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GENERAL SOLVABILITY THEOREMS

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A further development of the method of contractor directions is presented. In order to make the method applicable to applied problems, certain conditions are imposed on the sets of contractor directions. This in turn requires a more sophisticated transfinite induction argument. In terms of contractor directions, sufficient conditions are given for existence of solutions of nonlinear operator equations in Banach spaces.

Introduction. The general method of contractor directions seems to be a natural generalization of the method of directional contractors. However, in order to make the method applicable to nonlinear differential and integral equations, a further development of the concept of contractor directions is necessary. Therefore, certain conditions are imposed on the sets of contractor directions. In this way, a special class of increasing continuous functions is involved in the definition of specialized contractor directions. This class is closely connected with the classical Cauchy integral test for infinite series. By using the method of specialized contractor directions sufficient conditions are obtained for general existence theorems of solutions of nonlinear equations. This method does not require the operator to have closed range. However, if the range of the operator in question is closed, then no additional conditions are imposed on the sets of contractor directions. In this particular case, the method is much simpler. Thus, the unified theory also yields a generalization of results obtained by Pohožaev, Browder, Zabreiko and Krasnosel'skii, Kirk and Caristi. All the methods used by the authors mentioned above are different and applicable only in the case of an operator with closed range.

1. A general solvability principle. Let $P: D(P) \subset X \to Y$ be a nonlinear mapping, where $D(P)$ is a vector space and X, Y are real or complex Banach spaces. Denote by $\mathbf{B}$ the class of increasing continuous functions B such that

(i) $$B(0)=0, \quad B(s)>0 \text{ for } s>0;$$

(ii) $$\int_0^a s^{-1}B(s)ds<\infty \quad \text{for some } a>0;$$

and

(iii) $$0<\gamma<1 \text{ implies } B(e^{-\gamma t})t \to 0 \text{ as } t\to\infty.$$

Let g be a continuous function such that $g(t)>0$ for $t>0$.

DEFINITION 1.1. $\Gamma_x(P)=\Gamma_x(P,q)$ is a set of contractor directions at $x\in D(P)$ for $P\colon D(P)\subset X\to Y$, which has the (B,g)-property, if for arbitrary $y\in\Gamma_x(P)$ there exist a positive number $\epsilon=\epsilon(x,y)\leqq 1$ and an element $h\in X$ such that

$$\|P(x+\epsilon h)-Px-\epsilon y\|\leqq q\epsilon\|y\| \tag{1.1}$$

$$\|h\|\leqq B(g(\|x\|)\cdot\|y\|), \tag{1.2}$$

where $x+\epsilon h\in D(P)$ and $q=q(P)<1$ is some positive constant independent of $x\in D(P)$.

DEFINITION 1.2. The nonlinear mapping $P\colon D(P)\subset X\to Y$ is (B,g)-differentiable at $x\in D(P)$ if there is a dense subset $V\subset Y$ such that for arbitrary $y\in V$ there exists an element $h\in X$ which satisfies condition (1.2) and $\|P(x+\epsilon h)-Px-\epsilon y\|/\epsilon\to 0$ as $\epsilon\to 0+$.

Given an element $x_0\in D(P)$, a continuous function g such that $g(t)>0$ for $t>0$, and a function $B\in\mathbf{B}$, let us put

$$\bar r=2(1-q)^{-1}\int_0^b s^{-1}B(s)ds,\quad b=e^{1-q},$$

where $0<q<1$ is the same as in Definition 1.1. We can find a polynomial $\bar p$ such that

$$|g(t)-\bar p(t)|\leqq 1 \text{ for } 0\leqq t\leqq\|x_0\|+\bar r.$$

Then we have

$$g(t)\leqq p(t)=\sum_{i=0}^{N}c_it^i \quad\text{for } 0\leqq t\leqq\|x_0\|+\bar r, \tag{iv}$$

where p dominates $\bar p+1$ and has nonnegative coefficients. Now put

$$W(\|x_0\|)=\frac{1}{2}\sum_{i=0}^{N}c_i\sum_{k=0}^{i}\binom{i}{k}\|x_0\|^{2k} \tag{v}$$

$$C=\sum_{i=0}^{N}2^ic_i \tag{vi}$$

$$\rho=1/2N\quad\text{and}\quad\beta=1/2C. \tag{vii}$$

Let $M\geqq 1$ be such that

(viii) $$B(e^{W(\|x_0\|)+1}\|Px_0\|e^{-\frac{1}{2}(1-q)t})t < \rho\beta(1-q) \quad \text{for } t \geqq M.$$

A local existence theorem. Let $S = S(x_0, r)$ be an open ball with center $x_0 \in X$ and radius r, and put $U = D(P) \cap \bar{S}$, where $\bar{S}$ is the closure of S.

