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SURJECTIVITY RESULTS FOR ϕ-ACCRETIVE SET-VALUED MAPPINGS

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Let X and Y be Banach spaces. A mapping $f: X \to 2^Y$ is said to be locally strongly ϕ-accretive if for each $y_0 \in Y$ and $r > 0$ there exists a constant $c > 0$ such that if $z \in f(X) \cap B_r(y_0)$ and $x \in f^{-1}(z)$, then for all u sufficiently near x and $w \in f(u)$: $(\phi(x-u), (w-z)) \geqq c\|x-u\|^2$, where ϕ: $X \to Y^*$ is a suitably restricted mapping. A number of surjectivity results are obtained for this class of mappings, along with some other basic results.

In this continuation of the study of surjectivity results obtainable by application of refined versions of the fixed point approach of Caristi [5], we turn our attention to analogues of the results of Kirk [11] for set-valued mappings.

1. Preliminary results. One of the applications we have in mind requires the following minor reformulation of the fixed point theorem of Downing-Kirk [7]. (Recall that a mapping $f: X \to Y$, where X and Y are metric spaces, is said to be *closed* if the conditions $x_n \to x$, $x_n \in X$, and $f(x_n) \to y$ imply $f(x) = y$.)

THEOREM 1. *Let (X, d_1) and (Y, d_2) be complete metric spaces, $g: X \to X$ an arbitrary mapping, and $f: X \to Y$ a closed mapping. Suppose there exists a closed subset S of Y for which $f^{-1}(S) \neq \varnothing$ and $g: f^{-1}(S) \to f^{-1}(S)$, and suppose there exists a lower semicontinuous mapping $\varphi: f(X) \cap S \to R^+$ (the nonnegative reals) and a constant $c > 0$ such that*

$$\max\{d_1(x, g(x)), cd_2(f(x), f(g(x)))\} \leqq \varphi(f(x)) - \varphi(f(g(x))), \ x \in f^{-1}(S).$$

Then g has a fixed point in X.

The above theorem reduces to the theorem of Downing-Kirk [7] if $S = Y$ and to the original formulation of Caristi [5] if $S = Y = X$ and f is the identity. Indeed, because f is a closed mapping, the set $f^{-1}(S)$ is complete relative to the metric ρ defined by

$$\rho(u, v) = \max\{d_1(u, v), cd_2(f(u), f(v))\}, \quad u, v \in f^{-1}(S).$$

Thus Theorem 1 is a direct consequence of the original version of Caristi's theorem applied to the space $(f^{-1}(S), \rho)$. This observation is due to W. L. Bynum.

We also remark that Caristi's original theorem is essentially equivalent to a theorem of Ekeland [9] and that the proof given in [7] is along general lines implicit in Brøndsted [2]. (For an excellent discussion of the relation between the work of Brøndsted, Caristi, Ekeland, and others, along with a simple constructive proof of Caristi's theorem, see Brezis-Browder [1].)

Before turning to our applications of Theorem 1 we recall some definitions: a set-valued mapping $f: X \to 2^Y$ is said to be *upper semicontinuous* (u.s.c) [resp., *lower semicontinuous* (l.s.c)] if for each $x_0 \in X$ and each open set G in Y such that $f(x_0) \subseteqq G$ $[f(x_0) \cap G \neq \emptyset]$ there exists a neighborhood N of x_0 such that $x \in N$ implies $f(x) \subseteqq G$ $[f(x) \cap G \neq \emptyset]$; f is said to be *demi-closed* if the conditions $x_n \to x$ weakly in X and $y_n \to y$ with $y_n \in f(x_n)$, imply $y \in f(x)$.

We derive our results for ϕ-accretive mappings from the following more abstract result.

THEOREM 2. *Let X be a complete metric space, Y a Banach space, and $f: X \to 2^Y$ a mapping of X into the nonempty closed convex subsets of Y with $f(X)$ dense in Y. Suppose that one of the following two conditions holds:*

(i) *Y is reflexive and f is either demi-closed or* u.s.c.

