

ON PROSUPERSOLVABLE GROUPS

B. C. OLTIKAR AND LUIS RIBES

Let G be a prosupersolvable group (projective limit of finite supersolvable groups), whose order involves only finitely many primes; then we show that G is topologically finitely generated iff its Frattini subgroup is open in G . If a prosupersolvable group G is topologically finitely generated, so is each Sylow p -subgroup of G . If G is a topologically finitely generated prosupersolvable group, then every subgroup H of finite index is open.

1. Introduction. A profinite group is called *strongly complete* (cf. [1]) if it coincides with its profinite completion, i.e., if its subgroups of finite index are open. A profinite group is said to be *topologically finitely generated* if it contains a dense subgroup which is finitely generated as an abstract group. It is not difficult to find examples of (nontopologically finitely generated) profinite groups which are not strongly complete (cf. e.g., [6]).

J.-P. Serre has shown (in a letter to A. Pletch) that if G is a topologically finitely generated pro- p group, then G is strongly complete. Since a pronilpotent group is the cartesian product of its Sylow p -subgroups (cf. [8]), one deduces easily the same property for pronilpotent groups whose Sylow p -subgroups are topologically finitely generated. M. P. Anderson (cf. [1]), has extended this result to prove that topologically finitely generated profinite groups which are extensions of a proabelian group by a pronilpotent group are also strongly complete.

In this paper we show that topologically finitely generated prosupersolvable groups are strongly complete. First we study the structure of the Frattini subgroup and the Sylow p -subgroups of a prosupersolvable group. One knows that if P is a pro- p group, then it is topologically finitely generated iff its Frattini subgroup is open. In Theorem 3.8 we prove that the same characterization is valid for prosupersolvable groups whose orders involve only finitely many primes. Let \mathcal{C} be a class of finite groups closed under the operations of taking subgroups, homomorphic images and extensions, and suppose that there are at least two distinct primes p and q dividing the order of some group in \mathcal{C} . Let X be a set with at least two elements; then, the Sylow p -subgroups of the free pro- \mathcal{C} -group $F(X)$ on X , are topologically infinitely generated (cf. [4], Prop. 1.12). In contrast we prove here (Corollary 3.9) that the Sylow p -subgroups of a topologically finitely generated prosupersolvable group are topologically finitely generated.

2. The Frattini subgroup of a profinite group.

2.1. Given a profinite group G , its *Frattini subgroup* G^* (cf. [5]) is defined to be the intersection of all the maximal closed subgroups of G . It is clear that a maximal closed subgroup of G is in fact open. As in the discrete case, G^* is the characteristic subgroup of G consisting of the nongenerators (an element $x \in G$ is a nongenerator, if whenever G is the closed subgroup generated by x and a subset Y of G , then G is the closed subgroup generated by Y). The following easy property will be used often: if $\varphi: G_1 \rightarrow G_2$ is an epimorphism of profinite groups, then

$$\varphi(G_1^*) \subseteq G_2^* .$$

PROPOSITION 2.2. *Let $\{G_i | i \in I\}$ be an inverse system of profinite groups, where (I, \leq) is a directed set, and where the canonical maps $\varphi_{ij}: G_i \rightarrow G_j$, $i \geq j$, are epimorphisms. Then*

$$(\varprojlim G_i)^* = \varprojlim G_i^* .$$

Proof. Put $G = \varprojlim G_i$, and let $\pi_i: G \rightarrow G_i$ be the canonical epimorphism (cf. [3], p. 89). As we have pointed out in 2.1, $\pi_i(G^*) \subseteq G_i^*$; and since $G^* = \varprojlim \pi_i(G^*)$, we get $G^* \subseteq \varprojlim G_i^*$. Conversely, let $x = (x_i) \in \varprojlim G_i^*$, and suppose $x \notin G^*$. Then there is a maximal open subgroup \overleftarrow{M} of G with $x \notin \overleftarrow{M}$. Hence, there is some $i \in I$ with $x_i \notin \pi_i(\overleftarrow{M})$. Since $\pi_i(\overleftarrow{M})$ is maximal in G_i , one has $x_i \notin G_i^*$; a contradiction. Thus $x \in G^*$, i.e., $\varprojlim G_i^* \subseteq G^*$.

The following results can now be easily deduced from the corresponding properties of finite groups.

COROLLARY 2.3. *Let G be a profinite group. Then*

$$G/G^* = \varprojlim (G/U)/(G/U)^* ,$$

where U runs through the open normal subgroups of G .