THEOREM 1.1. *Suppose that the following hypotheses are satisfied*:

(1.3) $$P\colon U \to Y \text{ is closed on } U;$$

(1.4) *for each $x \in U_0 = D(P) \cap S$, a set $\Gamma_x(P)$ of contractor directions with the (B, g)-property exists, being dense in some ball with center 0 on Y*;

(1.5) $$r \geqq 2(1-q)^{-1}\int_0^a s^{-1}B(s)ds,\ a = e^{1-q}e^{W(\|x_0\|)}\|Px_0\|;$$

(1.6) $$B(e^{W(\|x_0\|)+1}\|Px_0\|)M < \rho\beta(1-q) \text{ and } e^{W(\|x_0\|)}\|Px_0\| < 1,$$

where $W(\|x_0\|), M$ and ρ, β are defined by (i)–(viii), *respectively. Then the equation $Px = 0$ has a solution $x \in U$.*

Proof. The proof is based upon a further development of the transfinite induction argument.

We construct well-ordered sequences of positive numbers t_α and elements $x_\alpha \in D(P)$ as follows. Put $t_0 = 0$, and let x_0 be the given element. Suppose that t_γ and x_γ have been constructed for all $\gamma < \alpha$, provided, for arbitrary ordinal numbers $\gamma < \alpha$, inequalities

(1.7_γ) $$\|Px_\gamma\| \leqq e^{-(1-q)t_\gamma}\|Px_0\|,$$

(1.8_γ) $$\|x_\gamma\| \leqq \|x_0\| + \beta(1-q)t_\gamma^\rho,$$

are satisfied, where $\rho = 1$ if $t_\gamma < 1$, and $0 < \rho < 1$ if $t_\gamma \geqq 1, 0 < \beta < 1$. The constants ρ and β will be determined below.

For first kind ordinal numbers, $\beta = \gamma + 1 < \alpha$, the following inequalities are satisfied:

$(1.9_{\gamma+1})$ $$0 < t_{\gamma+1} - t_\gamma \leqq 1$$

$(1.10_{\gamma+1})$ $$\|x_{\gamma+1} - x_\gamma\| \leqq (t_{\gamma+1} - t_\gamma)B(e^{W(\|x_0\|)}\|Px_0\|e^{-\frac{1}{2}(1-q)t_\gamma}),$$

and

$(1.11_{\gamma+1})$ $$\|Px_{\gamma+1} - Px_\gamma\| \leqq (1+q)\|Px_0\|e^{-(1-q)t_\gamma}(t_{\gamma+1} - t_\gamma);$$

and, for second kind (limit) ordinal numbers, $\gamma < \alpha$, the following relations hold:

$$(1.12_\gamma)\qquad t_\gamma = \lim_{\beta \nearrow \gamma} t_\beta, \quad x_\gamma = \lim_{\beta \nearrow \gamma} x_\beta, \quad Px_\gamma = \lim_{\beta \nearrow \gamma} Px_\beta.$$

Then it follows from (1.10), (1.12), Lemmas 1.3 and 1.4 **[2]** that, for arbitrary $\lambda < \gamma < \alpha$, we have

$$\begin{aligned}
\|x_\gamma - x_\gamma\| &\leq \sum_{\lambda \leqq \beta < \gamma} \|x_{\beta+1} - x_\beta\| \leqq \sum_{\lambda \leqq \beta < \gamma} (t_{\beta+1} - t_\beta) B(e^{W(\|x_0\|)} \|Px_0\| e^{-\frac{1}{2}(1-q)t_\beta}) \\
&= \sum_{\lambda \leqq \beta < \gamma} (t_{\beta+1} - t_\beta) B(e^{W(\|x_0\|)} \|Px_0\| e^{\frac{1}{2}(1-q)(t_{\beta+1}-t_\beta)} e^{-\frac{1}{2}(1-q)t_{\beta+1}}) \\
&< \sum_{\lambda \leqq \beta < \gamma} (t_{\beta+1} - t_\beta) B(e^{1-q} e^{W(\|x_0\|)} \|Px_0\| e^{-\frac{1}{2}(1-q)t_{\beta+1}}) \\
&\leqq \sum_{\lambda \leqq \beta < \gamma} \int_{t_\beta}^{t_{\beta+1}} B(e^{1-q} e^{W(\|x_0\|)} \|Px_0\| e^{-\frac{1}{2}(1-q)t}) dt \\
&= \int_{t_\gamma}^{t_\gamma} B(e^{1-q} e^{W(\|x_0\|)} \|Px_0\| e^{-\frac{1}{2}(1-q)t}) dt.
\end{aligned}$$

Hence, we obtain the following estimate

$$(1.13)\qquad \|x_\gamma - x_\lambda\| \leqq \int_{t_\lambda}^{t_\gamma} B(e^{1-q} \cdot e^{W(\|x_0\|)} \|Px_0\| e^{-\frac{1}{2}(1-q)t}) dt.$$

In the same way, we obtain from (1.9), (1.11), Lemmas 1.3 and 1.4 **[2]** that

$$(1.14)\qquad \|Px_\gamma - Px_\lambda\| \leqq (1+q) e^{1-q} \|Px_0\| \int_{t_\lambda}^{t_\gamma} e^{-(1-q)t} dx.$$

Suppose that α is a first kind ordinal number. If $Px_{\alpha-1} = 0$, then the proof of the theorem is completed.

