maps to the dimension-index of q under the canonical isomorphism $Z^{\cdot}\to Z/2Z$. Next, α_1 maps to the discriminant of q under the canonical isomorphism $k_1\to F^{\cdot}/F^{\cdot 2}$ (namely, $l(a)\to aF^{\cdot 2}$). Finally, α_2 maps to the Witt symbol of $-q$ under the canonical homomorphism $g\colon k_2\to B(F)$ ([**7**, p. 209]; $B(F)$ denotes the Brauer group of F and g carries each $l(a)l(b)$ to the quaternion algebra $(a,b/F)$). Thus the construction of w from $\hat{w}$ might be thought of as a generalization of the familiar process by which the classical invariants for isometry classes of quadratic forms (dimension, determinant, and Hasse-symbol) are modified to give the classical invariants for Witt classes of quadratic forms (dimension-index, discriminant, and Witt symbol).

Proof. Write $q=\langle a_1,\cdots,a_n\rangle$, $a_i\in F^{\cdot}$. By inspection i and α_1 correspond to the dimension-index and discriminant of q. (Note that if $n>0$, then $[n/2]$ and $n(n-1)/2$ are congruent modulo 2.) Let $c\colon W(F)\to B(F)$ be the Witt symbol. We show $c(-q)=g(\alpha_2)$ by induction on n. Write $q'=\langle a_1,\cdots,a_{n-1}\rangle$ and $w(q')=(i',1+\alpha_1'+\alpha_2'+\cdots)$. Let $p\colon Z^{\cdot}\times k^{\cdot}_{\pi}\to k_2$ be the projection. Then

$$\begin{aligned}
g(\alpha_2)&=gpw(q)=gp(w(q')w(\langle a_n\rangle))\\
&=gp((-1)^n,(1+\alpha_1'+\alpha_2')(1+l(a_n))(1+(n-1)(l(-1)+l(-1)^2)))\\
&=g(\alpha_2'+(n-1)l(-1)^2+(n-1)l(-1)l(a_n)+\alpha_1'l(a_n)\\
&\quad+(n-1)l(-1)\alpha_1')\\
&=g(\alpha_2'+l((-1)^{n+1}a_n)(\alpha_1'+l((-1)^{n-1})))
\end{aligned}$$

which by our induction hypothesis equals

$$c(-q')c(\langle -a_n\rangle)(\mathrm{disc}(-q'),(-1)^n\mathrm{disc}\langle -a_n\rangle/F)$$

(disc = discriminant), which equals $c(-q)$ [**12**, formula 3.13, p. 121].

Incidentally, it is not hard to use the above remark to read off ("on the spot") a formula for $c(q)$ as a product of quaternion algebras (cf. [**12**, p. 121, line -1]).

This paper is mainly concerned with computing the images of $\hat{w}$ and w. The next proposition records the fact that the problems of computing these two images are equivalent.

PROPOSITION 1.3. $\operatorname{Im} w = Z^{\cdot} \times \operatorname{Im} \hat{w}$. *That is, for any* $\alpha \in k_\pi^{\cdot}$ *the following are equivalent*: (i) $\alpha \in \operatorname{Im} \hat{w}$, (ii) $(1,\alpha) \in \operatorname{Im} w$, (iii) $(-1,\alpha) \in \operatorname{Im} w$.

Proof. That (ii) implies (iii) follows from the identity: $w(1)(1,\alpha) = (-1,\alpha)$. Formula (1) above shows (iii) implies (i) (note that $\hat{w}(\langle -1\rangle) = 1 + l(-1)$). That (i) implies (ii) follows from the fact that for any $q \in \hat{W}(F)$ we have

$$w(q - (\dim q)\cdot 1) = (1, \hat{w}(q)). \tag{2}$$

To prove (2) note that we can write $q - (\dim q)\cdot 1 = q' - n\langle 1, -1\rangle$ where $q' \in M(F)$ and $2n = \dim q'$. But then both sides of (2) equal

$$w(q') = (1, \hat{w}(q')(1 + l(-1))^{-n}).$$

PROPOSITION 1.4. *w is injective if and only if I^3 is torsion-free. In general,* $\ker w \subset I^3 \cap W_t(F)$.

Proof. $\ker w$ consists precisely of those elements of $W(F)$ which can be represented by elements of $\hat{I} \cap \ker \hat{w}$. The proposition therefore reduces to a theorem of Elman and Lam [**8**, Theorem 2.15 (and proof)].

2. Regular elements of $k_\pi^{\cdot}$. We now refine slightly [**13**, Remark 3.4], which gives a condition satisfied by elements of $\hat{w}(M(F))$. (Compare with [**15**, Corollary 2.2.2].) For any $\alpha \in k_\pi$ and $n \geqq 0$, we denote the projection of α into k_n by α_n and call it the *term of α of degree n*. We say α has *degree n* when $\alpha_n \neq 0$ but $\alpha_m = 0$ for all $m > n$.

DEFINITION 2.1. Let $\alpha \in k_\pi^{\cdot}$. We call α *regular* when

(A) if $n > 0$, then $\alpha_n = \alpha_{2^m}\alpha_{n-2^m}$ where $m = [\log_2 n]$, and

(B) there exist positive integers N, M such that if $k \geqq N$, then $\alpha_{k+2^M} = \alpha_k l(-1)^{2^M}$.

The set of regular elements of $k^{\cdot}_{\pi}$ will be denoted by $k_{\text{reg}} = k_{\text{reg}}(F)$.

PROPOSITION 2.2. *All elements of* $\operatorname{Im}\hat{w}$ *are regular.*

$\operatorname{Im}\hat{w}$ is precisely the subgroup of $k^{\cdot}_{\pi}$ generated by $1 + k_1$. Hence the above proposition follows from the next lemma (take $r = 0$, so $L_r = k_{\text{reg}}$).

LEMMA 2.3. *Let* $r \geqq 0$ *and* $L_r = \{\alpha \in k_{\text{reg}}: \alpha_i = 0$ *whenever* $0 < i < 2^r\}$. *Then* L_r *is the subgroup of* k_{reg} *generated by the set* $G_r = \{1 + l(a_1)\cdots l(a_{2^s}): s \geqq r$ *and* $a_i \in F^{\cdot}$ *for all* $i \leqq 2^s\}$. *Further, each element of* L_r *can be written in the form*

$$(1 + l(-1))^{-2^t}\prod_{i=r}^{s}(1 + \delta_{2^i}) \tag{3}$$

where $t \geqq r, s \geqq r$, *and* $\delta \in k_{\pi}$.

