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SETS WITH $(d-2)$-DIMENSIONAL KERNELS

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This work is about the dimension of the kernel of a starshaped set, and the following result is obtained: Let S be a subset of a linear topological space, where S has dimension at least $d \geqq 2$. Assume that for every $(d+1)$-member subset T of S there corresponds a collection of $(d-2)$-dimensional convex sets $\{K_T\}$ such that every point of T sees each K_T via S, $(\text{aff } K_T) \cap S = K_T$, and distinct pairs aff K_T either are disjoint or lie in a d-flat containing T. Furthermore, assume that when T is affinely independent, then the corresponding set K_T is exactly the kernel of T relative to S. Then S is starshaped and the kernel of S is $(d-2)$-dimensional.

We begin with some preliminary definitions: Let S be a subset of a linear topological space, S having dimension at least $d \geqq 2$. For points x, y in S, we say x sees y via S if and only if the corresponding segment $[x, y]$ lies in S. Similarly, for $T \subseteqq S$, we say x sees T (and T sees x) via S if and only if x sees each point of T via S. The set of points of S seen by T is called the kernel of T relative to S and is denoted $\ker_S T$. Finally, if $\ker_S S = \ker S$ is not empty, then S is said to be starshaped.

This paper continues a study in [1] concerning sets having $(d-2)$-dimensional kernels. Foland and Marr [2] have proved that a set S will have a zero-dimensional kernel provided S contains a noncollinear triple and every three noncollinear members of S see via S a unique common point. In [1], an analogue of this result is obtained for subsets S of R^d having $(d-2)$-dimensional kernels. Here it is proved that, with suitable hypothesis, these results may be extended to include subsets S of an arbitrary linear topological space.

As in [1], the following terminology will be used: Conv S, aff S, cl S, bdry S, rel int S and ker S will denote the convex hull, affine hull, closure, boundary, relative interior and kernel, respectively, of the set S. If S is convex, dim S will represent the dimension of S.

2. Proof of the theorem.

THEOREM. *Let S be a subset of a linear topological space, where S has dimension at least $d \geqq 2$. Assume that for every $(d+1)$-member subset T of S there corresponds a collection of $(d-2)$-dimen-*

sional convex sets $\{K_T\}$ *such that every point of* T *sees each* K_T *via* S, $(\operatorname{aff} K_T) \cap S = K_T$, *and distinct pairs* $\operatorname{aff} K_T$ *either are disjoint or lie in a* d*-flat containing* T. *Furthermore, assume that when* T *is affinely independent, then the corresponding set* K_T *is exactly the kernel of* T *relative to* S. *Then* S *is starshaped and the kernel of* S *is* $(d-2)$*-dimensional.*

Proof. The proof of the theorem is motivated by an argument in [2, Lemma 3], and it will be accomplished by a sequence of lemmas.

LEMMA 1. *Assume that* $\operatorname{conv}(K \cup \{x\}) \cup \operatorname{conv}(K \cup \{y\}) \subseteq S$, *where* K *is a convex set of dimension* $d-2$, $x \notin \operatorname{aff} K$ *and* $y \notin \operatorname{aff}(K \cup \{x\})$. *Then the set* $S \cap \operatorname{aff}(K \cup \{x, y\})$ *is starshaped, and its kernel is a* $(d-2)$*-dimensional set containing* K.

Proof. The argument is identical to the proof of the main theorem in [1].

LEMMA 2. *Assume that* $\operatorname{conv}(K \cup \{x\}) \cup \operatorname{conv}(K \cup \{y\}) \subseteq S$, *where* K *is a convex set of dimension* $d-2$, $x \notin \operatorname{aff} K$ *and* $y \notin \operatorname{aff}(K \cup \{x\})$. *Assume there exists some* $q \in S \sim \operatorname{aff}(K \cup \{x, y\})$ *such that* q *does not see* K *via* S. *Then if* z *sees* $d-1$ *affinely independent points of* K *via* S, $z \in \operatorname{aff}(K \cup \{x, y\})$.