If $Px_{\alpha-1} \neq 0$, then we put

$$(1.15)\qquad t_\alpha = t_{\alpha-1} + \tau_\alpha,$$

and

$$(1.16)\qquad x_\alpha = x_{\alpha-1} + \tau_\alpha h_\alpha,$$

where h_α and $\tau_\alpha = \epsilon \leqq 1$ are chosen so as to satisfy (1.1) and (1.2) with $x = x_{\alpha-1}$, $h = h_\alpha$ and $y = -Px_{\alpha-1}$. Hence, we obtain, by virtue of $(1.7_{\alpha-1})$ and (1.1) with $x = x_{\alpha-1}$, $y = -Px_{\alpha-1}$, $h = h_\alpha$,

$$\begin{aligned}
\|Px_\alpha\| &\leqq (1 - \tau_\alpha)\|Px_{\alpha-1}\| + q\tau_\alpha \|Px_{\alpha-1}\| \\
&= (1 - (1-q)\tau_\alpha)\|Px_{\alpha-1}\| < e^{-(1-q)\tau_\alpha}\|Px_{\alpha-1}\| \leqq e^{-(1-q)t_\alpha}\|Px_0\|,
\end{aligned}$$

$$(1.17_\alpha) \qquad \|Px_\alpha\| \leqq e^{-(1-q)t_\alpha}\|Px_0\|.$$

Now we consider two cases; case (a), where $t_\alpha < 1$ and case (b), where $t_\alpha \geqq 1$. In both cases we have

$$p(\|x_\alpha\|) = \sum_{i=0}^{N} c_i \|x_\alpha\|^i \leqq \sum_{i=0}^{N} c_i(\|x_0\| + \beta(1-q)t_\alpha^\rho)^i$$

$$= \sum_{i=0}^{N} c_i \sum_{k=0}^{i} \binom{i}{k} \|x_0\|^k [\beta(1-q)t_\alpha^\rho]^{i-k} \leqq \frac{1}{2}\sum_{i=0}^{N} c_i \sum_{k=0}^{i}\binom{i}{k}\|x_0\|^{2k}$$

$$+\frac{1}{2}\sum_{i=0}^{N} c_i \sum_{k=0}^{i}\binom{i}{k}[\beta(1-q)t_\alpha^\rho]^{2(i-k)} \leqq W(\|x_0\|) + \frac{1}{2}\beta(1-q)Ct_\alpha.$$

Hence, we obtain

$$(1.18_\alpha) \qquad p(\|x_\alpha\|) \leqq W(\|x_0\|) + \frac{1}{2}(1-q)t_\alpha$$

provided (1.18_α) holds true, where

$$(1.19) \qquad W(\|x_0\|) = \frac{1}{2}\sum_{i=0}^{N} c_i \sum_{k=0}^{i}\binom{i}{k}\|x_0\|^{2k}$$

$$(1.20) \qquad C = \sum_{i=0}^{N} c_i \sum_{k=0}^{i}\binom{i}{k} = \sum_{i=0}^{N} 2^i c_i,$$

and where ρ and β are chosen so as to satisfy

$$(1.21) \qquad \rho 2N = 1 \quad \text{and} \quad \beta C = 1/2.$$

In case (a), we have, by (1.15), (1.16), (1.2), $(1.8_{\alpha-1})$ and (iv)–(viii),

$$\|x_\alpha\| \leqq \|x_{\alpha-1}\| + \tau_\alpha B(g(\|x_{\alpha-1}\|)\cdot\|y_{\alpha-1}\|)$$

$$\leqq \|x_0\| + \beta(1-q)t_{\alpha-1}^\rho + \tau_\alpha B(e^{W(\|x_0\|)+\frac{1}{2}(1-q)t_{\alpha-1}}\|Px_0\|e^{-(1-q)t_{\alpha-1}}).$$

Hence,

$$(1.22) \qquad \|x_\alpha\| \leqq \|x_0\| + \beta(1-q)t_{\alpha-1}^\rho + \tau_\alpha B(e^{W(\|x_0\|)}\|Px_0\|e^{-\frac{1}{2}(1-q)t_{\alpha-1}}),$$

and

$$(1.23) \qquad \|x_\alpha\| \leqq \|x_0\| + \beta(1-q)t_{\alpha-1}^\rho + \tau_\alpha B(e^{W(\|x_0\|)}\|Px_0\|).$$

Hence, it follows that

$$\|x_\alpha\| \leqq \|x_0\| + \beta(1-q)(t_{\alpha-1}+\tau_\alpha)$$

or condition (1.8_α) will be satisfied with $\rho = 1$ if

$$B(e^{W(\|x_0\|)}\|Px_0\|) \leqq \beta(1-q) \tag{1.24}$$

is satisfied, where β is defined by (1.21) and (1.20).