Proof. We begin by showing that if $\delta, \gamma \in k^{\cdot}_{\pi}$ satisfy condition (A) of 2.1, then $\beta = \delta\gamma$ also satisfies (A). So suppose $m = [\log_2 n]$ (as in (A)); we show $\beta_n = \beta_{2^m}\beta_{n-2^m}$. We may suppose that γ and δ each have degree less than 2^{m+1} (nothing essential is lost by deleting all terms of degree $\geqq 2^{m+1}$). Note that for any $s \geqq 0$ and $\eta, \nu \in k_{2^s}$ we have $(1 + \eta)(1 + \nu) = (1 + \eta + \nu)(1 + \eta\nu)$. If we apply this identity repeatedly to the right-hand side of

$$\beta = \prod_{i=0}^{m}(1 + \delta_{2^i})(1 + \gamma_{2^i}),$$

we see we can write $\beta = \Pi_{i=0}^{m+1}(1 + \beta_{2^i})$ (note that β has degree less than 2^{m+2}). That $\beta_n = \beta_{2^m}\beta_{n-2^m}$ is now clear.

We next show that if $\alpha = \beta(1 + l(-1))^{-n}$ where $n \geqq 0$ and $\beta \in k^{\cdot}_{\pi}$ has finite degree, then α satisfies (B) of 2.1. There exists $M > 0$ with $2^M \geqq n$; we may as well assume $2^M = n$ (replace β by $\beta(1 + l(-1))^{2^M - n}$). Let $N = n + \text{degree}(\beta)$. Then

$$\alpha = (\beta_0 + \cdots + \beta_{N-1})(1 + l(-1)^{2^M\cdot 1} + l(-1)^{2^M\cdot 2} + l(-1)^{2^M\cdot 3} + \cdots)$$

$$= \sum_{r \geqq 0}\ \sum_{i+2^M j = r} \beta_i l(-1)^{2^M j}.$$

Hence for all $k \geqq N$,

$$l(-1)^{2^M}\alpha_k = \sum_{i+2^M j=k} \beta_i l(-1)^{2^M(j+1)}$$

$$= \sum_{i+2^M j=k+2^M} \beta_i l(-1)^{2^M j} = \alpha_{k+2^M},$$

as required. (The sums above are only over $i \geqq 0$, $j \geqq 0$.)

Now suppose α is in the subgroup of $k_\pi^{\cdot}$ generated by the set G_r. For any $\delta \in 1 + k_{2^n}$ we have $\delta^{-1} = (\delta + l(-1)^{2^n})(1 + l(-1))^{-2n}$. Hence α can be written as a product of a finite number of elements of $1 + \bigcup_{n \geqq 0} k_{2^n}$ and a power of $(1 + l(-1))^{-1}$. It follows from the previous two paragraphs that $\alpha \in L_r$. (All elements of $1 + \bigcup_{n \geqq 0} k_{2^n}$ and $(1 + l(-1))^{-1} = 1 + l(-1) + l(-1)^2 + \cdots$ are clearly regular.) Conversely, suppose $\alpha \in L_r$. Let M, N be as in (B) of 2.1. We may suppose $N = 2^M - 1 > 2^r$ (increasing if necessary the values of M and N). Let $\delta = \sum_{i=0}^N \alpha_i$, $\rho = \sum_{i=0}^N \alpha_{i+N+1}$, and $\beta = \rho + \delta(1 + l(-1))^{N+1}$. Then by (B) of 2.1,

$$\alpha = \delta + \rho(1 + l(-1)^{N+1} + l(-1)^{2N+2} + l(-1)^{3N+3} + \cdots)$$

which equals $\beta(1 + l(-1))^{-2^M}$. The first paragraph thus shows β is regular. Since β has finite degree, $\beta = \prod_{i=0}^s (1 + \beta_{2^i})$ for s sufficiently large. Since $\alpha \in L_r$, we have $\beta_{2^i} = 0$ for $0 \leqq i < r$. This shows α is in the group generated by $1 + \bigcup_{n \geqq r} k_{2^n}$ and α can be written in the form (3). It remains to show that each element $1 + \rho$ of $1 + \bigcup_{n \geqq r} k_{2^n}$ is in the group generated by G_r. We can write $\rho = \sum_{i=1}^s \delta_i$ where $s \geqq 1$ and each δ_i is a product of 2^n elements of k_1 (for fixed $n \geqq r$). If $s = 1$, then $1 + \rho \in G_r$ by definition. If $s > 1$, we can write

$$1 + \rho = \left(1 + \delta_1\right)\left(1 + \sum_{i=2}^s \delta_i\right)\left(1 + \delta_1 \sum_{i=2}^s \delta_i\right)^{-1}.$$

Our result now follows by induction on s.

COROLLARY 2.4. *The factor group $k_{\text{reg}}/\text{Im}\,\hat{w}$ is a 2-primary group.*

The corollary follows from Lemma 2.3 and [**13**, Lemma 3.2]. In §4 we study the exponent of $k_{\text{reg}}/\text{Im}\,\hat{w}$.

REMARK 2.5. Elman and Lam have computed $\hat{w}(M(F))$ for all fields with $k_4 = k_1 l(-1)^3$ and such that for all $\beta \in k_\pi$ there exists $q \in M(F)$ with $\dim q \leqq 3$ and with β_1 and β_2 the first two Stiefel–Whitney invariants of q [**8**, Proposition 2.23]. (These conditions are fairly restrictive, e.g. they imply $W_{\text{red}}(F)$ is 1-stable.) Their result shows $\operatorname{Im}\hat{w} = k_{\text{reg}}$ for these fields.

EXAMPLE 2.6. The structure of $\hat{w}(M(F))$ is probably a good bit more subtle than that of $\operatorname{Im}\hat{w}$. For example, for any $a, b \in F^{\cdot}$ we have

$$1 + l(a)l(b) = \hat{w}((\langle a\rangle - 1)(1 - \langle b\rangle)) \in \operatorname{Im}\hat{w}. \tag{4}$$

However, if $F = R((x))((y))$ (a field with $\operatorname{Im}\hat{w} = k_{\text{reg}}$, incidentally) and $x = a$ and $y = b$, then $1 + l(a)l(b) \notin \hat{w}(M(F))$.

