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A VANISHING THEOREM FOR THE MOD p MASSEY-PETERSON SPECTRAL SEQUENCE

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A vanishing theorem and periodicity theorem for the classical mod 2 Adams spectral sequence were originally proved by Adams [1]. The results were extended to the unstable range by Bousfield [2]. The purpose of this paper is to show the analogue of Bousfield's work for the mod p unstable Adams spectral sequence of Massey-Peterson type (called the mod p Massey-Peterson spectral sequence), where p is an odd prime. The results generalized those obtained by Liulevicius [5], [6] to the unstable range. As an immediate topological application we have the estimation of the upper bounds of the orders of elements in the p-primary component of the homotopy groups of, for example, an odd dimensional sphere, Stiefel manifold, or H-space.

1. The vanishing theorem. Let A denote the mod p Steenrod algebra. Let $A\mathscr{M}$ the category of unstable left A-modules and $\mathscr{M}A$ thecategory of unstable right A-modules. We may define $\mathrm{Ext}^s_{A\mathscr{M}}$, $s \geqq 0$, as the sth right derived functor of $\mathrm{Hom}_{A\mathscr{M}}$, and similarly define $\mathrm{Ext}^s_{\mathscr{M}A}$, since $A\mathscr{M}$ and $\mathscr{M}A$ are abelian categoriesw ith enough projectives. Note that, if $M \in A\mathscr{M}$ is of finite type, then

$$\mathrm{Ext}_{A\mathscr{M}}(M, Z_p) = \mathrm{Ext}_{\mathscr{M}A}(Z_p, M^*) .$$

Recall the mod p Massey-Peterson spectral sequence (see, for example, [4]). Let X be a simply connected space with $\pi_*(X)$ of finite type. Suppose that $H^*(X; Z_p) \cong U(M)$, $M \in A\mathscr{M}$, where $U(M)$ is the free unstable A-algebra generated by M. Then there is a spectral sequence $\{E_r(X)\}$ with

$$d_r \colon E_r^{s,t}(X) \longrightarrow E_r^{s+r,t+r-1}(X) ,$$

such that

$$E_2^{s,t}(X) \cong \mathrm{Ext}^{s,t}_{A\mathscr{M}}(M, Z_p) ,$$

and

$$E_\infty(X) \cong \mathrm{Gr}\, \pi_*(X)/(\text{torsion prime to } p) .$$

Let Λ be the bigraded differential algebra over Z_p introduced by Bousfield et al [3], which has multiplicative generators λ_i of bidegree $(1, 2i(p-1)-1)$ for $i>0$ and μ_i of bidegree $(1, 2i(p-1))$ for $i \geqq 0$.

For $N \in \mathscr{M}A$, let $V(N)^s$ be the subspace of $N \otimes \Lambda^s$ generated by all $x \otimes \nu_I$ with $\nu_I = \nu_{i_1} \cdots \nu_{i_s}$ allowable and $\deg x \geqq 2i$ if $\nu_{i_1} = \lambda_{i_1}$ and $\deg x \geqq 2i_1 + 1$ if $\nu_{i_1} = \mu_{i_1}$. Then $V(N)$ is the cochain complex with

$$\delta(x \otimes \nu_I) = (-1)^{\deg x} \sum_{i>0} x\rho^i \otimes \lambda_i \nu_I + \sum_{i \geqq 0} x\beta\rho^i \otimes \mu_i \nu_I + (-1)^{\deg x} x \otimes \partial \nu_I .$$

Here $x \otimes \nu_I$ is of bidegree (s, t) with $t = s + \deg x + \deg \nu_I$. Recall that for $N \in \mathscr{M}A$

$$\mathrm{Ext}^{s,t}_{\mathscr{M}A}(Z_p, N) \cong H^s(V(N))_{t-s} .$$

Let $O(N)$ be the subcomplex of $V(N)$ generated by all $x \otimes \nu_I \in V(N)^s$ with $\nu_I = \nu_{i_1} \cdots \nu_{i_s}$ allowable and $\nu_{i_s} = \lambda_{i_s}$. Let $T(N)$ be the quotient complex of $V(N)$ such that

$$T(N)^s = \begin{cases} N \otimes \mu_0^s & \text{for } s = 0, 1 , \\ N \otimes \mu_0^s + \sum_{t>0} N_{2t}^s \otimes \lambda_t \mu_0^{s-1} & \text{for } s \geqq 2 . \end{cases}$$