Now let us consider case (b). It is easy to see that (1.8_α) will be satisfied if

$$\beta(1-q)t_{\alpha-1}^{\rho} + \tau_\alpha B(e^{W(\|x_0\|)}\|Px_0\|e^{-\frac{1}{2}(1-q)t_{\alpha-1}}) \leqq \beta(1-q)(t_{\alpha-1}+\tau_\alpha)^{\rho}$$

or

$$B(e^{W(\|x_0\|)}\|Px_0\|e^{-\frac{1}{2}(1-q)t_{\alpha-1}}) < \rho\beta(1-q)(t_{\alpha-1}+\tau_\alpha)^{\rho-1}, \tag{1.25}$$

by virtue of (1.22). Therefore, replacing $t_{\alpha-1}+\tau_\alpha$ in (1.25) by t, let $M \geqq 1$ be such that

$$B(e^{W(\|x_0\|)+1}\|Px_0\|e^{-\frac{1}{2}(1-q)t})t < \rho\beta(1-q) \quad \text{for } t \geqq M, \tag{1.26}$$

by virtue of (viii), and let

$$B(e^{W(\|x_0\|)+1}\|Px_0\|)M < \rho\beta(1-q), \tag{1.27}$$

where ρ and β are defined by (1.20), (1.21). Then, obviously (1.25) will be satisfied, and consequently, (1.8_α) will be satisfied, too. Thus, in both cases, conditions (1.26) and (1.27) imply that (1.8_α) will be satisfied, where α is an ordinal number of first kind.

It follows from (1.15), (1.16), (1.2), $(1.7_{\alpha-1})$, $(1.18_{\alpha-1})$ and (iv) that

$$\|x_\alpha - x_{\alpha-1}\| \leqq (t_\alpha - t_{\alpha-1})B(e^{W(\|x_0\|)}\|Px_0\|e^{-\frac{1}{2}(1-q)t_{\alpha-1}}). \tag{1.28_α}$$

Thus, we have shown that if α is a first kind ordinal number, then the induction assumptions hold true for α. Now suppose that α is an ordinal number of second kind and put $t_\alpha = \lim_{\gamma\nearrow\alpha} t_\gamma$. Let $\{\gamma_n\}$ be an increasing sequence convergent to α. It follows from (1.13) and (1.14) that $\{x_{\gamma_n}\}$ and $\{Px_{\gamma_n}\}$ are Cauchy sequences and so are $\{x_\gamma\}$ and $\{Px_\gamma\}$. Denote by x_α and y_α their limits, respectively. Since P is closed on U, we infer that $x_\alpha \in U$ and $y_\alpha = Px_\alpha$, provided $x_\gamma \in U$. If $t_\alpha < \infty$, then the limit passage in (1.7_{γ_n}) and (1.8_{γ_n}) yields (1.7_α) and (1.8_α), respectively. The relationships (1.12_α) are satisfied by definition of t_α and x_α, since $y_\alpha = Px_\alpha$. This process will terminate if $t_\alpha = \infty$, where α is of second kind. In this case, $Px_\alpha = 0$, by virtue of (1.7_α). The limit x_α exists, by (1.13), since

$$\int_0^\infty B(e^{1-q}e^{W(\|x_0\|)}\|Px_0\|e^{-\frac{1}{2}(1-q)t})dt = 2(1-q)^{-1}\int_0^a s^{-1}B(s)ds < \infty, \tag{1.29}$$

where $a = e^{1-q}e^{W(\|x_0\|)}\|Px_0\|$. Finally, it follows from (1.13), (1.29) and (1.5) that all $x_\gamma \in U_0$. It results from (1.5) and (1.6) that

$$2(1-q)^{-1}\int_0^a s^{-1}B(s)ds \leqq 2(1-q)^{-1}\int_0^{e^{1-q}} s^{-1}B(s)ds. \tag{1.30}$$

Hence, it follows that $g(\|x_\alpha\|) \leqq p(\|x_\alpha\|)$, i.e., condition (iv) is valid for $t = \|x_\alpha\|$, and as a consequence, we also obtain that (1.18_α) is true. This completes the proof.

Consider now a particular case of Theorem 1.1.