Proof. Just suppose $1 + l(a)l(b) = \hat{w}(q)$ for some $q \in M(F)$. We can write $q = \Sigma i_c\langle c\rangle$ where the i_c are nonnegative integers and c ranges over the set $\{1, -1, a, -a, b, -b, ab, -ab\}$. (This set maps bijectively to $F^{\cdot}/F^{\cdot 2}$.) Clearly $i_c \neq 0$ for some $c \notin \{1, -1\}$. There exists an ordering P of F which excludes c but includes either a or b (F is superpythagorean). Clearly $\hat{w}((q - \langle c\rangle)\otimes F_p)$ has finite degree (it is in $\hat{w}(M(F_p))$; F_p is the real closure of F at P). On the other hand

$$\begin{aligned}\hat{w}((q - \langle c\rangle)\otimes F_p) &= (1 + l(a)l(b))(1 + l(c))^{-1}\\ &= (1 + l(-1))^{-1} \quad (\text{in } k_\pi(F_p))\end{aligned}$$

which surely does not have finite degree [**13**, p. 320]. This contradiction completes the proof.

3. When does $\operatorname{Im}\hat{w} = k_{\text{reg}}$? We show in this section that $\operatorname{Im}\hat{w} = k_{\text{reg}}$ if and only if $W(F)$ is 3-stable, i.e. $I^4 = 2I^3$ [**8**, Definition 3.8]. We begin with a general fact about m-stability. Note that $\hat{w}(\hat{I}^m) \subset L_{m-1}$ for all $m > 0$ (see Lemma 2.3 for notation, and [**13**, Lemma 3.2] and Proposition 2.2 for the reason). For all $a_1, \cdots, a_m \in F^{\cdot}$ we have

$$\langle\langle a_1, \cdots, a_m\rangle\rangle = 2^{m-1}\langle 1, -1\rangle + (-1)^m \prod_{i=1}^{m} (\langle -a_i\rangle - 1)$$

(in $\hat{W}(F)$), so that

$$w(\langle\langle a_1, \cdots, a_m\rangle\rangle) = (1, 1 + l(-a_1)\cdots l(-a_m)l(-1)^{2^{m-1}-m})^{-1} \tag{5}$$

([**13**, Lemmas 3.1 and 3.2] and formula (1)). This shows $w(I^m) \subset \{1\} \times L_{m-1}$.

THEOREM 3.1. *Let $m \geqq 3$ and $t = 2^{m-1}$. The following statements are equivalent*:

(i) $I^t = 2^{t-m}I^m$,
(ii) $w(I^m) = \{1\} \times L_{m-1}$,
(iii) $\hat{w}(\hat{I}^m) = L_{m-1}$.

Proof. (i) $\Rightarrow$ (ii). Consider any generator $\alpha = 1 + l(a_1)\cdots l(a_{2^s})$ of L_{m-1} (cf. Lemma 2.3; $s \geqq m-1$ and $a_i \in F^{\cdot}$ for each i). Our hypothesis implies $I^{2^s} = 2^{2^s-m}I^m$. Hence there exist $b_1,\cdots,b_m$ with $\langle\langle -a_1,\cdots,-a_{2^s}\rangle\rangle = 2^{2^s-m}\langle\langle b_1,\cdots,b_m\rangle\rangle$ [**8**, Theorem 2.1]. Hence

$$l(a_1)\cdots l(a_{2^s}) = l(-b_1)\cdots l(-b_m)l(-1)^{2^s-m}$$

[**7**, Theorem 3.2]. Formula (5) therefore shows

$$\begin{aligned} &w(-2^{s-m+1}\langle\langle b_1,\cdots,b_m\rangle\rangle) \\ &\quad = (1, 1 + l(-b_1)\cdots l(-b_m)l(-1)^{t-m})^{2^{s-m+1}} = (1,\alpha). \end{aligned}$$

(ii) $\Rightarrow$ (iii). Let $\alpha \in L_{m-1}$. By hypothesis there exists $q \in \hat{I}^m$ with $w(q) = (1,\alpha)$. Then $\hat{w}(q) = \alpha$, by (2) (in §1).

(iii) $\Rightarrow$ (i). For each generator $\langle\langle a_1,\cdots,a_t\rangle\rangle$ of I^t ($a_i \in F^{\cdot}$) there exists $q \in \hat{I}^m$ with $\hat{w}(q) = 1 + l(-a_1)\cdots l(-a_t)$. Thus

$$l(-a_1)\cdots l(-a_t) \in \operatorname{Im} w_t = \operatorname{Im} w_t s_m = l(-1)^{t-m}k_m$$

(see [**13**, proof of Theorem 4.1] for the two equalities above and the definitions of the maps $s_m\colon k_m \to I^m/I^{m+1}$ and $w_t\colon I^m/I^{m+1} \to k_t$). Hence $\langle\langle a_1,\cdots,a_t\rangle\rangle \in 2^{t-m}I^m$ [**8**, Theorem 2.1], which completes the proof.

Note that in the proof "(iii) $\Rightarrow$ (i)" above we only need the fact that $L_{m-1} \subset \hat{w}(\hat{I}^m)L_m$. If $W(F)$ is m-stable, then 3.1 (i) holds.

COROLLARY 3.2. $k_{\text{reg}} = \operatorname{Im}\hat{w}$ *if and only if* $W(F)$ *is 3-stable.*

Proof. By Proposition 2.2, $k_{\text{reg}} \supset \operatorname{Im}\hat{w}$. If $W(F)$ is 3-stable, then the reverse inclusion follows from Lemma 2.3, Theorem 3.1 and formula (4) (in §2). Now suppose that $k_{\text{reg}} = \operatorname{Im}\hat{w}$. If $\alpha \in L_2$, there exists $q \in \hat{W}(F)$ with $\hat{w}(q) = \alpha$. We may suppose $q \in \hat{I}$ (replace q by $q - (\dim q)\cdot 1$). Since the first two Stiefel–Whitney invariants of q are trivial, $q \in \hat{I}^3$ (use the injectivity of the maps $w_i\colon \hat{I}^i/\hat{I}^{i+1} \to k_i$, $i = 1$ or 2, as in the proof of [**8**, Theorem 2.15]). Hence $\hat{w}(\hat{I}^3) = L_2$. Theorem 3.1 now shows that $W(F)$ is 3-stable.

COROLLARY 3.3. $\operatorname{Im} w = Z^{\cdot} \times k_{\text{reg}}$ *if and only if* $W(F)$ *is 3-stable.* w *maps* $W(F)$ *isomorphically onto* $Z^{\cdot} \times k_{\text{reg}}$ *if and only if* I^3 *is torsion-free and* $W_{\text{red}}(F)$ *is 3-stable.*

Proof. The first sentence follows from Corollary 3.2 and Proposition 1.3. The second follows from the first and Proposition 1.4 (see Remark 5.1 E below).