COROLLARY 2.4. *For any profinite group G , G^* is pronilpotent.*

COROLLARY 2.5. *If H is a closed normal subgroup of a profinite group G , then $H^* \subseteq G^*$.*

3. Prosupersolvable groups. A projective limit of finite supersolvable groups is called *prosupersolvable*.

A finite group is supersolvable iff its maximal subgroups have

prime index (cf. [10], 9.3.8). From this one deduces immediately the following

PROPOSITION 3.2. *A profinite group G is prosupersolvable iff every maximal closed subgroup of G has prime index.*

From Corollary 3.2 and the fact that the Frattini factor of a finite supersolvable group is metabelian (cf. [10], 7.4.15 and 7.2.13), one gets the following.

PROPOSITION 3.3. *Let G be a prosupersolvable group. Then G/G^* is profinite metabelian.*

3.4. Let π denote a set of prime numbers, and π' the set of prime numbers not in π . A supernatural number δ (cf. [11], p. I-3) is a π -number if the prime numbers dividing δ are in π . A closed subgroup H of a profinite group G is a π -subgroup if the order of H is a π -number. If H is a maximal π -subgroup of G , it is called a *Sylow π -subgroup* of G , and it is called a *Hall π -subgroup* if it is a Sylow π -subgroup and $(G:H)$ is a π' -number. If G is prosupersolvable (and hence prosolvable), every Sylow π -subgroup is a Hall π -subgroup, and any two are conjugate (cf. [2]).

PROPOSITION 3.5. *Let π be the set of prime numbers greater than a fixed natural number n . Let G be a prosupersolvable group. Then*

- (1) *there is a (unique) normal Sylow π -subgroup K of G ;*
- (2) *there is a split exact sequence of prosupersolvable groups*

$$1 \longrightarrow K \longrightarrow G \xrightarrow{\varphi} H \longrightarrow 1$$

with φ an open map (i.e., G is the topological semidirect product of K and H), where H is a Sylow π' -subgroup of G .

Proof. For each open normal subgroup U of G , let K_U be the normal Sylow π -subgroup of G/U (cf. [9], p. 232). Set $K = \varprojlim K_U$. Then (cf. [2], Th. 7), K is a normal Sylow π -subgroup of G . Now, let H be a Sylow π' -subgroup such that $G = KH$ (cf. [2], Th. 13). Then G is, as an abstract group, the semidirect product of K and H . Since all the groups involved are compact, G is also the topological semidirect product of K and H . In particular G/K and H are topologically isomorphic and φ is open.

PROPOSITION 3.6. *Let G be a profinite group, and H a normal*

Hall π -subgroup of G , where π is a set of primes. Then $H^ = H \cap G^*$.*

Proof. By Proposition 2.2 we may assume that G is finite. Since $H^* \subseteq G^*$ (cf. [10], 7.3.17), we have $H^* \subseteq H \cap G^* = N \triangleleft G$. We may assume $H^* = 1$ (otherwise we reason with G/H^*), and we shall show $N = 1$. Since N is a normal nilpotent subgroup of H , it is contained in the Fitting subgroup of H . But $\text{Fitt}(H)$ is abelian since $H^* = 1$ (cf. [10], 7.4.15). So N is abelian, and since $H^* = 1$, H splits over N (cf. [10], 7.4.14). Thus (cf. [9], IV. 7.a) G splits over N . Say $G = NL$ with $N \cap L = 1$. If $L \neq G$, let $M \supseteq L$ be a maximal subgroup of G . Then $G = NM = M$, and this is impossible. Hence $L = G$ and $N = 1$.

COROLLARY 3.7. *Let G be a prosupersolvable group whose order is divisible only by a finite number of primes. If p is the largest prime dividing the order of G , and P is the unique Sylow p -subgroup of G (Proposition 3.5), then $P^* = P \cap G^*$.*

THEOREM 3.8. *Let G be a prosupersolvable group whose order is divisible by only finitely many primes p_1, p_2, \dots, p_r . Then G is topologically finitely generated iff G^* is open.*

Proof. By Proposition 3.2, all maximal closed subgroups of G have prime index (in particular they are open). Suppose G is topologically finitely generated; then for each prime p dividing the order of G , there are only finitely many open subgroups of index p . Therefore G^* is an intersection of finitely many open subgroups, and thus it is open. The converse is obvious.