THEOREM 1.1a. *Suppose that the hypotheses* (1.3)–(1.5) *are satisfied, where* $B(s) = s$, $g(t) = C(t+1)$ *for some constant* $C > 0$ *and*

$$r \geqq 2C(\|x_0\|+1)\|Px_0\|e^{(1-q)/2}/(1-q), \tag{1.5a}$$

where $\|x_0\|$ *should be replaced by* 1 *if* $\|x_0\| \leqq 1$.

$$C\|Px_0\| \leqq (1-q)/4. \tag{1.6a}$$

Then, equation $Px = 0$ *has a solution* $x \in U$.

Proof. The general method of proof is similar to that of Theorem 1.1. However, we have to replace the induction assumption (1.8_γ) by the following one.

$$\|x_\gamma\| \leqq \|x_0\|e^{\beta(1-q)t_\gamma}, \quad \beta = 1/2, \tag{$1.8'_\gamma$}$$

where $\|x_0\|$ should be replaced by 1 if $\|x_0\| \leqq 1$. Then we obtain the following estimate,

$$\|x_\gamma - x_\lambda\| \leqq \sum_{\lambda \leqq \beta < \gamma} \|x_{\beta+1} - x_\beta\| \leqq \sum_{\lambda \leqq \beta < \gamma} (t_{\beta+1} - t_\beta)C(\|x_\beta\|+1)\|Px_\beta\|$$

$$\leqq \sum_{\lambda \leqq \beta < \gamma} (t_{\beta+1} - t_\beta)C(\|x_0\|e^{\beta(1-q)t_\beta}+1)\|Px_0\|e^{-(1-q)t_\beta}$$

$$= \sum_{\lambda \leqq \beta < \gamma} (t_{\beta+1} - t_\beta)C(\|x_0\|\,\|Px_0\|e^{-\frac{1}{2}(1-q)t_\beta} + \|Px_0\|e^{-(1-q)t_\beta})$$

$$\leqq \sum_{\lambda \leqq \beta < \gamma} (t_{\beta+1} - t_\beta)C(\|x_0\|+1)\|Px_0\|e^{-\frac{1}{2}(1-q)t_\beta}$$

$$\leqq \sum_{\lambda \leqq \beta < \gamma} (t_{\beta+1} - t_\beta)C(\|x_0\|+1)\|Px_0\|e^{(1-q)/2}e^{-\frac{1}{2}(1-q)t_{\beta+1}}$$

$$\leqq \sum_{\lambda \leqq \beta < \gamma} C(\|x_0\|+1)\|Px_0\|e^{(1+q)/2}\int_{t_\beta}^{t_{\beta+1}} e^{-\frac{1}{2}(1-q)t}dt.$$

Hence, we obtain

$$\|x_\gamma - x_\lambda\| \leqq Ce^{(1-q)/2}(\|x_0\|+1)\|Px_0\|\int_{t_\lambda}^{t_\gamma} e^{-\frac{1}{2}(1-q)t}dt$$
$$\leqq 2Ce^{(1-q)/2}(\|x_0\|+1)\|Px_0\|/(1-q) \leqq r.$$

It remains to show the induction passage for $(1.8'_\gamma)$. Suppose that α is an ordinal number of first kind. We have in case of $\|x_0\|>1$,

$$\|x_\alpha\| \leqq \|x_{\alpha-1}\| + \tau_\alpha C(\|x_{\alpha-1}\|+1)\|Px_{\alpha-1}\|$$
$$\leqq \|x_{\alpha-1}\| + \tau_\alpha C(\|x_0\|e^{\beta(1-q)t_{\alpha-1}}+1)\|Px_0\|e^{-(1-q)t_{\alpha-1}}$$
$$\leqq \|x_{\alpha-1}\| + \tau_\alpha 2C\|x_0\|\cdot\|Px_0\|e^{-\frac{1}{2}(1-q)t_{\alpha-1}}.$$

Hence, it follows that

$$\|x_\alpha\| \leqq \|x_0\|e^{\beta(1-q)t_\alpha}, \quad \beta = 1/2$$

if

$$\|x_0\|e^{\beta(1-q)t_{\alpha-1}} + \tau_\alpha 2C\|x_0\|\cdot\|Px_0\|e^{-\frac{1}{2}(1-q)t_{\alpha-1}} \leqq \|x_0\|e^{\beta(1-q)t_\alpha},$$

where $t_\alpha = t_{\alpha-1}+\tau_\alpha$. Consider the function

$$\varphi(\tau) = e^{\beta(1-q)(t+\tau)} - \tau 2C\|Px_0\|e^{-\frac{1}{2}(1-q)t} - e^{\beta(1-q)t}.$$