Then we have a long exact sequence

$$\cdots \longrightarrow H^{s-1}(T(N)) \xrightarrow{\delta} H^s(O(N)) \xrightarrow{j^*} H^s(V(N)) \xrightarrow{q^*} H^s(T(N)) \longrightarrow \cdots ,$$

which is induced from the natural isomorphism

$$H^*(O(N)) \cong H^*(\mathrm{Ker}\, q) ,$$

where $j: O(N) \to V(N)$ and $q: V(N) \to T(N)$ are the natural maps. Remark that $H^*(T(N))$ consists of towers in the sense that

$$H^s(T(N)) \cong H^{s+1}(T(N)) ,$$

for $s \geqq 2$, and thus $H^*(T(N))$ is easily determined.

Definition. A function $\varphi_n(k)$, $n \geqq 2$, $k \geqq 0$, is defined as follows. If $n = 2, 3, 4$,

$$\varphi_n(k) = \begin{cases} [(k+2)/2(p-1)] & \text{for } k \geqq 2(p-1)-1 \\ 0 & \text{for } k < 2(p-1)-1 \end{cases}$$

where $[x]$ is the integer part of x, and if $n \geqq 5$,

$$\varphi(k) = \varphi_n(k) = i ,$$

where

$$\begin{array}{lll} 2i(p-1) \leqq k < 2(i+1)(p-1)-1 & \text{if} & i \not\equiv -1, 0 \bmod p\ , \\ 2i(p-1) \leqq k < 2(i+1)(p-1)-2 & \text{if} & i \equiv -1 \bmod p\ , \\ 2i(p-1)-2 \leqq k < 2(i+1)(p-1)-1 & \text{if} & i \equiv 0 \bmod p\ . \end{array}$$

Now we state our main theorem.

THEOREM 1 (*Vanishing*). *Let $N \in \mathscr{M}A$ with $N_i = 0$ for $i < n$, where $n \geqq 2$. Then*

$$\mathrm{Ext}^{s,s+k+n}_{\mathscr{M}A}(Z_p, N) \cong H^s(V(N))_{k+n} \xrightarrow{q^*} H^s(T(N))\ ,$$

is an isomorphism for $s > \varphi_n(k)$.

This will be proved in §4.

By virtue of our vanishing theorem the calculation of $H^*(V(N))$ is reduced to that of $H^*(O(N))$ in a large extent. Note that q^* is epimorphic when $U(M)$ is generated by a single element, where $M = N.^*$

As an immediate topological corollary we have.

COROLLARY 2. *Let X be a simply connected space with $\pi_*(X)$ of finite type. Suppose that $H^*(X; Z_p) \cong U(M)$, where M is an unstable A-module. If $M^i = 0$ for $i < n$, then the orders of elements in the p-primary component of $\pi_{k+n}(X)$ are at most $p^{\varphi_n(k)}$.*

This may be applied, for example, when X is an odd dimensional sphere, Stiefel manifold, or H-space.

REMARK. If $N_i = 0$, $i > m$, for some m, then $H^s(N)_t$ is zero for dimensional reason when t is large with respect to s. Hence in this case Corollary 2 is slightly improved.

2. Periodicity theorems.

For a module $M \in A\mathscr{M}$ we define the β-cohomology by $H_\beta(M) = \mathrm{Ker}\,\beta/\mathrm{Im}\,\beta$.

DEFINITION. A module $M \in A\mathscr{M}$ is called β-trivial if

$$\rho^i\colon M^{2i} \longrightarrow H^{2ip}_\beta(M)\ ,$$

is an isomorphism for all i and $H^k_\beta(M) = 0$ for $k \not\equiv 0 \bmod 2p$.

Remark that $M \in A\mathscr{M}$ is β-trivial if and only if $N = M^* \in \mathscr{M}A$ is towerless, i.e., $H^s(T(N)) = 0$ for $s > 0$.