Elman and Lam have shown that any field F of transcendence degree at most two over R has I^3 torsion-free and $W(F)$ 3-stable [**8**, Example 2.17 (4), and Example 2 on p. 1177].

We now develop results for $W_{\mathrm{red}}(F)$ analogous to those above. Let T denote the torsion subgroup of k_{reg} and let $T' = \{1\} \times T$ be the corresponding subgroup of $Z^{\cdot} \times k_{\mathrm{reg}}$.

THEOREM 3.4. *w induces an injective homomorphism*

$$w_{\mathrm{red}}\colon W_{\mathrm{red}}(F) \to Z^{\cdot} \times k_{\mathrm{reg}}/T'.$$

Moreover, for all $m \geqq 3$ the following are equivalent:

(i) $W_{\mathrm{red}}(F)$ *is m-stable,*

(ii) $w_{\mathrm{red}}(I^m) = (\{1\} \times L_{m-1}) \cdot T'/T'$,

(iii) $T \cdot \operatorname{Im} \hat{w} \supset L_{m-1}$.

Proof. The nil radical of $W(F)$ is $I \cap W_t(F)$ [**12**, Theorem 6.1, p. 248], which equals $w^{-1}(T')$ (Proposition 1.4). Hence w_{red} is well-defined and injective.

(i) $\Rightarrow$ (ii). Let $\alpha = 1 + l(a_1) \cdots l(a_{2^s})$ be a generator of L_{m-1} (so $s \geqq m-1$ and each a_i is in $F^{\cdot}$, cf. Lemma 2.3). By hypothesis $I^{2^s} \equiv 2^{2^s-m} I^m (\operatorname{mod} W_t(F))$. Hence there exists $q \in I^m$ and $r \geqq 0$ with $2^r\langle\langle -a_1, \cdots, -a_{2^s}\rangle\rangle = 2^{r+2^s-m} q$. We may suppose $q = \langle\langle b_1, \cdots, b_m\rangle\rangle$ for some $b \in F^{\cdot}$ [**8**, Theorem 2.1]. Then, as in the proof of 3.1, we have $w(-2^{s-m+1}q)^{2^r} = (1, \alpha)^{2^r}$, so $(1, \alpha)T' \in w_{\mathrm{red}}(I^m)$.

(ii) $\Rightarrow$ (iii). Let $\alpha \in L_{m-1}$. There exists $q \in \hat{I}^m$ with $(1, \alpha)T' = w_{\mathrm{red}}(q) = (1, \hat{w}(q))T'$ (cf. (2) of §1). Hence $\alpha \in T \cdot \operatorname{Im} \hat{w}$.

(iii) $\Rightarrow$ (i). Let $t = 2^{m-1}$. It suffices to show $I^t \equiv 2^{t-m} I^m$ (modulo $W_t(F)$). Let $a_1, \cdots, a_t \in F^{\cdot}$. There exists by hypothesis $q \in \hat{I}$ and $r \geqq 0$ with

$$w(q)^{2^r} = (1, \hat{w}(q)^{2^r}) = (1, (1 + l(-a_1) \cdots l(-a_t))^{2^r})$$

(cf. (2) of §1). By (5) we have

$$\begin{aligned} w(-2^r\langle\langle a_1, \cdots, a_t\rangle\rangle) &= (1, 1 + l(-a_1) \cdots l(-a_t))^{2^{t-m+r}} \\ &= w(2^{t-m+r} q). \end{aligned}$$

Since w_{red} is injective, we therefore have $2^n\langle\langle a_1,\cdots,a_t\rangle\rangle = 2^n 2^{t-m}(-q)$ for some $n \geqq r$. We may suppose $-q \in I^m$ [**8**, Theorem 2.1]. Hence $\langle\langle a_1,\cdots,a_t\rangle\rangle \in 2^{t-m}I^m + W_t(F)$. Finished.

The above proof shows that the three conditions of Theorem 3.4 are equivalent to: (ii′) $\operatorname{Im} w_{\text{red}} \supset (\{1\} \times L_{m-1})T'/T'$, and (iii′) $T\hat{w}(\hat{I}^m) = TL_{m-1}$.

COROLLARY 3.5. *w_{red} is an isomorphism if and only if $W_{\text{red}}(F)$ is 3-stable.*

The corollary follows from Theorem 3.4 (together with Lemma 2.3 and formula (4)).

4. The exponent of $k_{\text{reg}}/\operatorname{Im} \hat{w}$. Recall that $k_{\text{reg}}/\operatorname{Im}\cdot\hat{w}$ is a 2-primary abelian group (Corollary 2.4). Let $e = e(F)$ denote the infimum of the set of integers $n \geqq 0$ such that all elements of $k_{\text{reg}}/\operatorname{Im} \hat{w}$ have order dividing 2^n. (Thus 2^e is the exponent of $k_{\text{reg}}/\operatorname{Im} \hat{w}$ if this group has finite exponent and $e = \infty$ otherwise.) Theorem 3.1 says that $e = 0$ if and only if $W(F)$ is 3-stable.

It is convenient for us to write $[-\log_2 m] = 1$ if $m = 0$ or ∞.

THEOREM 4.1. *Suppose $m \geqq 0$, and $W(F)$ is m-stable. Then $e \leqq m - 1 + [-\log_2 m]$.*

Proof. We may suppose $m \geqq 3$ (Corollary 3.2). Let $g = m - 1 + [-\log_2 m]$. By Lemma 2.3 it suffices to show $(1+\alpha)^{2^g} \in \operatorname{Im} \hat{w}$ where $\alpha = l(a_1)\cdots l(a_{2^s})$, $s \geqq 1$, $a_i \in F^{\cdot}$. Let $t = 2^s - 1$ and $r = \max\{0, m - s - 1\}$. Then $(1+\alpha)^{2^r} \in L_{m-1} \subset \operatorname{Im} \hat{w}$ (Proposition 3.1) and

$$(1+\alpha)^{2^{t-s}} = 1 + l(a_1)\cdots l(a_{2^s})l(-1)^{2^t-2^s} \in \operatorname{Im} \hat{w}$$

[**13**, Lemma 3.2]. Hence it suffices to show $g \geqq \min\{r, t-s\}$ for all $s \geqq 1$. But just suppose $g < \min\{r, t-s\}$ for some $s \geqq 1$. Then

$$(6) \qquad 2^s - s > m + [-\log_2 m] \quad \text{and} \quad -[-\log_2 m] > s.$$

The second inequality shows $\log_2 m > s$. Using this and the first inequality of (6) we obtain

$$1 + m + [-\log_2 m] \geqq m - \log_2 m > 2^s - s > m + [-\log_2 m]$$

which is impossible. The theorem is proven.