The derivative $\varphi'(\tau)>0$ if $C\|Px_0\| \dot{<} (1-q)/4$. Hence, it follows that (1.8_α) holds true. Consider now the case, where $\|x_0\|\leqq 1$. Then we have

$$\|x_\alpha\| \leqq \|x_{\alpha-1}\| + \tau_\alpha C(\|x_{\alpha-1}\|+1)\|Px_{\alpha-1}\|$$
$$\leqq e^{\beta(1-q)t_{\alpha-1}} + \tau_\alpha C(e^{\beta(1-q)t_{\alpha-1}}+1)\|Px_0\|e^{-(1-q)t_{\alpha-1}}$$
$$\leqq e^{\beta(1-q)t_{\alpha-1}} + \tau_\alpha 2C\|Px_0\|e^{-(1-q)t_{\alpha-1}}.$$

Thus, we have to show that

$$e^{\beta(1-q)t_{\alpha-1}} + \tau_\alpha 2C\|Px_0\|e^{-\frac{1}{2}(1-q)t_{\alpha-1}} \leqq e^{\beta(1-q)(t_{\alpha-1}+\tau_\alpha)}.$$

But this inequality is exactly the same as in the case where $\|x_0\|>1$. Hence, it follows that condition (1.8_α) holds true in both cases. The further reasoning is the same as in the proof of Theorem 1.1.

REMARK 1.1. In Theorem 1.1, if $B(s)=s$ for $s\geqq 0$, then the first inequality in (1.6) can be replaced by the following one.

(1.31) $$e^{W(\|x_0\|)}\|Px_0\| < \rho\beta(1-q)^2/2e^{(1-q)/2},$$

where $W(\|x_0\|), \rho, \beta$ are defined by (1.19), (1.20), (1.21), respectively.

Proof. Put $a = e^{W(\|x_0\|)}\|Px_0\|$, then, by virtue of (1.22), we have to prove that

(1.32) $$\beta(1-q)t_{\alpha-1}^{\rho} + \tau_\alpha a e^{-\frac{1}{2}(1-q)t_{\alpha-1}} \leqq \beta(1-q)(t_{\alpha-1}+\tau_\alpha)^\rho.$$

Let us consider case (b), where $0<\rho<1, t \geqq 1$, since case (a) yields the same condition (1.24). Consider the function

$$\varphi(\tau) = \beta(1-q)(t+\tau)^\rho - \tau a e^{-\frac{1}{2}(1-q)t} - \beta(1-q)t^\rho,$$

which satisfies the condition $\varphi(0) = 0$, and its derivative is positive if

$$\varphi'(\tau) = \rho\beta(1-q)(t+\tau)^{\rho-1} - ae^{-\frac{1}{2}(1-q)t} > 0,$$

that is, if

(1.33) $$\alpha < \rho\beta(1-q)e^{\frac{1}{2}(1-q)t}(t+\tau)^{\rho-1}, \quad 0<\tau\leqq 1, \quad t \geqq 1.$$

But it is easy to see that

$$\rho\beta(1-q)^2/2e^{(1-q)/2} < \rho\beta(1-q)e^{\frac{1}{2}(1-q)t}(t+\tau)^{\rho-1}.$$

Thus, it follows from the last inequality that if (1.31) is satisfied, then so is (1.33) and, consequently, condition (1.32) holds true.

REMARK 1.2. Condition (1.4) can be replaced by the requirement that P is (B, g)-differentiable at every $x \in U_0$.

Proof. It follows from Definitions 1.1 and 1.2 that, for every $x \in U_0$, P has a set $\Gamma_x(P)$ of contractor directions with the (B, g)-property and $\Gamma_x(P)$ is dense in Y.

THEOREM 1.2. *Suppose that the hypotheses* (1.3), (1.4) *and* (1.5), *where* $a = e^{1-q}$, *and, in addition,* $Px_0 = y_0$. *Then there exists a ball K with center* y_0 *such that for every* $y \in K$, *the equation* $Px = y$ *has a solution* $x \in U$.

Proof. Denote by $\bar{P}$ the operator with values $\bar{P}x = Px - y$. Then there exists a ball K with center y_0 such that condition (1.6) will be satisfied for $\bar{P}$ if $y \in K$. Thus, all hypotheses of Theorem 1.1 are satisfied.

A global existence theorem. We now assume that $P\colon D(P) \subset X \to Y$ satisfies the following condition.

(α) If the sequence $\{x_n\} \subset D(P)$ is not bounded, then $\{Px_n\}$ contains no Cauchy sequence.

A mapping which satisfies condition (α) will be briefly called a Cauchy mapping.

THEOREM 1.3. *Suppose that the following hypotheses are satisfied.*

(1) *The graph of the nonlinear mapping $P\colon D(P) \subset X \to Y$ is closed in $X \times Y$.*

(2) *For each $x \in D(P)$, a set $\Gamma_x(P)$ of contractor directions with the (B, g)-property exists, which is dense in some ball with center 0 in Y.*

(3) *P is a Cauchy mapping.*

Then P is a mapping onto Y.

