PACIFIC JOURNAL OF MATHEMATICS
Vol. 79, No. 1, 1978

ALMOST PERIODIC FUNCTIONS ON SEMIDIRECT PRODUCTS OF TRANSFORMATION SEMIGROUPS

H. D. JUNGHENN

The notion of semidirect product of two transformation semigroups is introduced, and its space of almost periodic functions is expressed as a tensor product. The general techniques developed are applied to the special case of a semidirect product $S \circledcirc T$ of two semigroups. As a consequence new results are obtained on the characterization of the almost periodic compactification of $S \circledcirc T$ as a semidirect product of compact semigroups. A related result is the splitting of the enveloping semigroup of a semidirect product of certain flows into a semidirect product of enveloping semigroups.

0. Introduction. Let S and T be semitopological semigroups and $S \circledcirc T$ a semidirect product of S and T. In an earlier paper [10] we showed that, under certain conditions, the almost periodic (a.p.) compactification $(S \circledcirc T)'$ of $S \circledcirc T$ is a semidirect product of the a.p. compactification of T and a certain compact topological semigroup containing a dense homomorphic image of S. A simple corollary of this result is that the space of a.p. functions on $S \circledcirc T$ is a tensor product of the space of a.p. functions on T and a subspace of a.p. functions on S.

In this paper we introduce the notion of semidirect product of transformation semigroups and determine exactly when its space of a.p. functions may be expressed as a tensor product in analogy with the semigroup case described above. Cast in this general setting the problem of characterizing the space of a.p. functions on a semidirect product of semigroups becomes clear, and the techniques developed lead to elegant necessary and sufficient conditions for $(S \circledcirc T)'$ to be a semidirect product. As a consequence we are able to show that $(S \circledcirc T)'$ is a semidirect product for *all* semitopological semigroups S with identity and *all* semitopological groups T, thus generalizing results of [10, 11, 12]. The same conclusion holds if T merely contains a dense subgroup. In a similar vein, but using different techniques, we show that in a wide variety of cases the enveloping semigroup of the semidirect product of two equicontinuous flows is (canonically isomorphic to) a semi- direct product of the original enveloping semigroups.

1. Preliminaries. Let S and T be semitopological semigroups [1] and let $\tau: T \times S \to S$ be a separately continuous map satisfying

$$\tau(t, ss') = \tau(t, s)\tau(t, s') , \qquad \tau(tt', s) = \tau(t, \tau(t', s)) .$$

Thus $t \to \tau(t, \cdot)$ is a homomorphism from T into $\mathrm{Hom}(S)$, the semigroup of all homomorphisms on S. We shall assume that the map $(s, t) \to s\tau(t, s'): S \times T \to S$ is continuous for each $s' \in S$. The *semidirect product* $S \circledS T$ of S and T is the topological space $S \times T$ with the multiplication[1]

$$(s, t)(s', t') = (s\tau(t, s'), tt') .$$

The above conditions on τ imply that $S \circledS T$ is a semitopological semigroup. If S (respectively T) has an identity 1, we shall require that $\tau(t, 1) = 1$ (respectively, $\tau(1, \cdot)$ is the identity mapping).

A *transformation semigroup* is a triple (S, X, π) where S is a semitopological semigroup, X is a (Hausdorff) topological space, and $\pi: S \times X \to X$ is a separately continuous mapping (called an *action*) which satisfies $\pi(ss', x) = \pi(s, \pi(s', x))$. Usually we suppress the symbol π and write sx for $\pi(s, x)$. The *orbit* of $x \in X$ is the set $Sx = \{sx: s \in S\}$. In case S has an identity 1 we require that $1x = x$ for all $x \in X$. Note that every semitopological semigroup is a transformation semigroup, where the action is left multiplication.

If X is compact, the transformation semigroup (S, X) is called a *flow*. The *enveloping semigroup* of a flow (S, X) is the closure in the product space X^X of the set $\pi(S, \cdot) = \{\pi(s, \cdot): s \in S\}$ [8]; it is denoted by $E(S, X)$, or simply E_X. If the flow (S, X) is *equicontinuous* (i.e., $\pi(S, \cdot)$ is an equicontinuous family of mappings) then E_X is a (compact) topological semigroup with respect to the relativized product topology and composition of mappings.