Proof. By Lemma 1, the d-dimensional set $S \cap \operatorname{aff}(K \cup \{x, y\})$ is starshaped, and its kernel K' is a $(d-2)$-dimensional set containing K. Hence without loss of generality we may assume that $K = K'$. Let $\pi = \operatorname{aff}(K \cup \{x\})$, $\pi' = \operatorname{aff}(K \cup \{y\})$, and let $k_1, \cdots, k_{d-1}$ be $d-1$ affinely independent points in K seen by z. The affinely independent points $k_1, \cdots, k_{d-1}, q, x$ see via S a unique $(d-2)$-dimensional convex set $A = (\operatorname{aff} A) \cap S$, and $A \subseteq \pi$ by [1, Corollary 1 to Lemma 1]. Similarly $k_1, \cdots, k_{d-1}, q, y$ see a $(d-2)$-dimensional set A', and $A' \subseteq \pi'$. Clearly each of A, A' sees K via S. There are two cases to consider.

Case 1. If K, z, and q are not in a $(d-1)$-dimensional flat, then the affinely independent points $k_1, \cdots, k_{d-1}, z, q$ see a unique $(d-2)$-dimentional set R, $(\operatorname{aff} R) \cap S = R$, and R must lie in $\operatorname{aff}(K \cup \{z\})$: Otherwise, $\{k_1, \cdots, k_{d-1}, z\} \cup R$ would contain a set T of $d+1$ affinely independent points with corresponding segments in S, contradicting the fact that K_T is a convex set of dimension $d-2$. Again by Lemma 1, the d-dimensional set $S \cap \operatorname{aff}(K \cup \{z, q\})$ is starshaped, and its kernel must be R. Thus K sees R via S, so R,

A, A' all see $K \cup \{q\}$ via S. Hence $R \cup A \cup A'$ cannot contain $d+1$ affinely independent points, and $R \subseteq \operatorname{aff}(A \cup A') \subseteq \operatorname{aff}(\pi \cup \pi')$. Since q sees R but not K via S, $R \neq K$, and $\operatorname{aff}(K \cup R)$ is $(d-1)$-dimensional. Then $\operatorname{aff}(K \cup \{z\}) = \operatorname{aff}(K \cup R)$, and $z \in \operatorname{aff}(K \cup R) \subseteq \operatorname{aff}(\pi \cup \pi')$, the desired result.

Case 2. If K, z, and q lie in a $(d-1)$-dimensional flat, then since $q \notin \operatorname{aff}(K \cup \{x\}) \cup \operatorname{aff}(K \cup \{y\})$, neither x nor y is in that flat. However, K, z, q, x lie in a d-dimensional flat, and this flat is exactly $\operatorname{aff}(K \cup A \cup \{z, q\}) = \operatorname{aff}(K \cup A \cup \{q\})$. Since $\operatorname{conv}(K \cup A) \cup \operatorname{conv}(A \cup \{q\}) \subseteq S$, by Lemma 1, A is the kernel of $S \cap \operatorname{aff}(K \cup A \cup \{q\})$, and z sees A via S. Since S cannot contain $d+1$ affinely independent points with corresponding segments in S, $K \cup A \cup \{z\}$ must lie in a $(d-1)$-dimensional flat, and $z \in \operatorname{aff}(K \cup A) \subseteq \operatorname{aff}(\pi \cup \pi')$. (In fact, $z \in K$.) This completes Case 2 and finishes the proof of Lemma 2.

LEMMA 3. *Assume that* $\operatorname{conv}(K \cup \{x\}) \cup \operatorname{conv}(K \cup \{y\}) \subseteq S$, *where* K *is a convex set of dimension* $d-2$, $x \notin \operatorname{aff} K$, *and* $y \notin \operatorname{aff}(K \cup \{x\})$. *If* $q \in S \sim \operatorname{aff}(K \cup \{x, y\})$, *then* q *sees* K *via* S.