Proof. If the range $P(D(P))$ is closed, then a more general theorem is true (see **[2]**). Thus suppose there exists an element y_0 that is not in $P(D(P))$ and a sequence $\{x_n\} \subset D(P)$ such that

$$\|Px_n - y_0\| \to 0 \text{ as } n \to \infty \tag{1.34}$$

$$\|x_n\| \leqq c \text{ for } n = 1, 2, \cdots, \tag{1.35}$$

where c is some constant, by virtue of condition (3). Consider the operator $\bar{P}$ with values $\bar{P}x = Px - y_0$. With $\bar{r}$ defined by condition (iv), we approximate $g(t)$ on the closed interval $0 \leqq t \leqq c + \bar{r}$ by the polynomial $\bar{p}(t)$ and define $p(t)$ as in (iv). Then we put $\bar{x}_0 = x_m$, where x_m is an element of the sequence $\{x_n\}$ which satisfies inequalities (1.6) with x_0 replaced by x_m. Such an element exists for $\bar{P}$, since $\bar{P}x_n \to 0$ as $n \to \infty$. With such a choice of $\bar{x}_0$ for $\bar{P}$, all hypotheses of Theorem 1.1 are satisfied and the equation $\bar{P}x = Px - y_0 = 0$ has a solution. Therefore, our assumption that y_0 is not in $P(D(P))$ leads to a contradiction which proves the theorem.

As a consequence of Theorem 1.3, we obtain the following.

THEOREM 1.4. *A closed Cauchy mapping $P\colon D(P) \subset X \to Y$ which is (B, g)-differentiable is a mapping onto Y.*

Proof. The proof follows from Remark 1.2.

As a consequence of Theorem 1.1a, we obtain the following two theorems which do not require P to be a Cauchy mapping.

THEOREM 1.5. *Suppose that conditions* (1) *and* (2) *of Theorem* 1.3 *are satisfied, where* $B(s)=s$ *and* $g(t)=C(t+1)$ *with some constant* $C>0$. *Then* $P\colon D(P)\subset X\to Y$ *is a mapping onto* Y.

Proof. The proof follows from that of Theorem 1.3 and from Theorem 1.1a.

THEOREM 1.6. *A closed mapping* $P\colon D(P)\subset X\to Y$ *which is* (B,g)*-differentiable, where*

$$B(s)=s \quad \textit{and} \quad g(t)=C(t+1)$$

for some constant $C>0$, *is a mapping onto* Y.

Proof. The proof follows from that of Theorem 1.6, from Theorem 1.1a and Remark 1.2.

2. Nonbounded directional contractors. The global existence theorems proved in Section 1 can be applied to nonlinear operators having directional contractors which may not be bounded. For the definition of a directional contractor, see **[1]**. Denote by $L(Y\to X)$ the set of all linear continuous mappings from the Banach space Y into the Banach space X.

THEOREM 2.1. *A nonlinear closed Cauchy mapping* $P\colon D(P)\subset X\to Y$ *which has a directional contractor* $\Gamma\colon D(P)\to L(Y\to X)$ *such that*

$$\|\Gamma(x)\| \leqq g(\|x\|) \quad \textit{for all } x\in D(P), \tag{2.1}$$

where g *is some continuous function, is a mapping onto* Y.

Proof. The proof follows from Theorem 1.3, where $B(s)=s$.

We consider now nonlinear operators which are differentiable in the Gâteaux sense.

THEOREM 2.2. *Let* $P\colon D(P)\subset X\to Y$ *be a nonlinear closed Cauchy mapping. Suppose that for each* $x\in D(P)$, *the Gâteaux derivative* $P'(x)$ *is an additive and homogeneous operator which has a continuous inverse* $\Gamma(x)=P'(x)^{-1}$ *such that*

$$\|\Gamma(x)\| \leqq g(\|x\|) \quad \textit{for all } x \in D(P),$$

where g is some continuous function. Then P is a mapping onto Y.

Proof. The proof follows from Theorem 2.1, since $\Gamma: D(P) \to L(Y \to X)$ is a directional contractor for P and satisfies condition (2.1).

THEOREM 2.3. *Let $P: D(P) \subset X \to Y$ be a nonlinear closed Cauchy mapping which is differentiable in the sense of Fréchet. Suppose that for each $x \in D(P)$, the Fréchet derivative $P'(x)$ is a mapping onto Y. If there exists a continuous function g such that*

$$\|[P'(x)^*]^{-1}\| \leqq g(\|x\|) \quad \textit{for all } x \in D(P),$$

*where * indicates the adjoint, then P is a mapping onto Y.*

Proof. The proof follows from Theorem 1.4 and Lemma 3.4 **[2]**, since P is (B, g)-differentiable with $B(s) = s$.