(ii) *Y is arbitrary and f is a single-valued closed mapping. Suppose in addition that for each $y_0 \in Y$ there exist constants $r > 0$, $c > 0$ such that:*

(a) If *$x \in X$ with* $\operatorname{dist}(y_0, f(x)) \leqq r$, *then for each $z \in f(x)$, there exists $\delta = \delta(z) > 0$ such that $y \in B_\delta(z) \cap f(X)$ implies*

$$\operatorname{dist}(x, f^{-1}(y)) \leqq c\,||z - y|| .$$

Then $f(X) = Y$.

Proof. Fix $\eta > 1$ and suppose there exists $y_0 \in Y$ such that $y_0 \notin f(X)$. Let r be the constant of condition (a) relative to y_0, let $x \in X$ be such that $\operatorname{dist}(y_0, f(x)) \leqq r$, and suppose $z \in f(x)$ satisfies

$$||y_0 - z|| = \operatorname{dist}(y_0, f(x)) > 0 .$$

(Such a choice is always possible under assumption (i) because Y is reflexive and $f(x)$ is weakly closed.) Select $w \in \operatorname{seg}[z, y_0]$ so that $z \neq w \neq y_0$ and $0 < ||w - z|| < \delta(z)$. Since $w \in \overline{f(X)}$, there exists a sequence $\{y_j\} \subseteqq f(X)$ with $y_j \to w$ and since

$$0 < ||w - z|| = ||z - y_0|| - ||w - y_0||$$

we may choose j sufficiently large so that

$$(1) \qquad 0 < ||z - y_j|| \leqq \eta[||z - y_0|| - ||y_j - y_0||]$$

and

$$\|z - y_j\| \leqq \delta(z) .$$

By (a), there exists $v \in f^{-1}(y_j)$ with

$$(2) \qquad d(x, v) \leqq c\|z - y_j\| \leqq c\eta[\|z - y_0\| - \|y_j - y_0\|] .$$

We note by (1), that since $\operatorname{dist}(y_0, f(x)) = \|z - y_0\| \leqq r$, $\operatorname{dist}(y_0, f(v)) \leqq r$ also. Moreover, (2) implies

$$(3) \qquad d(x, v) \leqq c\|z - y_j\| \leqq c\eta[\operatorname{dist}(y_0 - f(x)) - \operatorname{dist}(y_0, f(v))] .$$

Case (i): We claim the mapping $\varphi: X \to R^+$ given by $\varphi(x) = \operatorname{dist}(y_0, f(x))$ is l.s.c. This is well-known to be the case if f is u.s.c. (cf [12]), so we suppose f is demi-closed. Let $x_0 \in X$ and suppose $x_n \to x_0$. Choose a sequence $z_n \in f(x_n)$ such that

$$\|y_0 - z_n\| = \varphi(x_n) .$$

If $\{z_n\}$ is unbounded, then for all n sufficiently large, $\operatorname{dist}(y_0, f(x_0)) \leqq \|y_0 - z_n\|$ which implies $\varphi(x_0) \leqq \liminf_n \varphi(x_n)$. Hence we may assume $\{z_n\}$ is bounded. In this case, because Y is reflexive, we may choose a weakly converging subsequence $\{z_j\} \subseteq \{z_n\}$ such that

$$\lim_{j\to\infty} \|z_j - y_0\| = \liminf_{n\to\infty} \|z_n - y_0\|$$

where $\{z_j\}$ converges weakly to z_0, $z_0 \in Y$. Since f is demi-closed, $z_0 \in f(x_0)$. Now weak-l.s.c. of the norm implies

$$\begin{aligned} \operatorname{dist}(y_0, f(x_0)) &\leqq \|y_0 - z_0\| \leqq \liminf_{j\to\infty} \|z_j - y_0\| \\ &= \lim_{j\to\infty} \|z_j - y_0\| \\ &= \liminf_{n\to\infty} \|z_n - y_0\| \end{aligned}$$

establishing l.s.c. of φ at x_0.