REMARK 4.2. In Theorem 4.1 it would have been sufficient to assume that $I^t = 2^{t-m}I^m$, where $t = 2^{m-1}$ (and not that $W(F)$ was actually m-stable). For example if $F = Q((x_1))((x_2))((x_3))$, then $I^8 = 2^4I^4$, but $W(F)$ is not 4-stable. (See Example 5.4 D.)

Note that $k_{\mathrm{reg}}/\mathrm{Im}\,\hat{w}$ and $Z^{\cdot} \times k_{\mathrm{reg}}/\mathrm{Im}\,w$ are isomorphic, and hence have the same exponent. (The isomorphism carries $\alpha\,\mathrm{Im}\,\hat{w}$ to $(1, \alpha)\mathrm{Im}\,w$, cf. Proposition 1.3.) The cokernel of w_{red} (cf. Theorem 3.4) is a factor group of $Z^{\cdot} \times k_{\mathrm{reg}}/\mathrm{Im}\,w$, so its exponent is bounded by that of $k_{\mathrm{reg}}/\mathrm{Im}\,\hat{w}$. We now compute the exponent of $\mathrm{coker}\,w_{\mathrm{red}}$. Let $s_{\mathrm{red}}(F)$ denote the infimum of the set of integers $m \geqq 0$ with $W_{\mathrm{red}}(F)$ m-stable. ($\mathrm{Inf}\,\varnothing = \infty$.) It is usually easy to compute $s_{\mathrm{red}}(F)$ (see §5, especially Lemma 5.3).

THEOREM 4.3. *Let $m = s_{\mathrm{red}}(F)$ and let 2^f be the exponent of $\mathrm{coker}\,w_{\mathrm{red}}$ (so $f = \infty$ if $\mathrm{coker}\,w_{\mathrm{red}}$ does not have finite exponent). Then $f = m + [-\log_2 m] - 1$. In particular, $f < \infty$ if and only if $m < \infty$.*

Proof. Coker w_{red} may be identified with $k_{\mathrm{reg}}/(\mathrm{Im}\,\hat{w})T$. With this identification, the proof of Theorem 4.1 adapts (using 3.4 in place of 3.1) to show that if $m < \infty$, then $f \leqq m + [-\log_2 m] - 1$. It would therefore suffice to show that if $f < \infty$, then $W_{\mathrm{red}}(F)$ is $(n-1)$-stable for any $n \geqq 1$ with $n + [-\log_2 n] - 1 > f$. For such n we have $n \geqq 4$. For any $a_1, \cdots, a_n \in F^{\cdot}$ we have by (5),

$$w(\langle\langle a_1, \cdots, a_n\rangle\rangle) = (1, 1 + l(-a_1)\cdots l(-a_n)l(-1)^{2^{n-1}-n})^{-1}$$

which equals (setting $b = -[-\log_2 n]$, so $2^b \geqq n$)

$$= (1, 1 + l(-a_1)\cdots l(-a_n)l(-1)^{2^b - n})^{-2^{n-b-1}}.$$

Since by hypothesis $n - b - 1 > f$, there exists $q \in W(F)$ with

$$w(\langle\langle a_1, \cdots, a_n\rangle\rangle) \equiv w(2q) \pmod{T'}$$

Thus $\langle\langle a_1, \cdots, a_n\rangle\rangle \in 2W(F) + W_t(F)$ (Proposition 1.4), so $W_{\mathrm{red}}(F)$ is $(n-1)$-stable [**1**, Satz 3.17].

COROLLARY 4.4. *Let $m \geqq 0$. If $W(F)$ is m-stable and $s_{\mathrm{red}}(F) = m$, then $e = m - 1 + [-\log_2 m]$.*

This corollary follows immediately from 4.1 and 4.3. It applies, for example, to any formally real algebraic function field in $m \geqq 1$ variables over R (see Example 5.4 B below, or else [**1**, Satz 4.8], [**8**, Example 2, p.

1177]). This shows that Theorem 4.1 cannot in general be improved. However, if F is a field whose level is small relative to the values of m for which $W(F)$ is m-stable, then Theorem 4.1 gives a poor estimate of e. For example, the next theorem shows that if F is an algebraic function field in $n \geqq 1$ variables over C, then $e \leqq 1$. However such a field is n-stable but not $(n-1)$-stable (see Example 5.4 B below).

THEOREM 4.5. *Suppose F has level $2^r < \infty$. Then $e \leqq \max\{1, r+1+[-\log_2(r+2)]\}$.*

Proof. We adapt the proof of Theorem 4.1. Let $g = \max\{1, r+1+[-\log_2(r+2)]\}$. We must show $(1+\alpha)^{2^g} \in \operatorname{Im}\hat{w}$ for all $\alpha = l(a_1)\cdots l(a_{2^s}), s \geqq 1$. Our hypothesis implies $l(-1)^{2^r} = 0$ [13, p. 320]. If $r < s$, then clearly $(1+\alpha)^2 = 1$, so $(1+\alpha)^{2^g} \in \operatorname{Im}\hat{w}$. Suppose $r \geqq s$. Then $(1+\alpha)^{2^{r-s+1}} = 1 + \alpha l(-1)^{2^{r+1}-2^s} = 1 \in \operatorname{Im}\hat{w}$. But also $(1+\alpha)^{2^{t-s}} \in \operatorname{Im}\hat{w}$ where $t = 2^s - 1$ [13, Lemma 3.2]. Thus it suffices to show $g \geqq \min\{r-s+1, t-s\}$ for all s with $1 \leqq s \leqq r$. This can be done by the argument in the proof of Theorem 4.1, replacing m by $r+2$.

5. Stability in $W(F)$ and $W_{\text{red}}(F)$.

Before considering more concrete examples we collect, and somewhat refine, some known results on n-stability in $W(F)$ and $W_{\text{red}}(F)$. (See especially [1], [8].) It will be useful to have in mind the facts in the following remark.