Let (S, X) be a transformation semigroup, B a Banach space, and $C(X; B)$ the Banach space (uniform norm) of continuous bounded B-valued functions on X. A function $f \in C(X; B)$ is *almost periodic* with respect to the action of S on X if the set $\{f_s: s \in S\}$ is relatively compact in $C(X; B)$, where $f_s(x) = f(sx)$. The (closed) subspace of all a.p. functions in $C(X; B)$ is denoted by $AP(S, X; B)$. If B is the complex field then we shall suppress this symbol from the notation. Thus the usual space of a.p. functions on S is denoted by $AP(S)$. (S, X) is called *almost periodic* if $AP(S, X) = C(X)$. The reader is referred to [1, 2, 4, 5] for the general theory of a.p. functions on semigroups.

[1] The definition of semidirect product given here agrees with the classical definition for groups (see, for example, [**9**, p. 6]), but differs from the definition given in [**10**, **12**]. By considering "reverse" multiplication is S, T and $S \circledS T$, however, the two definitions may be shown to be equivalent.

Let X and Y be topological spaces, F and G closed linear subspaces of $C(X)$ and $C(Y)$ respectively. For $f \in F$ and $g \in G$ define $f \otimes g \in C(X \times Y)$ by $(f \otimes g)(x, y) = f(x)g(y)$. The closed linear span in $C(X \times Y)$ of all such functions is denoted by $F \otimes G$ and is called the *tensor product* of F and G. Note that $F \otimes G$ may be identified with a subspace of $C(X; G)$ via the isometric isomorphism $h \to h'$, where $h'(x) = h(x, \cdot)$. More generally, let B be any Banach space, and define $(f \otimes b)(x) = f(x)b (f \in F, b \in B)$. The *tensor product* of F and B, denoted $F \otimes B$, is defined as the closed linear span in $C(X; B)$ of all functions $f \otimes b$.

2. Semidirect products of transformation semigroups. Let (S, X), (T, Y) be transformation semigroups and $S \circledT T$ a semidirect product of S and T. Let σ: $T \times X \to X$ be a separately continuous mapping satisfying

$$\sigma(tt', x) = \sigma(t, \sigma(t', x)) , \qquad \sigma(t, sx) = \tau(t, s)\sigma(t, x) .$$

(In particular, (T, X, σ) is a transformation semigroup.) We shall also require that the mapping $(s, t) \to s\sigma(t, x)$: $S \times T \to X$ be continuous for each $x \in X$ and that $\sigma(1, \cdot)$ is the identity map if T has an identity 1. The *semidirect product* $(S \circledT T, X \times_\sigma Y)$ of (S, X) and (T, Y) is the transformation semigroup $(S \circledT T, X \times Y)$, where the action on $X \times Y$ is defined by

$$(s, t)(x, y) = (s\sigma(t, x), ty) .$$

Note that $(S \circledT T, X \times_\sigma Y)$ reduces to the direct product $(S \times T, X \times Y)$ if for each $t \in T$, $\tau(t, \cdot)$ and $\sigma(t, \cdot)$ are the identity functions.

Taking $\sigma = \tau$ one immediately sees that $S \circledT T$, when considered as a transformation semigroup (with respect to the usual action of left multiplication), is a semidirect product of the transformation semigroups S and T. We shall examine the a.p. properties of this kind of semidirect product in §4.

Another interesting class of semidirect products can be gotten as follows: Let G be a topological group, S and T closed subgroups of G with S normal in G, $G = ST$, and $S \cap T = \{1\}$. Let $X = G/S$ and $Y = G/T$ (left coset spaces), and consider the usual actions of S on X and T on Y (e.g., if $x = s'S$ then $sx = ss'S$). Define an action G on $X \times Y$ by $st(x, y) = (stxt^{-1}, ty)$. Then $(G, X \times Y)$ is a semidirect product of (S, X) and (T, Y) (where $\tau(t, s) = tst^{-1}$ and $\sigma(t, x) = txt^{-1}$).