REMARK 2.1. In Theorems 2.1–2.3, the condition that P is a Cauchy mapping can be omitted if $g(t) = C(t + 1)$ for some constant $C > 0$.

THEOREM 2.4. *Let $P: D(P) \subset X \to Y$ be a nonlinear closed Cauchy mapping which is differentiable in the Fréchet sense with Hölder continuous derivative $P'(x)$, i.e., there exist positive numbers K, $\alpha \leqq 1$ such that*

$$\|P'(x) - P'(\bar{x})\| \leqq K\|x - \bar{x}\|^{\alpha} \quad \textit{for all } x, \bar{x} \in D(P). \tag{2.2}$$

Moreover, for every $x \in D(P)$, let $A(x): X \to Y$ be a bounded linear nonsingular operator such that

$$\|A(x)^{-1}\| \leqq g(\|x\|) \quad \textit{and} \quad \|P'(x) - A(x)\| \leqq c(\|x\|), \tag{2.3}$$

for all $x \in D(P)$, where g and c are some functions, g being continuous. Suppose that there exist positive constants r and $q < 1$ such that

$$(1 + \alpha)^{-1}K[g(\|x\|)]^{1+\alpha}r + c(\|x\|)g(\|x\|) \leqq q < 1 \quad \textit{for all } x \in D(P). \tag{2.4}$$

Then P is a mapping onto Y.

Proof. The proof follows from Theorem 2.1, since $\Gamma(x) = A(x)^{-1}$ is a directional contractor satisfying condition (2.1). In fact, we have, by (2.2)–(2.4),

$$\begin{aligned}\|P(x+\Gamma(x)y) - Px - y\| &\leqq \|P(x+\Gamma(x)y) - Px - P'(x)\Gamma(x)y\| \\ &\quad + \|P'(x)\Gamma(x)y - A(x)\Gamma(x)y\| \\ &\leqq (1+\alpha)^{-1}K\|\Gamma(x)y\|^{1+\alpha} + c(\|x\|)\|\Gamma(x)y\| \\ &\leqq (1+\alpha)^{-1}K[g(\|x\|)]^{1+\alpha}\|y\|^{1+\alpha} + c(\|x\|)g(\|x\|)\|y\| \\ &\leqq q\|y\| \quad \text{if} \quad \|y\| \leqq r^{1/\alpha}.\end{aligned}$$

THEOREM 2.5. *Let $P\colon D(P) \subset X \to Y$ be a nonlinear closed Cauchy mapping and let $T\colon D(P) \subset X \to Y$ be an operator differentiable in the Fréchet sense with Hölder continuous Fréchet derivative $T'(x)$, i.e., there exist positive constants K and $\alpha \leqq 1$ such that*

$$\|T'(x) - T'(\bar{x})\| \leqq K\|x - \bar{x}\|^{\alpha}, \quad 0 < \alpha \leqq 1, \tag{2.5}$$

for all $x \in D(P)$. Moreover, suppose that $T'(x)$ is nonsingular and that there exist a continuous function g and a function c such that

$$\begin{aligned}&\|T'(x)^{-1}\| \leqq g(\|x\|) \quad \textit{and} \\ &\|(Px - Tx) - (P\bar{x} - T\bar{x})\| \leqq c(\|x\|)\|x - \bar{x}\|,\end{aligned} \tag{2.6}$$

for all $x, \bar{x} \in D(P)$. If there exists a positive constant $q < 1$ such that condition (2.4) is satisfied, then P is a mapping onto Y.

Proof. The proof follows from Theorem 2.1, since $\Gamma(x) = T(x)^{-1}$ is a directional contractor satisfying condition (2.1). In fact, we have, by (2.5), (2.6) and (2.4),

$$\begin{aligned}\|P(x+\Gamma(x)y) - Px - y\| &\leqq \|T(x+\Gamma(x)y) - Tx - T'(x)\Gamma(x)y\| \\ &\quad + \|[P(x+\Gamma(x)y - T(x+\Gamma(x)y)] - [Px - Tx]\| \\ &\leqq (1+\alpha)^{-1}K\|\Gamma(x)y\|^{1+\alpha} + c(\|x\|)\|\Gamma(x)y\| \\ &\leqq (1+\alpha)^{-1}K[g(\|x\|)]^{1+\alpha}\|y\|^{1+\alpha} + c(\|x\|)g(\|x\|)\|y\| \\ &\leqq [(1+\alpha)^{-1}K[g(\|x\|)]^{1+\alpha}r + c(\|x\|)g(\|x\|)]\|y\| \\ &\leqq q\|y\|| \quad \text{if} \quad \|y\| \leqq r^{1/\alpha}.\end{aligned}$$

REMARK 2.2. Remark 2.1 applies also to Theorems 2.4 and 2.5.

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