COROLLARY 3.9. *Let G be a topologically finitely generated prosupersolvable group. Then for each prime number p , every Sylow p -subgroup of G is topologically finitely generated.*

Proof. Let p be a prime number. We use the notation of Proposition 3.5 with $n = p$. Then since π' is a finite set, by Theorem 3.8 H^* is open in H . Let S_p be the unique Sylow p -subgroup of H . By Corollary 3.7, $S_p^* = S_p \cap H^*$. Then S_p^* is open in S_p , and hence S_p is topologically finitely generated. Finally, it is clear that S_p is also a Sylow p -subgroup of G .

4. Subgroups of finite index of prosupersolvable groups.

LEMMA 4.1. *Let G be a profinite group and n natural number*

relatively prime to the order of G . Then $G^n = G$, where G^n is the set of n th powers of the elements of G .

Proof. The result is obviously true if G is finite. Hence $(G/U)^n = G/U$ for each open normal subgroup U of G ; i.e., the map

$$G/U \longrightarrow G/U,$$

given by $x \rightarrow x^n$ is onto. Since $G = \varprojlim G/U$, and \varprojlim preserves onto-ness (cf. [7], p. 35), we have that the map

$$G \longrightarrow G$$

given by $x \rightarrow x^n$, is also onto, i.e., $G = G^n$.

THEOREM 4.2. *Let G be a topologically finitely generated pro-supersolvable group. Then every subgroup V of G of finite index is open, i.e., G is strongly complete.*

Proof. We may assume that V is normal in G (otherwise we work with the intersection of the conjugates V in G). Say $(G:V) = n$. Let π consist of the prime numbers greater than n . Let K be the Sylow π -subgroup of G ; then we have a split exact sequence of profinite groups

$$1 \longrightarrow K \longrightarrow G \begin{array}{c} \xrightarrow{\varphi} \\ \xleftarrow{\quad} \end{array} H \longrightarrow 1$$

where H is a Sylow π' -subgroup of G (Proposition 3.5). Then n and the order of K are relatively prime. Hence (Lemma 4.1) $K = K^n$, and so $K \subseteq V$. Therefore $V = \varphi^{-1}(\varphi V)$. Thus it suffices to show that φV is open in H . Now, H is topologically finitely generated pro-supersolvable and its order involves finitely many primes only. Hence, by Theorem 3.8 its Frattini subgroup H^* is open in H . Therefore H^* is topologically finitely generated. On the other hand H^* is pro-nilpotent (Corollary 2.4). As we have pointed out in the Introduction (also, cf. [1], p. 26), it follows then from Serre's result that every subgroup of H^* of finite index is open. Therefore, $(\varphi V) \cap H^*$ is open in H^* , which in turn is open in H ; hence $(\varphi V) \cap H^*$ is open in H . Since φV contains an open subgroup of H , it is also open in H .

REFERENCES

1. M. P. Anderson, *Subgroups of finite index in profinite groups*, Pacific J. Math., **62** (1976), 19-28.
2. E. D. Bolker, *Inverse limits of solvable groups*, Proc. Amer. Math. Soc., **14** (1963), 147-152.
3. N. Bourbaki, *General Topology*, Addison-Wesley, Reading, Mass., 1966.

4. D. Gildenhuys and C.-K. Lim, *Free pro- \mathcal{S} -groups*, Math. Z., **125** (1972), 233-254.
5. K. W. Gruenberg, *Projective profinite groups*, J. London Math. Soc., (3) **42** (1967), 155-165.
6. H. L. Peterson, *Discontinuous characters and subgroups of finite index*, Pacific J. Math., **44** (1973), 683-691.
7. L. Ribes, *Introduction to profinite groups and Galois cohomology*, Queen's Papers in Pure and Applied Math., Kingston, Canada, 1970.
8. ———, *Productos amalgamados de grupos pronilpotentes*, Rev. Mat. Hisp.-Amer., **33** (1973), 49-53 and 133-138.
9. E. Schenkman, *Group Theory*, Van Nostrand, Princeton, N. J., 1965.
10. W. R. Scott, *Group Theory*, Prentice Hall, Englewood Cliffs, N. J., 1964.
11. J.-P. Serre, *Cohomologie Galoisienne*, Lect. Notes Math., no. 5, Springer, Berlin, 1965.

Received May 6, 1977 and in revised form December 8, 1977.

CARLETON UNIVERSITY
OTTAWA, ONT., CANADA