Let $\mathscr{G}$ denote the category of graded Z_p-modules. Let L_sF denote the sth left derived functor of a functor $F\colon A\mathscr{M} \to \mathscr{G}$.

THEOREM 3 (*Periodicity*). *Let $F: A\mathscr{M} \to \mathscr{G}$ be a functor such that $F(M) = M/\beta M + \rho^1 M$. If $M \in A\mathscr{M}$ is β-trivial, then there is a natural map*

$$P: L_s F(M)^t \longrightarrow L_{s+p} F(M)^{t+2p(p-1)+p}\,,$$

such that P is an isomorphism for $s \geqq 2$ and a monomorphism for $s = 1$.

This will be proved in §3.

Additionally, we give here such a kind of periodicity theorems.

THEOREM 4. *Let $G_i(M) = M/\rho^i M$ for $M \in A\mathscr{M}$, where $0 < i < p$. Then there is a natural map*

$$Q: L_s G_i(M)^t \longrightarrow L_{s+2} G_i(M)^{t+2p(p-1)}\,,$$

such that Q is an isomorphism for $s \geqq 2$ and a monomorphism for $s = 1$.

THEOREM 5. *Let $G_i(M) = M/\beta M + \rho^i M$ for $M \in A\mathscr{M}$, where $0 < i < p$. If $M \in A\mathscr{M}$ is β-trivial, then there is a natural map*

$$Q: L_s G_i(M)^t \longrightarrow L_{s+2} G_i(M)^{t+2p(p-1)}\,,$$

such that Q is an isomorphism for $s \geqq 2$ and a monomorphism for $s = 1$.

THEOREM 6. *Let $G(M) = M/\rho^1 + \beta\rho^1 M$ for $M \in A\mathscr{M}$. If $M \in A\mathscr{M}$ is β-trivial, then there is a natural map*

$$R: L_s G(M)^t \longrightarrow L_{s+2} G(M)^{t+4p-2}\,,$$

such that R is an isomorphism for $s \geqq 2$ and a monomorphism for $s = 1$.

3. Proofs of periodicity theorems.

Suppose given a circular sequence of functors from $A\mathscr{M}$ to $\mathscr{G}$ and natural transformations,

(♯)
$$\begin{tikzcd}
A_{k-2} \arrow[r, "R_{k-2}"] & A_{k-3} \arrow[r] & \cdots \arrow[r] & A_1 \arrow[r, "R_1"] & A_0 = A_k \arrow[dll, "R_0 = R_k"] \\
 & & A_{k-1} \arrow[ull, "R_{k-1}"] & &
\end{tikzcd}$$

satisfying $R_i R_{i+1} = 0$ for $i = 0, \cdots, k-1$. Define functors $\operatorname{Ker} R_i$, $\operatorname{Im} R_i$, $\operatorname{Coker} R_i$, $H_i = \operatorname{Ker} R_i / \operatorname{Im} R_{i+1}$ in a usual way.

DEFINITION. A module $M \in A\mathscr{M}$ is called trivial for the diagram (♯) if

$$L_s A_i(M) = L_s H_i(M) = 0\ ,$$

for all $s > 0$ and $i = 0, \cdots, k-1$.

LEMMA. *If $M \in A\mathscr{M}$ is trivial for the diagram* ($\sharp$), *then there is a natural map*

$$P\colon L_s \operatorname{Coker} R_0(M)^t \longrightarrow L_{s+k} \operatorname{Coker} R_0(M)^{t+h}\ ,$$

such that P is an isomorphism for $s \geqq 2$ and a monomorphism for $s = 1$. Here $h = \sum_{i=0}^{k-1} h_i$, $h_i = \deg R_i$.

Proof. Let $h(a) = \sum_{i=a}^{k-1} h_i$. Since M is trivial for ($\sharp$), we have the following natural isomorphism

$$\begin{aligned} L_{s+k} \operatorname{Coker} R_0(M)^{t+h} &\cong L_{s+k-1} \operatorname{Im} R_0(M)^{t+h} \\ &\cong L_{s+k-2} \operatorname{Ker} R_0(M)^{t+h(1)} \cong \cdots \\ &\cong L_s \operatorname{Ker} R_{k-2}(M)^{t+h(k-1)} \cong L_s \operatorname{Im} R_{k-1}(M)^{t+h(k-1)}\ . \end{aligned}$$

On the other hand the natural map

$$L_s \operatorname{Coker} R_0(M)^t \longrightarrow L_s \operatorname{Im} R_{k-1}(M)^{t+h(k-1)}\ ,$$

is an isomorphism for $s \geqq 2$ and a monomorphism for $s = 1$.