Proof. Assume on the contrary that q does not see K via S to reach a contradiction. As in the previous lemma, we may assume that K is the kernel of $S \cap \operatorname{aff}(K \cup \{x, y\})$. Let $\pi = \operatorname{aff}(K \cup \{x\})$, $\pi' = \operatorname{aff}(K \cup \{y\})$, and let A, A' denote the $(d-2)$-dimensional subsets of π, π' seen by $k_1, \cdots, k_{d-1}, q, x$ and by $k_1, \cdots, k_{d-1}, q, y$, respectively, where $k_1, \cdots, k_{d-1}$ are affinely independent points in K. Then A and A' see $K \cup \{q\}$ via S, so $A \cup A'$ cannot contain $d+1$ affinely independent points, and $A \cup A'$ lies in a $(d-1)$-dimensional flat. By hypothesis, since A and A' both correspond to $K \cup \{q\}$ and $K \cup \{q\} \cup A \cup A'$ does not lie in a d-flat, the distinct sets $\operatorname{aff} A$ and $\operatorname{aff} A'$ are disjoint, and these sets must be parallel in $\operatorname{aff}(A \cup A')$. Furthermore, since K and A' lie in π', $\operatorname{aff} K \cap \operatorname{aff} A \subseteq \operatorname{aff}(K \cup A') \cap \operatorname{aff}(A \cup A') = \operatorname{aff} A'$, and $\operatorname{aff} K \cap \operatorname{aff} A \subseteq \operatorname{aff} A' \cap \operatorname{aff} A = \emptyset$. Hence $\operatorname{aff} K$ and $\operatorname{aff} A$ are parallel in π. Similarly, $\operatorname{aff} K$ and $\operatorname{aff} A'$ are parallel in π', and it is easy to see that $\operatorname{aff} K \cap \operatorname{aff}(A \cup A') = \emptyset$.

Select some point u in rel int $\operatorname{conv}(A \cup \{q\})$, and examine the d-dimensional flat $\operatorname{aff}(A \cup A' \cup \{u\})$, which contains q. Clearly $\operatorname{aff}(A \cup A' \cup \{u\})$ intersects $\operatorname{aff}(\pi \cup \pi')$ in exactly $\operatorname{aff}(A \cup A')$. Hence for any point v in rel int $\operatorname{conv}(A' \cup \{q\}) \subseteq \operatorname{aff}(A \cup A' \cup \{u\})$, the line $L(u, v)$ determined by u and v does not intersect $\operatorname{aff} K$, and K, u, v affinely span a full d-dimensional set. Furthermore, for any point k in $\operatorname{aff} K$, the plane $\operatorname{aff}(k, u, v)$ intersects $\operatorname{aff}(\pi \cup \pi')$ in a line containing k, and this line cannot intersect $\operatorname{aff}(A \cup A')$: Otherwise k would lie in $\operatorname{aff}(A \cup A' \cup \{u, v\}) \cap \operatorname{aff}(\pi \cup \pi') = \operatorname{aff}(A \cup A')$, impossible. Hence $\operatorname{aff}(K \cup \{u, v\}) \cap \operatorname{aff}(A \cup A') = \emptyset$, and the d-dimensional flats $\operatorname{aff}(K \cup \{u, v\})$ and $\operatorname{aff}(\pi \cup \pi')$ intersect in a $(d-1)$-dimensional flat in $\operatorname{aff}(\pi \cup \pi')$ parallel to $\operatorname{aff}(A \cup A')$.

To complete the proof, we will find some nonempty subset F of S contained in $\operatorname{aff}(A \cup A') \cap \operatorname{aff}(K \cup \{u, v\})$, giving the desired contradiction. Let $E \equiv (\operatorname{aff} E) \cap S$ denote the $(d-2)$-dimensional subset of S seen by $k_1, \cdots, k_{d-1}, u$, and v. By Lemma 2, each point of E lies in $\operatorname{aff}(\pi \cup \pi')$, and since K is the kernel of $S \cap \operatorname{aff}(\pi \cup \pi')$, each point of E sees K via S. Hence $E \cup K$ cannot contain $d+1$ affinely independent points, and $\dim \operatorname{aff}(E \cup K) \leqq d-1$. Clearly $K \neq E$: Otherwise u and v would see K via S and by Lemma 2, $u, v \in \operatorname{aff}(K \cup \{x, y\})$, impossible by our choice of u and v. Therefore $\operatorname{aff}(E \cup K)$ is a $(d-1)$-dimensional subset of $\operatorname{aff}(\pi \cup \pi')$, and $E, K, \{q\}$ affinely span a d-flat. By selecting d affinely independent points in $E \cup K$, these points together with q see a $(d-2)$-dimensional subspace F of S, and it is easy to see that $F \subseteqq \operatorname{aff}(E \cup K) \subseteqq \operatorname{aff}(\pi \cup \pi')$. Hence F sees K via S. We conclude that F, A, A' all see $K \cup \{q\}$ via S, so $F \cup A \cup A'$ cannot contain $d+1$ affinely independent points, and $F \subseteqq \operatorname{aff}(A \cup A')$.