As a third example, let (S, X) and (T, X) be transformation semigroups, where S and T are subsemigroups of a topological semigroup and $st = ts$ for all $s \in S$, $t \in T$. Define an action of the direct product $S \times T$ on $X \times X$ by $((s, t), (x, y)) \to (stx, ty)$. Then $(S \times T, X \times X)$ is a semidirect product. If the elements of S and T fail to commute with one another, but if S and T are subgroups of a topological group G with S normal in G, then the same mapping defines a semidirect product action of $S \circledS T$ on $X \times X$, where $\tau(t, s) = tst^{-1}$. (In each case, $\sigma(t, x) = tx$.)

Recall that a *homomorphism* from a transformation semigroup (T, Y) into a transformation semigroup (T, X) is a continuous map $\theta: Y \to X$ such that $\theta(ty) = t\theta(y)$ for all $t \in T$, $y \in Y$.

THEOREM 1. *Let (S, X) and (T, Y) be equicontinuous flows and $(S \circledS T, X \times_\sigma Y)$ a semidirect product. Suppose that (T, X, σ) is a homomorphic image of (T, Y). Then $(S \circledS T, X \times_\sigma Y)$ is equicontinuous, and if T is a group then $E = E((S \circledS T, X \times_\sigma Y))$ is canonically isomorphic (as a topological semigroup) to a semidirect product $E_X \circledS E_Y$ of E_X and E_Y.*

Proof. We omit the straightforward verification that $(S \circledS T, X \times_\sigma Y)$ is equicontinuous. Assume T is a group, let $\theta: Y \to X$ denote the given homomorphism onto X, and let $\bar{\theta}: E_Y \to E(T, X, \sigma)$ be the unique continuous semigroup homomorphism satisfying $\bar{\theta}(\zeta)(\theta(y)) = \theta(\zeta y)$ $(\zeta \in E_Y, y \in Y)$ [8, p. 20]. Let $\xi \in E_X$, $\zeta \in E_Y$ and define $\Psi(\xi, \zeta): X \times Y \to X \times Y$ by $\Psi(\xi, \zeta)(x, y) = (\xi\bar{\theta}(\zeta)(x), \zeta y)$. If (s_i) and (t_j) are nets in S and T, respectively, such that $s_i x \to \xi x (x \in X)$ and $t_j y \to \zeta y (y \in Y)$, then for all $y, z \in Y$, $(s_i, t_j)(\theta(z), y) = (s_i\theta(t_j z), t_j y) \to (\xi\theta(\zeta z), \zeta y) = \Psi(\xi, \zeta)(\theta(z), y)$, hence $\Psi(\xi, \zeta) \in E$. A similar argument shows that every member of E is of this form, hence $\Psi: E_X \times E_Y \to E$ is a surjection. Note that Ψ is also injective (by the surjectivity of θ and the members of E_Y) and continuous (since E_X is equicontinuous). Thus Ψ is a homeomorphism of $E_X \times E_Y$ onto E.

Next, for $\xi \in E_X$ and $\zeta \in E_Y$ define $\rho(\zeta, \xi): X \to X$ by $\rho(\zeta, \xi) = \bar{\theta}(\zeta)\xi\bar{\theta}(\zeta^{-1})$ (recalling that T, hence E_Y, is a group). Identifying $s \in S$ with the map it defines in E_X, and doing the same for $t \in T$, we see that for all $x \in X$,

$$(1) \qquad \rho(t, s)x = \bar{\theta}(t)s\bar{\theta}(t^{-1})x = \sigma(t, s\sigma(t^{-1}, x)) = \tau(t, s)x .$$

It follows from (1) and the equicontinuity of (S, X) and (T, Y) that $\rho(\zeta, \xi)x = \lim_{i,j} \tau(t_j, s_i)x$ $(x \in X)$ whenever $\xi = \lim_i s_i$ and $\zeta = \lim_j t_j$ (pointwise limits). Therefore $\rho(\zeta, \cdot): E_X \to E_X$. It is readily verified that $\zeta \to \rho(\zeta, \cdot)$ is a homomorphism from E_Y into $\mathrm{Hom}(E_X)$, that ρ is continuous, and finally that Ψ is a homomorphism from $E_X \circledS E_Y$ onto E.