REMARK 5.1. (A) $W(F)$ is n-stable if and only if $k_{n+1} = l(-1)k_n$. More generally, for any $n \geqq m \geqq 0, I^n = 2^{n-m}I^m$ if and only if $k_n = l(-1)^{n-m}k_m$. (This follows easily from [8, Theorem 2.1].)

(B) $W(F)$ is n-stable if F is a direct limit of fields K with $W(K)$ n-stable.

(C) If K is a field extension of F with $K^{\cdot} = F^{\cdot} \cdot K^{\cdot 2}$ and $W(F)$ is n-stable, then $W(K)$ is n-stable. (After all, the canonical map $W(F) \to W(K)$ is surjective and carries $I(F)$ onto $I(K)$.)

(D) If $W(F)$ is n-stable, then it is $(n+1)$-stable.

(E) If I^{n+1} is torsion-free and $W_{\text{red}}(F)$ is n-stable, then $W(F)$ is n-stable.

We leave to the interested reader the (easy) task of modifying each of the above criteria for n-stability of $W(F)$ so as to give an analogous criterion for n-stability of $W_{\text{red}}(F)$.

LEMMA 5.2 (see [8, Example 4, p. 1178]). *Suppose τ is a place on F with $\tau^{-1}(1) \subset F^{\cdot 2}$ and $\tau(2) \neq 0$. Let K be the residue class field of τ and let Λ be the square factor group of the value group of τ. Suppose $|\Lambda| = 2^m < \infty$. Then $W(F)$ is $(n+m)$-stable if and only if $W(K)$ is n-stable.*

Proof. Let $U = \tau^{-1}(K^{\cdot})$. There exist additive homomorphisms ψ: $W(K) \to W(F)$ and ϕ: $W(F) \to W(K)$ such that $\psi(\langle\tau(a)\rangle) = \langle a\rangle$ and $\phi(\langle a\rangle) = \langle\tau(a)\rangle$ for all $a \in U$, and $\phi(\langle b\rangle) = 0$ for all $b \notin U \cdot F^{\cdot 2}$. (It is easy to see that any relation in $W(K)$ satisfied by the generators $\langle\tau(a)\rangle$, $a \in U$, is satisfied by the corresponding elements $\langle a\rangle$ in $W(F)$ [**14**, Lemma 1.1, p. 84]. This establishes the existence of ψ. The existence of ϕ can be established similarly, or one can obtain ϕ by composing one of the usual ring homomorphisms from $W(F)$ to the group ring $W(K)(\Lambda)$ [**10**], [**16**, Satz 3.1] with the appropriate projection $W(K)(\Lambda) \to W(K)$.) Now let $B = \{b_1, \cdots, b_m\}$ be a subset of $F^{\cdot}$ mapping bijectively onto a basis of Λ. Note that $k_*(F)$ is generated by $\{l(c)\colon c \in B \cup U\}$.

Now suppose $W(K)$ is n-stable. $k_{n+m+1}(F)$ is generated by elements of the form $\beta = l(c_1) \cdots l(c_{n+m+1})$ where $c_i \in B \cup U$ for all i. After re-indexing we may assume $c_1, \cdots, c_{n+1} \in U$. There exist $d_1, \cdots, d_n \in U$ with

$$\langle\langle -\tau(c_1), \cdots, -\tau(c_{n+1})\rangle\rangle = 2\langle\langle -\tau(d_1), \cdots, -\tau(d_n)\rangle\rangle$$

[**8**, Theorem 2.1]. Applying the map ψ gives

$$\langle\langle -c_1, \cdots, -c_{n+1}\rangle\rangle = 2\langle\langle -d_1, \cdots, -d_n\rangle\rangle.$$

Hence $l(c_1) \cdots l(c_{n+1}) \in l(-1)k_n(F)$ [**7**, Theorem 3.2]. Thus $\beta \in l(-1)k_{n+m}(F)$. Remark 5.1 A now shows that $W(F)$ is $(n+m)$-stable. Conversely, suppose $W(F)$ is $(n+m)$-stable. Let $a_1, \cdots, a_{n+1} \in U$. Then there exists $q \in W(F)$ with $\langle\langle a_1, \cdots, a_{n+1}, b_1, \cdots, b_m\rangle\rangle = 2q$. Applying the map ϕ shows

$$\langle\langle\tau(a_1), \cdots, \tau(a_{n+1})\rangle\rangle \in 2W(K)$$

(since $\langle\langle b_1, \cdots, b_m\rangle\rangle$ is a sum of 2^m one-dimensional forms whose determinants represent the 2^m elements of Λ). Thus $\langle\langle\tau(a_1), \cdots, \tau(a_{n+1})\rangle\rangle \in 2I(K)^n$ [**8**, Theorem 2.1]. Hence $W(K)$ is n-stable.

Note. The proof of 5.2 shows that for any $s \geqq r \geqq 0$, $I(K)^s = 2^{s-r}I(K)^r$ if and only if $I(F)^{s+m} = 2^{s-r}I(F)^{r+m}$. ($F, K, m$ are as in Lemma 5.2.)

Our second lemma is a consequence of Bröcker's computation of $s_{\mathrm{red}}(F)$ (cf. §4) [**1**, 3.18 and 3.19]. Let $\mathcal{M}(F)$ denote the set of places from F into R, and for each $\sigma, \tau \in \mathcal{M}(F)$ let $\Lambda_{\sigma\tau}$ denote the square factor group of the value group of the valuation ring $\sigma^{-1}(R) \cdot \tau^{-1}(R)$ [**3**; **4**].

LEMMA 5.3 (see [**5**, Theorem 4.3]). *$W_{\mathrm{red}}(F)$ is n-stable if and only if for all $\sigma, \tau \in \mathcal{M}(F)$ (not necessarily distinct),*

$$2^n \geqq |\Lambda_{\sigma\tau}|\,|\{\sigma, \tau\}|. \tag{7}$$

Proof. Following the notation of [**1**], let us write $s(F)$ for $s_{\mathrm{red}}(F)$. We may suppose F is formally real (otherwise the lemma is trivially true) and $n \geqq 1$. ($W_{\mathrm{red}}(F)$ is 0-stable if and only if F has at most one ordering [**1**, 3.14], i.e., F has at most one place σ into R and $|\Lambda_{\sigma\sigma}| = 1$ [**2**].) Now suppose $W_{\mathrm{red}}(F)$ is n-stable. Let $\sigma, \tau \in \mathcal{M}(F)$. We suppose $\Lambda_{\sigma\tau} \neq 1$ (otherwise (7) holds trivially). Let K, with residue class field E, denote the Henselization of F at $\sigma^{-1}(R) \cdot \tau^{-1}(R)$. Note $2^{s(E)} \geqq |\{\sigma, \tau\}|$ (if $\sigma \neq \tau$, then E has at least two orderings, so $s(E) \neq 0$). Hence [**1**, 3.18 and 3.19]

$$2^n \geqq 2^{s(F)} \geqq 2^{s(K)} = 2^{s(E)}|\Lambda_{\sigma\tau}| \geqq |\{\sigma, \tau\}|\,|\Lambda_{\sigma\tau}|.$$