Finally, we show that $F \subseteqq \operatorname{aff}(K \cup \{u, v\})$. Observe that $u \notin \operatorname{aff}(\pi \cup \pi')$, so the set $K \cup E \cup \{u\}$ contains $d+1$ affinely independent points, and by Lemma 1, the kernel of $S \cap \operatorname{aff}(K \cup E \cup \{u\})$ is E. Also, there exist points in $S \sim \operatorname{aff}(K \cup E \cup \{u\})$ which do not see E via S: In particular, at least one of the sets A, A' cannot lie in the d-flat $\operatorname{aff}(K \cup E \cup \{u\})$, for otherwise $u \in \operatorname{aff}(K \cup E \cup \{u\}) = \operatorname{aff}(K \cup A \cup A') = \operatorname{aff}(\pi \cup \pi')$, impossible. If $A \not\subseteqq \operatorname{aff}(K \cup E \cup \{u\})$, then A cannot see E via S (for otherwise $K \cup E \cup A$ would contain $d+1$ affinely independent points with corresponding segments in S). Similarly, if $A' \not\subseteqq \operatorname{aff}(K \cup E \cup \{u\})$, then A' cannot see E via S. Thus the set $\operatorname{conv}(K \cup E) \cup \operatorname{conv}(E \cup \{u\})$ satisfies the hypothesis of Lemma 2, and we may apply that lemma to conclude that $v \in \operatorname{aff}(K \cup E \cup \{u\})$. Therefore $K \cup E \cup F \cup \{u, v\}$ lies in a d-flat, and since $K \cup \{u, v\}$ contains $d+1$ affinely independent points, this flat must be exactly $\operatorname{aff}(K \cup \{u, v\})$. Hence $F \subseteqq \operatorname{aff}(K \cup \{u, v\})$.

We conclude that $F \subseteqq \operatorname{aff}(A \cup A') \cap \operatorname{aff}(K \cup \{u, v\}) = \emptyset$. This yields the desired contradiction, our opening assumption is false, and q sees K via S, finishing the proof of Lemma 3.

The rest of the proof is easy. Select a set T consisting of $d+1$ affinely independent points of S, and let $K = \ker_S T$. Since $\dim K = d-2$, we may select points x, y in T with $x \notin \operatorname{aff} K$ and $y \notin \operatorname{aff}(K \cup \{x\})$. Then K, x, y satisfy the hypotheses of Lemmas 1 and 3, and by the lemmas, $K \subseteqq \ker S$. Since $\ker S \subseteqq \ker_S T = K$, we conclude that $K = \ker S$. Hence S is a starshaped set whose kernel is $(d-2)$-dimensional, completing the proof of the theorem.

We conclude with the following analogue of [1, Corollary 3]:

COROLLARY. *The hypothesis of the theorem above provides a characterization of subsets S of a linear topological space, S having dimension at least $d \geqq 2$, for which $K \equiv \ker S$ has dimension $d-2$, $(\operatorname{aff} K) \cap S = K$, and the maximal convex subsets of S have dimension $d-1$.*

Proof. If S satisfies the properties above, then to each $(d+1)$-member subset T of S, the set $K \equiv \ker S$ will be a suitable K_T set. For K_1 and K_2 distinct K_T sets, we assert that T, K_1, and K_2 lie in a d-flat: At least one of the sets K_1, K_2 is not K, so without loss of generality assume that $K_1 \neq K$. Since maximal convex subsets of S have dimension $d-1$, clearly each K_i set lies in a $(d-1)$-dimensional flat containing K, $i = 1, 2$, and it is easy to see that each point of T lies in the $(d-1)$-flat $\operatorname{aff}(K_1 \cup K)$. Furthermore, if $T \not\subseteq K$, then K_2 must also lie in $\operatorname{aff}(K_1 \cup K)$, finishing the argument. In case $T \subseteq K$, then since both K_1 and K_2 lie in $(d-1)$-flats containing K, the set $K_1 \cup K_2 \cup K$ lies in a d-flat, and this flat contains $K_1 \cup K_2 \cup T$, again the desired result.

The remaining steps of the proof are identical to those of [1, Corollary 3].

REFERENCES

1. Marilyn Breen, *Sets in R^d having $(d-2)$-dimensional kernels*, Pacific J. Math., **75** (1977), to appear.

2. N. E. Foland and J. M. Marr, *Sets with zero dimensional kernels*, Pacific J. Math., **19** (1966), 429–432.

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