REMARK. If $s \in S$ and $t \in T$ are considered also as members of

E_X and E_Y respectively, then

$$\Psi(s, t)(x, y) = (s\sigma(t, x), ty) = (s, t)(x, y)\,. \tag{2}$$

It is in this sense that the word canonical is used in the statement of Theorem 1.

The following examples show that the requirement of equicontinuity cannot in general be relaxed. In the first example, (S, X) is equicontinuous, (T, Y) is not. In the second example, the reverse is true.

EXAMPLE 1. Let $X = Y = \{z \in \boldsymbol{C}: |z| = 1\}$ and $S = \{1, s\} \subset X^X$, where 1 is the identity mapping and s is conjugation. For each positive integer n define $f_n: X \to X$ by

$$f_n(e^{2\pi i r}) = \begin{cases} e^{\pi i (2r)^n}, & 0 \leqq r \leqq 1/2 \\ e^{-\pi i (2-2r)^n}, & 1/2 \leqq r \leqq 1\,. \end{cases}$$

Since s commutes with each f_n it commutes with every member of the group T of homeomorphisms of X generated by the f_n. Therefore the mapping $((s, t), (x, y)) \to (stx, ty)$ is an action of the direct product $S \times T$ on $X \times Y$ such that $(S \times T, X \times Y)$ is a semidirect product. Let f denote the pointwise limit of $\{f_n\}$. If $\Psi: E_X \times E_Y \to E$ is any continuous mapping satisfying (2), then $\Psi(s, f) = \Psi(1, f)$. Therefore E cannot be canonically isomorphic to a semidirect product of E_X and E_Y.

EXAMPLE 2. Let X and Y be as in Example 1, and take S to be the group of all homeomorphisms of X, and T the subgroup of all rotations. Define $\tau: T \times S \to S$ by $\tau(t, s) = tst^{-1}$ and $\sigma: T \times X \to X$ by $\sigma(t, x) = tx$. Then $((s, t), (x, y)) \to (stx, ty)$ is the action of $S \,ⓣ\, T$ on $X \times Y$ which defines $(S \,ⓣ\, T, X \times_\sigma Y)$. Let f_n be as in Example 1, and let g_n denote counterclockwise rotation by $\pi - 1/n$. Then $\lim_{m,n\to\infty} f_m(g_n(1))$ does not exist, hence there can be no continuous mapping $\Psi: E_X \times E_Y \to E$ satisfying (2).

Recall that a flow (S, X) is *distal* if $x \neq x'$ implies the existence of a net (s_i) in S such that $\lim_i s_i x$ and $\lim_i s_i x'$ exist and are unequal. Equivalently, (S, X) is distal if and only if $E(S, X)$ is a group [8]. The following result is immediate.

COROLLARY 1. *Let (S, X) and (T, Y) satisfy all of the hypotheses of the theorem. Then if (S, X) is distal, so is $(S \,ⓣ\, T, X \times_\sigma Y)$.*

The *proximal relation* in a flow (S, X) is the set $P \subset X \times X$ defined as follows: $(x, y) \in P$ if and only if there exists a net (s_i)

in S such that $\lim_i s_i x = \lim_i s_i y$. In general P is only reflexive and symmetric. It is transitive if and only if $E(S, X)$ has a unique minimal left ideal [8, p. 39].

COROLLARY 2. *Let (S, X) and (T, Y) satisfy all of the hypotheses of the theorem. If the proximal relation is transitive in (S, X) then it is transitive in $(S \circledS T, X \times_\sigma Y)$.*

Proof. Let J be the unique minimal left ideal in E_X. We show first that $J \times E_Y$ is a left ideal in $E = E_X \circledP E_Y$. Let $(\xi', \zeta') \in J \times E_Y$ and $(\xi, \zeta) \in E$. Since $\rho(\zeta, \cdot): E_X \to E_X$ is an isomorphism, $\rho(\zeta, J) = J$. Hence $(\xi, \zeta)(\xi', \zeta') = (\xi\rho(\zeta, \xi'), \zeta\zeta') \in J \times E_Y$.