Now let $X_1 = \{x \in X \mid \operatorname{dist}(f(x), y_0) \leqq r\}$. Note that since φ is a l.s.c. map, X_1 is a closed set. For each $x \in X_1$, define a function $g: X_1 \to X_1$ by $g(x) = v$, where v is selected as in (2). Then (3) implies

$$d(x, g(x)) \leqq c\eta[\varphi(x) - \varphi(g(x))]$$

so by Caristi's theorem (Theorem 1) g has a fixed point, which is a contradiction to (1).

Case (ii): Since f is single-valued, we may make the identifications $z = f(x)$ and $y_j = f(v)$. Define $g: X \to X$ by

$$g(x) = \begin{cases} v \text{ if } \|y_0 - f(x)\| \leqq r \\ x_0 \text{ if } \|y_0 - f(x)\| > r \end{cases}$$

where x_0 is some fixed element in X such that $\|y_0 - f(x_0)\| \leqq r$. Also define $\varphi: f(X) \cap B_r(y_0) \to R^+$ by

$$\varphi(f(x)) = c\eta \|f(x) - y_0\| .$$

Then by (2) we have $\max \{d(x, g(x)), c\|g(x) - f(g(x))\|\} \leqq \varphi(f(x)) - \varphi(f(g(x)))$, so by Theorem 1 with $S = B_r(y_0)$, we again have a fixed point of g and hence a contradiction. Hence, in either case, $y_0 \in f(X)$.

2. ϕ-Accretive mappings. Let X and Y be Banach spaces, ϕ: $X \to Y^*$ a mapping with $\phi(X) = Y^*$ which satisfies $\|\phi(x)\| = \|x\|$, and $\phi(\xi x) = \xi\phi(x)$ for all $x \in X$, $\xi \geqq 0$. A mapping $f: X \to 2^Y$ is said to be *locally strongly ϕ-accretive* if for each $y_0 \in Y$ and $r > 0$ there exists a constant $c > 0$ such that the following condition holds: If $z \in f(X) \cap B_r(y_0)$ and $x \in f^{-1}(z)$, then for all u sufficiently near x and $w \in f(u)$:

$$(\phi(x - u), (w - z)) \geqq c\|x - u\|^2 .$$

This is a localized version of a definition formulated by Browder [4] in an attempt to link the theory of strongly monotone mappings for which $Y = X^*$ with the theory of strongly accretive mappings for which $Y = X$. (The strongly K-monotone mappings introduced by Petryshyn, e.g., [14, 15], constitute a similar unifying class.)

In order to obtain surjectivity results for mappings of the above type by application of Theorem 2 it is necessary to determine conditions under which such mappings will have dense range and satisfy condition (a). The following theorem, based on Theorem 3 of [4], gives sufficient conditions for denseness of the range. In this theorem Y is assumed to *admit nearest points*. By this it is meant that for each closed subset A of Y the set $A_0 = \{y \in Y |$ there exists $a \in A$ with $\|y - a\| = \text{dist}(y, A)\}$ is dense in Y. Recall also that the *duality mapping* J: $Y \to Y^*$ (cf., [3]) satisfies $(Jy, y) = \|y\|^2 = \|Jy\|^2$.

THEOREM 3. *Let X be a topological space and Y a Banach space for which the (single-valued) duality mapping J: $Y \to Y^*$ is lipschitzian on bounded sets. Suppose Y admits nearest points and suppose $f: X \to 2^Y$ is any mapping such that for each $y_0 \in Y$ there exists a constant $r > 0$ with*

(i) *$B_r(y_0) \cap f(X) \neq \varnothing$;*

(ii) *If $y_0 \notin \overline{f(X)}$, then for each $y \in B_r(y_0) \cap f(X)$ there exists $v \in f(X)$ such that*

$$(J(y_0 - y), (v - y)) > M\|v - y\|^2$$

where M is the Lipschitz constant of J on $B_r(y_0)$. Then $f(X)$ is dense in Y.