We shall use the following circular sequence due to Toda [9] (see, also, Oka [8]) to prove the periodicity theorems.

(3.1)
$$\begin{tikzcd} M \arrow[r, "R_{p-2}"] & M \arrow[r] & \cdots \arrow[r] & M \arrow[r, "R_1"] & M \arrow[dll, "R"] \\ & & M/\beta M + M/\beta M \arrow[ull, "R'"] & & \end{tikzcd}$$

where $R_i = (i+1)\beta\rho^1 - i\rho^1\beta$, $R = (\rho^1\beta, \rho^1)$ and $R' = \rho^1\beta - \beta\rho^1\beta$.

(3.2)
$$M \overset{\rho^i}{\underset{\rho^{p-i}}{\rightleftarrows}} M \qquad \text{for} \quad 0 < i < p\ ,$$

(3.3)
$$M/R_i M \overset{\rho^i}{\underset{\rho^{p-i}}{\rightleftarrows}} M/\beta M \qquad \text{for} \quad 0 < i < p\ ,$$

(3.4)
$$M/\rho^1 M \overset{\rho^1}{\underset{\rho^1}{\rightleftarrows}} M/\rho^1 M\ .$$

Here $M \in A\mathscr{M}$ and the maps are induced from the left actions.

Proof of Theorem 3. We shall use the diagram (3.1). For convenience, we put $R_0 = R_p = R$, $R_{p-1} = R'$. Let $H_i(M)$ denote the cohomology $\operatorname{Ker} R_i/\operatorname{Im} R_{i+1}$. If M is a free unstable A-module, then:

(i) $\rho^s\colon (M/\beta M)^{2s} \cong H_0^{2sp}(M)$ if $s \equiv -1 \bmod p$,
$\rho^s\colon M \cong H_0^{2sp}(M)$ if $s \not\equiv -1 \bmod p$,
$\rho^s\colon (\beta M)^{2s+1} \cong H_0^{2tp+1}(M)$ if $s \not\equiv -1 \bmod p$,

(ii) for $i = 1, \cdots, p-2$,

$$\rho^s\colon (M/\beta M)^{2s} \cong H_i^{2sp}(M)\,,$$
$$\rho^s + \beta\rho^s\colon (\beta M)^{2s+2} + (M/\beta M)^{2s+1} \cong H_i^{2sp+2}(M)\,,$$
$$\rho^s\colon (M/\beta M)^{2s+2} \cong H_i^{2sp+3}(M)$$

(iii) $\rho^s + 0\colon (M/\beta M)^{2s} + 0 \cong H_{p-1}^{2sp}(M)$,

$$0 + \rho^s\colon 0 + (M/\beta M)^{2s+j} \cong H_{p-1}^{2sp+j+1}(M) \quad \text{for } j = 0, 1 \text{ if } s \equiv 0 \bmod p\,,$$
$$\rho^s + \rho^s\colon (\beta M)^{2s+j+1} + (M/\beta M)^{2s+j} \cong H_{p-1}^{2sp+j+1}(M) \quad \text{for } j = 0, 1 \text{ if } s \not\equiv 0 \bmod p\,,$$
$$0 + \rho^s\colon 0 + (M/\beta M)^{2s+2} \cong H_{p-1}^{2sp+3}(M)\,.$$

(iv) otherwise $H_i^k(M) = 0$.

This unstable version of Toda's exactness theorem is shown by long but straightforward computations. Now Theorem 3 is proved by applying lemma.

By using the diagrams (3.2), (3.3) and (3.4), Theorems 4, 5 and 6 follow in a similar way, and thus we only state the following facts.

Let $M \in A\mathscr{M}$ be a free unstable module. Fix i such that $0 < i < p$.