Let K be any minimal left ideal contained in $J \times E_Y$. If $(\xi, \zeta) \in K$, then $(\xi\rho(\zeta^{-1}, \xi), 1) = (\xi, \zeta^{-1})(\xi, \zeta) \in K$, hence the set $A = \{\xi \in J: (\xi, 1) \in K\}$ is nonempty. Since A is a left ideal, $A = J$. Therefore $(\xi, 1) \in K$ for every $\xi \in J$. Let $\xi \in J$, $\zeta \in E_Y$, and let e be any idempotent in J. Then $\rho(\zeta, e)$ is an idempotent in J and $J = J\rho(\zeta, e)$, so $(\xi, \zeta) = (\xi\rho(\zeta, e), \zeta) = (\xi, \zeta)(e, 1) \in K$. Therefore $J \times E_Y$ is a minimal left ideal.

Now let I be any minimal left ideal in E and set $B = \{\xi \in E_X: (\xi, 1) \in I\}$. Then B is a nonempty left ideal in E_X, hence $J \subset B$. It follows that $J \times E_Y \cap I \neq \phi$, so $J \times E_Y = I$.

3. Almost periodic functions on semidirect products.

Let (S, X), (T, Y) be transformation semigroups such that S has an identity 1, let $(S \circledS T, X \times_\sigma Y)$ be a given semidirect product, and let

$$F = AP((S \circledS T, X \times_\sigma Y)) .$$

Define an action of $S \circledS T$ on X by

$$(s, t)x = s\sigma(t, x) .$$

Clearly $AP(S \circledS T, X) \otimes AP(T, Y) \subset F$. We shall determine necessary and sufficient conditions for equality to hold.

To this end we define the following auxiliary actions on $X \times Y$:
$\alpha: (S \circledS T) \times (X \times Y) \to X \times Y$, $\alpha((s, t), (x, y)) = (s\sigma(t, x), y)$
$\beta: T \times (X \times Y) \to X \times Y$, $\beta(t, (x, y)) = (x, ty)$
$\gamma: T \times (X \times Y) \to X \times Y$, $\gamma(t, (x, y)) = (\sigma(t, x), y)$.

Consider the following statements:

(A) $F \subset AP(S \circledS T, X \times Y, \alpha)$
(B) $F \subset AP(T, X \times Y, \beta)$
(C) $F \subset AP(T, X \times Y, \gamma)$.

LEMMA 1. *If T contains a dense subgroup G, then* (A), (B), *and* (C) *are equivalent.*

Proof. That (A) implies (C) is clear, since $AP(S \circledT T, X \times Y, \alpha) \subset AP(T, X \times Y, \gamma)$.

To prove that (B) implies (A), let $f \in F$ and (s_n), (t_n) sequences in S and T respectively. There exist subsequences (p_n) of (s_n) and (q_n) of (t_n), and $g \in F$, such that $f(p_n\sigma(q_n, x), q_n y) \rightrightarrows g(x, y)$ (where $\rightrightarrows$ means uniform convergence in the free variables). Since $f \in F \subset AP(T, X \times Y, \beta)$ and $\bar{G} = T$, for each n we may choose $r_n \in G$ such that

$$(3) \qquad |f(p_n\sigma(r_n, x), r_n y) - f(p_n\sigma(q_n, x), q_n y)| < 1/n$$

and

$$(4) \qquad |f(x, r_n y) - f(x, q_n y)| < 1/n$$

for all $x \in X$, $y \in Y$. Replacing y in (3) and (4) by $r_n^{-1}y$, and x in (4) by $p_n\sigma(q_n, x)$, we see that $f(p_n\sigma(r_n, x), y) - g(x, r_n^{-1}y) \rightrightarrows 0$ and $|f(p_n\sigma(r_n, x), y) - f(p_n\sigma(q_n, x), y)| < 2/n$. Since $g \in AP(T, X \times Y, \beta)$ we may assume without loss of generality that $g(x, r_n^{-1}y) \rightrightarrows h(x, y)$ for some $h \in C(X \times Y)$. Then $f(p_n\sigma(q_n, x), y) \rightrightarrows h(x, y)$, so $f \in AP(S \circledT T, X \times Y, \alpha)$. The proof that (C) implies (B) is similar.

We omit the elementary proof (essentially a diagonalization argument) of our next lemma.