M. Edelstein has shown [8] that if Y is uniformly convex, then Y admits nearest points. By a result of J. Smulian (see, e.g., [6]) the fact that the duality mapping J is lipschitzian on bounded sets suffices to guarantee that Y is uniformly smooth. A Banach space Y is said to satisfy *property* (H) ([10]) if the following condition holds:

(H) If $\{x_n\}$ is a sequence in Y which converges weakly to x and if $\|x_n\| \to \|x\|$, then $\{x_n\}$ converges strongly to x.

D. E. Wulbert (cf. [16], [17]) has observed that every uniformly smooth Banach space which has property (H) admits nearest points.

We might also point out that the duality mapping $J: Y \to Y^*$ may be characterized by:

$$J(y) = \|y\|\, G(y, \cdot)$$

where $G(y, \cdot)$ is the derivative of the norm of Y at y given by

$$G(y, h) = \lim_{t \to 0+} t^{-1}[\|y + th\| - \|y\|] \quad (h \in Y)\,.$$

If the norm of Y is twice differentiable with bounded derivatives then it is easily seen that the mapping $y \mapsto G(y, \cdot)$ is lipschitzian on bounded sets (cf. [13, pg 175]); hence that the mapping J is lipschitzian with Lipschitz constant depending on the bounds for $\|y\|$, $\|G(y, \cdot)\|$ and on the Lipschitz constant for the mapping $y \mapsto G(y, \cdot)$ As a result of these facts, which are generally known, any space Y whose norm is twice differentiable with bounded derivatives and which has property (H) will satisfy the assumptions of Theorem 3.

We now state our generalization of Theorem 4 of [11]. The major improvement here lies in the fact that even in the single-valued case the assumptions on f are not sufficient to imply continuity.

THEOREM 4. *Let X and Y be Banach spaces with Y and Y^* uniformly convex, and suppose the duality mapping $J: Y \to Y^*$ is lipschitzian on bounded sets. Suppose $f: X \to 2^Y$ is locally strongly ϕ-accretive and maps points of X into the nonempty closed convex subsets of Y, suppose $f^{-1}: f(X) \to 2^X$ is lower semincontinuous, and suppose f satisfies either one of the following continuity assumptions:*

(i) *f is either demi-closed or* u.s.c.

(ii) *f is a single-valued closed mapping.*

Suppose in addition that f satisfies the condition:

(b) *For each u, $v \in X$ and $y \in f(u)$,*

$$\liminf_{t \to 0} \operatorname{dist}(y, f(x_t))/\|x_t - u\|^{1/2} = 0$$

where $x_t = u + tv$.

Then $f(X) = Y$.

Condition (b), which in the single-valued case reduces to a weak Hölder condition of order 1/2 along line segments, is the mildest condition we can state under which our proof of Theorem 4 carries through. It is not known, however, whether any such condition is necessary.

Proof of Theorem 4. It suffices to show that $f(X)$ is dense in Y and that condition (a) of Theorem 1 is satisfied. To obtain density, we let $y_0 \in Y$ be fixed, choose $y \in f(X)$, and apply Theorem 3.

Since $\phi(X) = Y^*$, we may choose $v \in X$ so that $\|v\| = \|y - y_0\|$ and $\phi(v) = J(y_0 - y)$. Condition (i) is satisfied with $r = \|y_0 - y\|$. Let $c > 0$ be the constant given by the definition of locally strongly ϕ-accretive with respect to y_0 and r, let $u \in f^{-1}(y)$, and let $x_t = u + tv$. Then for all $t > 0$ sufficiently small, rechoosing c if necessary,