If $H(M) = \operatorname{Ker}(\rho^i\colon M \to M)/\operatorname{Im}(\rho^{p-i}\colon M \to M)$, then:

(i)
$$\rho^s\colon M^{2s+j} \cong H^{2sp+j}(M) \quad \text{for } j = 0, 1\,,$$
$$\beta\rho^s + \rho^s\colon M^{2s+j-1} + M^{2s+j} \cong H^{2sp+j}(M) \quad \text{for } j = 2, \cdots, 2i-1\,,$$
$$\beta\rho^s\colon M^{2s+j-1} \cong H^{2sp+j}(M) \quad \text{for } j = 2i, 2i+1\,.$$

(ii) otherwise $H^k(M) = 0$.

If $H(M) = \operatorname{Ker}(\rho^{p-i}\colon M/\beta M \to M/R_iM)/\operatorname{Im}(\rho^i\colon M/R_iM \to M/\beta M)$, then:

(i) $\rho^s\colon (M/\beta M)^{2s+j} \cong H^{2sp+j}(M)$

for $j = 0, 1, \cdots, i-1$ and $s \equiv 0, 1, \cdots, i-1 \bmod p$,

(ii) otherwise $H^k(M) = 0$.

Next put $H(M) = \operatorname{Ker}(\rho^i\colon M/R_iM \to M/\beta M)/\operatorname{Im}(\rho^{p-i}\colon M/\beta M \to M/R_iM)$, then:

(i) if $s \equiv 0 \bmod p$,

$$\rho^s\colon (M/\beta M)^{2s+j} \cong H^{2sp+j}(M) \quad \text{for } j = 0, 1\,,$$
$$\beta\rho^s + \rho^s\colon (M/\beta M)^{2s+j-1} + (M/\beta M)^{2s+j} \cong H^{2sp+j}(M) \quad \text{for } j = 2, \cdots, 2i-1\,,$$
$$\beta\rho^s\colon (M/\beta M)^{2s+j-1} \cong H^{2sp+j}(M) \quad \text{for } j = 2i, 2i+1\,,$$

(ii) if $s \equiv 1, \cdots, p-i-1 \bmod p$,

$$\begin{aligned} \rho^s\colon (M/\beta M)^{2s+j} &\cong H^{2sp+j}(M) && \text{for} \quad j = 0, 1\,, \\ \beta\rho^s + \rho^s\colon M^{2s+j-1} + (M/\beta M)^{2s+j} &\cong H^{2p+i}(M) && \text{for} \quad j = 2, \cdots, 2i-1\,, \\ \beta\rho^s\colon M^{2s+j-1} &\cong H^{2sp+j}(M) && \text{for} \quad j = 2i, 2i+1\,, \end{aligned}$$

(iii) if $s \equiv p - i \bmod p$,

$$\beta\rho^s\colon M^{2s+j-1} \cong H^{2sp+j}(M) \qquad \text{for} \quad j = 2, \cdots, 2i-1\,,$$

(iv) otherwise $H^k(M) = 0$.

Finally, if $H(M) = \operatorname{Ker}(\beta\rho^1\colon M/\rho^1 M \to M/\rho^1 M)/\operatorname{Im}(\beta\rho^1\colon M/\rho^1 M \to M/\rho^1 M)$, then:

(i) $$\rho^{ps}\colon M^{2s+j} \cong H^{2sp^2+j}(M) \qquad \text{for} \quad j = 0, 1\,,$$
$$\beta\rho^{ps} + \rho^{ps}\colon (M/\beta M)^{2s+1} + M^{2s+2} \cong H^{2sp^2+2}(M)\,,$$

(ii) otherwise $H^k(M) = 0$.

4. **Proof of the vanishing theorem.** Let $F(n)$ denote a free unstable A-module on one generator ι_n. We define an unstable A-module $N(n)$ to be the quotient of $F(n)$ by the relation $\beta\iota_n = 0$. Next define $M(n)$ to be the subcomplex of $N(n)$ by ommitting the ι_n from $N(n)$ if n odd and ommitting the $\iota_n, (\iota_n)^p, \cdots, (\iota_n)^{p^t}, \cdots$ from $N(n)$ if n even. Note that $M(n)$ is β-trivial.

