PACIFIC JOURNAL OF MATHEMATICS
Vol. 78, No. 1, 1978

ON RELATIONS FOR REPRESENTATIONS OF FINITE GROUPS

C. L. MORGAN

Let G be a finite group, and suppose that

$$A\colon G \longrightarrow \mathrm{GL}(n, C)$$

is a (complex) representation of G with character χ. A (complex) linear relation for A is a formal complex linear combination $\sum_{g\in G} a_g g$ such that $\sum_{g\in G} a_g A(g) = 0$.

We prove the following theorem, which determines the linear relations in terms of the character χ.

THEOREM. **Let A be a representation for a finite group G, let χ be the character of A, and let $\{g_1, \cdots, g_k\}$ be a subset of G. Then $\sum_{j=1}^{k} a_j g_j$ is a relation for A if and only if $\sum_{j=1}^{k} \chi(g_i g_j^{-1}) a_j = 0$, for all $i = 1, \cdots, k$.**

NOTE 1. If C is the $k\times k$ matrix whose ij-entry is $\chi(g_i g_j^{-1})$ and a is the column vector whose jth entry is a_j, then the above conclusion can be rephrased as follows:

$$\sum_{j=1}^{k} a_j g_j \quad \text{is a relation for } A \iff Ca = 0 .$$

NOTE 2. The above theorem is a generalization of a result by Russell Merris [3]. His result may be stated in the following way. Let χ be an irreducible character of G, let M be the matrix obtained by applying χ to the entries of the multiplication table of G, let A be any representation of G affording χ, and let S be a subset of G. Then $\{A(g) \mid g \in S\}$ is linearly independent if and only if the rows of M corresponding to S are linearly independent. Our result strengthens Merris' result in three ways: (1) the condition about irreducibility is removed, (2) a way to determine the coefficients of any relation is given, and (3) smaller matrices are involved.

Proof of Theorem. Let $\chi_1, \cdots, \chi_r$ be the irreducible characters of G, and let CG denote the complex group algebra of G. For each $k = 1, \cdots, r$, let

$$c_k = (\chi_k(e)/|G|) \sum_{g\in G} \chi_k(g) g .$$

Then c_k is a central idempotent of CG and corresponds to a representation of G with character χ_k in the following way:

Let R_k denote the principal ideal of CG generated by c_k, and let Z_k be any minimal left ideal of CG contained in R_k. Then

$R_k \approx \mathrm{Hom}(Z_k, Z_k)$ and the irreducible group representation

$$A_k\colon G \longrightarrow \mathrm{GL}(Z_k)$$

given by left multiplication has character χ_k. Furthermore, $\{c_1, \cdots, c_k\}$ is a set of mutually annihilating central idempotents of G such that

$$c_1 + c_2 + \cdots + c_r = e\ .$$

See [1, pp. 233-236] and [2, p. 257].

We can write A in terms of these representations as follows:

$$A \approx n_1 A_1 \oplus n_2 A_2 \oplus \cdots \oplus n_r A_r$$

where n_k is a nonnegative number given by $n_k = (\chi, \chi_k)$.

Now let

$$L = \sum_{i=1}^{n} a_i g_i\ .$$

Then L is a relation for A if it is a relation for those irreducible representations A_k such that $(\chi, \chi_k) \neq 0$. It follows that L is a relation for A iff $c_k L = 0$, for those k such that $(\chi, \chi_k) \neq 0$.

Define

$$c = (\chi(e)/|G|) \sum_{g \in G} \chi(g) g\ .$$

A straightforward calculation shows that

$$c = \sum_{i=1}^{r} ((\chi, \chi_i)\chi(e)/\chi_i(e)) c_i\ .$$

Because of the mutual annihilation property,

$$cL = 0 \iff c_k L = 0 \quad \text{for all } k \text{ such that} \quad (\chi, \chi_k) \neq 0\ .$$

Thus L is a relation for A if and only if $cL = 0$.

Left multiplication by c is a linear transformation on the complex vector space $\boldsymbol{C}G$, and thus c has a matrix N with respect to the basis G for $\boldsymbol{C}G$. The g, h-entry of N is $(\chi(e)/|G|)\chi(gh^{-1})$. Also with respect to this basis, L corresponds to the column vector with a_j as the g_jth entry, and zero otherwise. With a slight abuse of notation, this becomes $a_{g_i} = a_i$ for $i = 1, \cdots, k$ and $a_g = 0$ if $g \neq g_i$ for all i.

Since G is finite, we have $\chi(g^{-1}) = \bar{\chi}(g)$, and thus N is hermitian. Since c is a positive linear combination of mutually annihilating idempotents, all eigenvalues of c and hence of N are nonnegative. Thus N is hermitian positive semidefinite, and so there exists a set of vectors $\{v_g \in \boldsymbol{C}^{|G|} \mid g \in G\}$ such that the g, h-entry of N is $\langle v_g, v_h \rangle$, where $\langle\ ,\ \rangle$ is the ordinary hermitian inner product on $\boldsymbol{C}^{|G|}$. Thus we can write:

$$cL = \sum_{g,h\in G} \langle v_g, v_h \rangle a_h g = \sum_{g\in G} \langle v_g, v \rangle g$$

where $v = \sum_{h\in G} a_h v_h = \sum_{i=1} a_i v_{g_i}$.

If $cL = 0$, then $\langle v_g, v \rangle = 0$, for all $g \in G$. This implies that $\langle v_{gj}, v \rangle = 0$, for $i = 1, \cdots, k$.

Conversely, if $\langle v_{g_i}, v \rangle = 0$, for $i = 1, \cdots, k$, then $\langle v, v \rangle = 0$. Since $\langle\, , \rangle$ is the usual hermitian inner product on $C^{|G|}$, this implies that $v = 0$. But then $\langle v_g, v \rangle = 0$, for all $g \in G$. Thus $cL = 0$.

Thus $cL = 0 \Leftrightarrow \langle v_{g_i}, v \rangle = 0$, for all $i = 1, \cdots, k$.

But $\langle v_{g_i}, v \rangle = 0 \Longleftrightarrow \sum_{j=1}^{n} \langle v_{g_i}, v_{g_j} \rangle a_j = 0$

$$\Longleftrightarrow (\chi(e)/|G|) \sum_{j=1}^{k} \chi(g_i g_j^{-1}) a_j = 0$$

$$\Longleftrightarrow \sum_{j=1}^{k} \chi(g_i g_j^{-1}) a_j = 0 \,.$$

Thus L is a relation for A iff

$$\sum_{j=1}^{k} \chi(g_i g_j^{-1}) a_j = 0, \quad \text{for all} \quad j = 1, \cdots, k \,.$$

REFERENCES

1. C. W. Curtis and I. Reiner, *Representation Theory of Finite Groups and Associative Algebras*, Interscience, New York, New York, 1962.

2. M. Hall, Jr., *The Theory of Groups*, The Macmillan Company, New York, New York, 1959.

3. R. Merris, *On Burnside's theorem*, J. Algebra, **48** (1977), No. 1, 214-215.

Received June 30, 1977 and in revised form January 21, 1978.

CALIFORNIA STATE UNIVERSITY
HAYWARD, CA 94542