Conversely, suppose (7) holds. There exists a place ρ on F with $s(F) = s(E) + \dim \Lambda$ where E is the formally real residue class field of ρ and Λ is the square factor group of the value group of ρ (e.g., take $E = F$). Suppose such ρ is chosen with $|\Lambda|$ maximal. Then $|\Lambda| \leqq 2^n$ (apply (7) to any $\sigma = \tau \in \mathcal{M}(F)$ factoring through ρ). The square factor group of the value group of every place from E into a formally real field is trivial (otherwise $|\Lambda|$ would not be maximal [**1**, 3.18 and 3.19]). Hence $s(E) \leqq 1$. If $s(E) = 0$, then $s(F) = \dim \Lambda \leqq n$, so $W_{\mathrm{red}}(F)$ is n-stable. If $s(E) = 1$, then E admits at least two distinct places into R, say σ and τ [**2**]. Then

$$2^{s(F)} = 2|\Lambda| \leqq |\Lambda_{\sigma\rho,\tau\rho}|\,|\{\sigma\rho, \tau\rho\}| \leqq 2^n,$$

so again $W_{\mathrm{red}}(F)$ is n-stable.

We now apply the above results to some familiar classes of fields. We wish to calculate $s_{\mathrm{red}}(F)$ and

$$s(F) = \inf\{n \geqq 0\colon W(F) \text{ is } n\text{-stable}\}.$$

(Warning: this is not the notation used in [**1**] or in the proof of Lemma 5.3.) Of course $W(F)$ is n-stable if and only if $n \geqq s(F)$ (Remark 5.1 D). In the following examples most of the values of $s_{\mathrm{red}}(F)$ are well-known (see especially [**1**]) and easily computed from Lemma 5.3. We leave these computations to the interested reader. Our remarks here about $s(F)$ substantially overlap [**8**, §5] and in many cases consist of showing that Elman and Lam's upper bounds for $s(F)$ actually equal $s(F)$.

EXAMPLES 5.4. (A) Suppose F is a finite algebraic extension of $R((x_1))\cdots((x_n))$ (iterated Laurent series). Then $s(F)=n$, and $s_{\text{red}}(F)$ is n if F is formally real and is 0 otherwise.

Proof. F is isomorphic to $F_0((x_1))\cdots((x_n))$ where F_0 is R or C [**11**, Theorem 6]. Note $s(F_0)=0$. Hence $s(F)\leqq n$ (Lemma 5.2). If $s(F)<n$, then $F_0((x_1))$ would be 0-stable (Lemma 5.2). But this is false since $1+\langle x_1\rangle$ is in $I(F_0((x_1)))$ but not in $2W(F_0((x_1)))$ [**12**, Springer's Theorem, p. 145].

(B) Let F be an algebraic function field in n variables over R. Then $s(F)=n$, and $s_{\text{red}}(F)$ is n if F is formally real and is 0 otherwise.

Proof. Elman and Lam show $s(F)\leqq n$ [**8**, Example 2, p. 1177]. There exists a place ρ on F whose residue class field is R or C and whose value group has 2^n square classes. Let E be a maximal immediate extension of F at ρ. Then E is isomorphic to a finite algebraic extension of $R((x_1))\cdots((x_n))$ [**11**, Theorem 6]. Thus by Example A above and Remark 5.1 C, $n=s(E)\leqq s(F)\leqq n$.

(C) Suppose F is an algebraic number field. Then $s(F)=2$; indeed, $I^3=4I$. $s_{\text{red}}(F)$ is 1 if F has more than one ordering and 0 otherwise.

Proof. Since I^3 is torsion-free [**14**, p. 81] and $W_{\text{red}}(F)$ is 1-stable, we have $I^3=4I$ (this requires a trivial extension of Remark 5.1 E). If we had $I^2=2I$, then every element of I^2 would have trivial Hasse–Witt invariant, contradicting [**14**, Lemma 4.4, p. 97]. Hence $s(F)=2$.

(D) Let F be a finite algebraic extension of $Q((x_1))\cdots((x_n))$. Then $s(F)=n+2$. Moreover, if $n=3$, then $I^8=2^4I^4$. $s_{\text{red}}(F)$ is 0, n, or $n+1$ according as the residue class field F_0 of the canonical place from F into an algebraic number field has zero, one, or more than one ordering.

Proof. F is isomorphic to $F_0((x_1))\cdots((x_n))$ (Kaplansky's theorem). Lemma 5.2 and Example C show $s(F)=n+2$. That $I^8=2^4I^4$ when $n=3$ follows from Example C and formula (7_n) of [**13**] (argue as in the proof of [**8**, Example 4, p. 1178]).

(E) Let F_0 be an algebraic number field. Then $s(F_0(x))=3$. If F is an algebraic function field in n variables over F_0, then $s(F)\geqq n+2$. $s_{\text{red}}(F)$ is 0 or $n+1$ depending on whether F has finite level or not.

Proof. Example C and [**13**, Lemma 5.7] show $s(F_0(x))\leqq 3$. That $s(F_0(x))\geqq 3$ follows by Remark 5.1 C and Example D: $s(F_0(x))\geqq s(F_0((x)))=3$. The same argument shows $s(F)\geqq n+2$.

Some final remarks. We can use Remark 5.1 B to get an upper bound on $s(F)$ for arbitrary algebraic extensions of the fields considered in Examples A, B, C, and D. The condition $k_{n+1} = l(-1)k_n$ is discussed in [7, §5]; in particular, Elman and Lam show $k_4 = l(-1)k_3$ (so $W(F)$ is 3-stabie, cf. Remark 5.1 A) when $|k_3| \leqq 8$ [7, Corollary 5.9]. They also show that if the quaternion algebras over F form a subgroup of $B(F)$, then $W(F)$ is 3-stable [9].