First we suppose that n is odd. Then by the long exact sequence induced from a short exact sequence

$$0 \longrightarrow M(n) \longrightarrow N(n) \longrightarrow Z_p \longrightarrow 0\,,$$

we have an isomorphism

$$E_2^{s,t+n}(S^n) = \operatorname{Ext}_{A\mathscr{M}}^{s,t+n}(Z_p, Z_p) \cong \operatorname{Ext}_{A\mathscr{M}}^{s-1,t+n}(M(n), Z_p)\,,$$

for $t \neq s$. Let $C(n)$ be a minimal resolution of $M(n)$. By virtue of Theorem 3 we can prove the vanishing theorem for Z_p by analysing $C(n)$. Namely, $\operatorname{Ext}_{A\mathscr{M}}^{s-1,t+n}(M(n), Z_p)(t \neq s)$ vanishes for $s > \varphi_n(t-s)$. Furthermore we can observe the periodicity phenomenon in a range near the vanishing line. In fact, by Theorems 3, 4 and 5 we have two periodicity operators P and Q of bidegree $(p, 2p(p-1)+p)$ and $(2, 2p(p-1))$, respectively.

For lower dimensional sphere we shall give periodic families. Let $1 < m \leqq p+1$. In $E_2^{s,t+2m-1}(S^{2m-1})$ there appear nontrivial elements when $(s, t-s)$ is as follows:

(i) $(1, q-1)$

$(1, pq-1)$ for $m = p+1$,

(ii) $(s, sq-1), (s, (m+s-2)q-2)$

for $s = 2, \cdots, p-m+1$ and $m \neq p, p+1$,

$$(s, sq-1), (s, pq-2), (s, pq-1)$$
$$\text{for} \quad s = p - m + 2 \quad \text{and} \quad m \neq p+1\,,$$
$$(s, sq-1), (s, pq-2), (s, pq-1), (s, (m+s-2)q-2)$$
$$\text{for } s = p-m+3, \cdots, p-1 \quad \text{and } p \neq 3\,,$$
$$(p, pq-2), (p, pq-1), (p, (p+m-2)q-2)\,,$$
$$(p+1, (p+1)q-1), (p+1, (p+m-1)q-2)\,,$$

where $q = 2(p-1)$. Applying the periodicity operators P and Q repeatedly, we can determine the behavior of all $E_2(S^{2m-1})$ near the vanishing line. (Possibly other elements appear in a range apart from the vanishing line when we apply the iteration of the operator Q.)

We next suppose that n is even. Let $L(n; t)(0 < t \leqq \infty)$ be an unstable A-module with elements $\sigma_n, (\sigma_n)^p, \cdots, (\sigma_n)^{p^t}$ where $\deg \sigma_n = n$. By the long exact sequence induced from short exact sequences

$$0 \longrightarrow M(n) + L(p^{t+1}; \infty) \longrightarrow N(n) \longrightarrow L(n; t) \longrightarrow 0\,,$$
$$0 \longrightarrow M(p^{t+1}n) \longrightarrow N(p^{t+1}n) \longrightarrow L(p^{t+1}n; \infty) \longrightarrow 0\,,$$

we have an isomorphism

$$\begin{aligned}&\mathrm{Ext}_{A\mathscr{M}}^{s,t+n}(L(n; t), Z_p)\\ &\quad\cong \mathrm{Ext}_{A\mathscr{M}}^{s-1,t+n}(M(n), Z_p) + \mathrm{Ext}_{A\mathscr{M}}^{s-2,t+n}(M(p^{t+1}n), Z_p)\,,\end{aligned}$$

for $t \neq s, s + (p^{t+1}-1)n - 1$. Thus in a similar way we have the required results for $L(n; t)$.

Now we have shown that

$$q^*\colon H^s(V(N))_{k+n} \longrightarrow H^s(T(N)_{k+n}\,,$$

is an isomorphism for $s > \varphi_n(k)$, when $N^* = H^*(S^n; Z_p) = Z_p(n \text{ odd})$ and $N^* = L(n; t)(n \text{ even})$. The general case follows inductively using the five lemma.

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