$$(\phi(x_t - u), (w - y)) \geqq c\|x_t - u\|^2 \quad (w \in f(x_t)) ,$$

which in turn implies

$$(J(y_0 - y), (w - y)) \geqq ct\|y - y_0\|^2 \quad (w \in f(x_t)) .$$

Now since $y \in f(u)$, (b) implies there exists $w \in f(x_t)$ such that

$$\begin{aligned} \|w - y\|^2 &\leqq \varepsilon(t)\|x_t - u\| \\ &= \varepsilon(t)t\|y_0 - y\| \end{aligned}$$

where $\liminf_{t \to 0} \varepsilon(t) = 0$. Hence given $M > 0$, we can select $t > 0$ so that $M\varepsilon(t) < c\|y_0 - y\|$; hence

$$M\|w - y\|^2 < (J(y_0 - y), (w - y)) ,$$

which implies (ii) of Theorem 3 and proves $f(X)$ is dense in Y.

To see that (a) holds, select $x \in X$ such that $\operatorname{dist}(y_0, f(x)) \leqq r$. Since f is locally strongly ϕ-accretive at x, then for $z \in f(x)$,

$$(\phi(u - x), (w - z)) \geqq c\|u - x\|^2$$

for all $w \in f(u)$ where $u \in X$ is sufficiently near x. Therefore

$$\begin{aligned} c\|x-u\|^2 &\leqq (\phi(u-x),\ (w-z)) \\ &\leqq \|\phi(u-x)\|\cdot\|w-z\| \\ &= \|w-z\|\cdot\|u-x\| \ . \end{aligned}$$

Thus $c\|x-u\| \leqq \|w-z\|$, where $w \in f(u)$ and u is sufficiently near x, say $u \in B_\varepsilon(x)$ for some $\varepsilon > 0$. Since f^{-1} is l.s.c., there exists $\delta = \delta(z) > 0$ such that if $w \in B_\delta(z) \cap f(X)$, there exists $u \in f^{-1}(w)$ such that $u \in B_\varepsilon(x)$. So for such w,

$$\|x-u\| \leqq c^{-1}\|w-z\|$$

which, since z is any element in $f(x)$, implies (a).

Finally, we append a proof of Theorem 3. The details differ only slightly from those of Browder [4].

Proof of Theorem 3. We claim that $\overline{f(X)}$ is dense in Y; hence that $\overline{f(X)} = Y$. To see this, let $Y_0 = \{y_0 \in Y \mid \exists y \in \overline{f(X)} \text{ with } \|y_0 - y\| = \operatorname{dist}(y_0, \overline{f(X)})\}$. Since Y_0 is dense in Y, we need only show that $Y_0 \subseteq \overline{f(X)}$. Suppose this is not the case and choose $y_0 \in Y_0$ with $y_0 \notin \overline{f(X)}$. Let $y \in \overline{f(X)}$ be the point such that

$$\|y - y_0\| \leqq \|y_0 - v\| \quad (v \in \overline{f(X)}) \ .$$

Then for all $v \in \overline{f(X)}$,

$$\begin{aligned} \|y_0 - v\|^2 + (J(y_0 - v),\ (v - y)) &= (J(y_0 - v),\ (y_0 - v)) \\ + (J(y_0 - v),\ (v - y)) &= (J(y_0 - v),\ (y_0 - y)) \\ &\leqq \|y_0 - v\|\cdot\|y_0 - y\| \\ &\leqq \|y_0 - v\|^2 \ . \end{aligned}$$

Hence

$$(J(y_0 - v),\ (v - y)) \leqq 0 \ .$$

This in turn implies

$$\begin{aligned} (J(y_0 - y),\ (v - y)) &\leqq (J(y_0 - y),\ (v - y)) - (J(y_0 - v),\ (v - y)) \\ &= (J(y_0 - y) - J(y_0 - v),\ (v - y)) \\ &\leqq \|J(y_0 - y) - J(y_0 - v)\|\ \|v - y\| \\ &\leqq M\|v - y\|^2 \end{aligned}$$

where M is the Lipschitz constant for J on $B_r(y_0)$. Since this is true for all $v \in \overline{f(X)}$, (ii) is contradicted. Hence $f(X)$ is dense in Y.

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