6. Superpythagorean fields. Suppose F is a superpythagorean field, i.e., a formally real field in which every subgroup of $F^{\cdot}$ of index two excluding -1 is an ordering of F [8, Definition 4.4]. Such fields play a special role (as "local objects") in a general theory of formally real fields [4]. In this section we compute $\operatorname{Im} \hat{w}$.

NOTATION 6.1. Suppose A is a finite subset of $F^{\cdot}$ whose cosets in $F^{\cdot}/Z^{\cdot} \cdot F^{\cdot 2}$ are linearly independent. The elements

$$l(-1)^t \prod_{a \in B} l(a) \qquad (t \geqq 0, B \subseteq A) \tag{8}$$

form a basis for a subspace $k(A)$ of k_* [8, Theorem 5.13 (2)]. (The empty product equals 1.) Hence for each $C \subseteq A$ there is a unique map $\phi_C \colon k(A) \to \{0,1\}$ (where $0, 1 \in Z$) which preserves addition modulo two (i.e., induces a homomorphism into $Z/2Z$) and carries each basis element (8) to 1 if $C \supseteq B$ and to 0 otherwise. Finally, set $V(C) = \{B \colon B \subseteq A$ and $|C \cap B|$ is even $\}$ for all $C \subseteq A$.

Now let $\alpha \in k_{\text{reg}}$. We give a computational procedure for determining whether $\alpha \in \operatorname{Im} \hat{w}$. Because α is regular, there exists a set A satisfying the hypotheses of (6.1) such that every term of α is in $k(A)$ (cf. Lemma 2.3). Fix such a set A.

Note. If $F^{\cdot}/F^{\cdot 2}$ is finite, we can take for A any subset of $F^{\cdot}$ representing a basis of $F^{\cdot}/Z^{\cdot} \cdot F^{\cdot 2}$.

THEOREM 6.2. *Let $n = |A|$. $\alpha \in \operatorname{Im} \hat{w}$ if and only if for all $C \subseteq A$,*

$$\sum_{k=1}^{m} \sum_{B \in V(C)} 2^k \phi_B(\alpha_{2^k}) \equiv 0 \pmod{2^{m+1}} \tag{9}$$

where $m = n-2$ if $C = \varnothing$ and $m = n-3$ otherwise.

Before sketching the proof of (6.2) we give some examples.

EXAMPLES 6.3. (1) If F has 16 or fewer square classes, then $W(F)$ is 3-stable. It is easy to check directly that (9) holds for all $C \subseteq A$. If F has 32 square classes, then (9) can be shown to hold for all $C \neq \emptyset$. (It suffices to show $\Sigma_{B \in V(C)} \phi_B(l(a)l(b))$ is even for all $a, b \in A \cup \{-1\}$. But $\{B \in V(C)\colon B \supseteq \{a, b\} \cap A\}$ has an even number of elements.) Thus in this case, $\alpha \in \operatorname{Im} \hat{w}$ if and only if

$$\sum_{B \subseteq A} \phi_B(\alpha_2) + 2\phi_B(\alpha_4) \equiv 0 \pmod 4.$$

(2) Let $F = R((x_1))((x_2))((x_3))((x_4))((x_5))$. Let $A = \{x_1, x_2, x_3, x_4, x_5\}$ (see the Note preceeding 6.2). Then $\alpha = 1 + l(x_1)l(x_2) + l(x_3)l(x_4)$ and $\beta = 1 + l(x_1)l(x_2)l(x_3)l(x_4)$ are not in $\operatorname{Im} \hat{w}$ (in both cases, (9) fails with $C = \emptyset$). However $\alpha\beta \in \operatorname{Im} \hat{w}$. Also, $\alpha + l(x_5)^2 \notin \operatorname{Im} \hat{w}$ (the congruence (9) holds with $C = \emptyset$ but not with $C = \{x_5\}$).

We now sketch the proof of Theorem 6.2. Arguing as in the proof of Lemma 2.3, we can write α in the form $(\Pi_{i=1}^{n-2} 1 + \alpha_{2^i})(\Pi_\beta 1 + \beta)$ where each β is one of the basis elements (8) for $k(A)$ with degree at least 2^{n-1}. Since each factor $1 + \beta$ is in $\operatorname{Im} \hat{w}$ [13, Lemma 3.2], we may assume without loss of generality that $\alpha_{2^i} = 0$ if $i = 0$ or $i \geqq n - 1$. Let $\mathcal{O}$ denote the space of orderings of F [12]. For each $P \in \mathcal{O}$, set $f(P) = \Sigma_{i=1}^{n-2} 2^{i+1} \phi_{A \setminus P}(\alpha_{2^i})$. Then f is continuous. Consider the diagram

$$\begin{array}{ccc} W(F) & \xrightarrow{\ t\ } & \Pi_{\mathcal{O}} W(F_p) \\ {\scriptstyle w}\big\downarrow & & \big\downarrow{\scriptstyle w'} \\ Z^{\cdot} \times k_\pi(F)^{\cdot} & \xrightarrow{\ t'\ } & \Pi_{\mathcal{O}} Z^{\cdot} \times k_\pi(F_p)^{\cdot} \end{array}$$

where for each $P \in \mathcal{O}$, F_p denotes the real closure of F at P. (t and t' are induced by the inclusions $F \to F_p$, and w' is the product of the "Stiefel–Whitney maps" $W(F_p) \to Z^{\cdot} \times k_\pi(F_p)^{\cdot}$ of Proposition 1.1.) We identify each $W(F_p)$ with Z (by the signature map) and check that $w'(-f) = t'((1, \alpha))$. Since t' and w' are injective ([8, Theorem 5.13 (6)] and Corollary 3.3), we have $f \in \operatorname{Im} t$ if and only if $\alpha \in \operatorname{Im} \hat{w}$ (Proposition 1.3). Thus $\alpha \in \operatorname{Im} w$ if and only if for all $b \in F^{\cdot}$,

$$\int_{V(b)} f(P)dP \in \mu(V(b))Z \tag{10}$$

(see [3, Theorem 15 (3)] for notation and the proof). If $C \subseteq A$ and $b = \Pi_{a \in C}\, a$, then the left hand side of (10) equals $2^{1-n} \Sigma_{B \in V(C)} \Sigma_{i=1}^{n-2} 2^i \phi_B(\alpha_{2^i})$. Also, $\mu(V(b))$ is 1 if $C = \emptyset$ and 1/2 otherwise. The necessity of (9) follows immediately. Its sufficiency is an easy consequence.

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University of Hawaii
Honolulu, HI 96822