LEMMA 2. *Let K be a relatively compact subset of a Banach space B, and A a uniformly bounded collection of linear operators on B such that $\{ux: u \in A\}$ is relatively compact in B for each $x \in K$. Then each sequence (u_n) in A has a subsequence (v_n) such that $(v_n x)$ converges uniformly for $x \in K$.*

LEMMA 3. *If (T, Y) has a dense orbit then F is isometric and isomorphic to $AP(T, Y; AP(S \circledT T, X))$ under the mapping $f \to f'$, where $f'(y) = (f\cdot, y)$, if and only if conditions* (A) *and* (B) *hold.*

Proof. Suppose conditions (A) and (B) hold, and let $f \in F$ and $B = AP(S \circledT T, X)$. Clearly, then, $f'(y) \in B$, $(y \in Y)$. Claim that $f': Y \to C(X)$ is continuous. For let $y' \in Y$ and $\{i\}$ the directed set of open neighborboods of y'. If f' is not continuous at y', then there exist $\varepsilon > 0$ and nets (y_i), (x_i), with $y_i \in i$, such that for all i, $|f(x_i, y_i) - f(x_i, y')| > 2\varepsilon$. For each i choose $t_i \in T$ such that $|f(x_i, t_i y_0) - f(x_i, y_i)| < \varepsilon$ and $t_i y_0 \in i$, where $y_0 \in Y$ has dense orbit. Then $t_i y_0 \to y'$, and for all i,

$$(5) \qquad |f(x_i, y') - f(x_i, t_i y_0)| > \varepsilon .$$

Choose a subnet (t'_j) of (t_i) and $h \in C(X \times Y)$ such that $f(x, t'_j y_0) \rightrightarrows h(x, y_0)$. Then $h(x, y_0) = f(x, y')$, hence $f(x, t'_j y_0) \rightrightarrows f(x, y')$, contradicting (5). Therefore $f' \in C(Y, B)$.

To see that $f' \in AP(T, Y; B)$, let (t_n) be a sequence in T and choose a subsequence (q_n) such that $f(x, q_n y) \rightrightarrows h(x, y)$, where $h \in C(X \times Y)$. Then $\|f'(q_n y) - h'(y)\| \to 0$ uniformly in $y \in Y$.

Since $f \to f'$ is clearly a linear isometry, it remains to show that if $g \in AP(T, Y; B)$ and if $f(x, y) = g(y)(x)$, then $f \in F$. Let (s_n) and (t_n) be sequences in S and T, respectively. For each $s \in S$, $t \in T$ define $u(s,t)\colon C(X) \to C(X)$ by $u(s,t)h(x) = h(s\sigma(t,x))$. Then $\{u(s,t)g(y)\colon s \in S, t \in T\}$ is relatively compact in B for each $y \in Y$. Furthermore, $g(Y) \subset \{g(ty_0)\colon t \in \bar{T}\}$, and the latter is compact in B. Therefore, by Lemma 2 there exists a subsequence (p_n, q_n) of (s_n, t_n) and $h \in C(X \times Y)$ such that $f(p_n\sigma(q_n, x), y) \rightrightarrows h(x, y)$. Since $h' \in AP(T, Y; B)$ we may assume without loss of generality that $h(x, q_n y) \rightrightarrows k(x, y)$ for some $k \in C(X \times Y)$. Thus $f(p_n\sigma(q_n, x), q_n y) \rightrightarrows k(x, y)$.

Conversely, if $f \to f'$ maps F onto $AP(T, Y; B)$, then (B) obviously holds, and the argument of the previous paragraph up to the last two sentences shows that (A) also holds.

The following lemma generalizes Corollary 1(iii) of [11].

LEMMA 4. *Let B be a Banach space. If (T, Y) contains a dense orbit then $AP(T, Y; B) = AP(T, Y) \otimes B$.*

Proof. Clearly $AP(T, Y) \otimes B \subset AP(T, Y; B)$. For each $y \in Y$ define $e(y)\colon AP(T, Y; B) \to B$ by $e(y)f = f(y)$. Let $L = L(AP(T, Y; B), B)$ denote the space of bounded linear operators from $AP(T, Y; B)$ into B, and give L the strong operator topology. Then $e\colon Y \to L$ is obviously continuous. Let Z denote the closure of $e(Y)$ in L. Since $e(Y) \subset \Pi\{f(Y)\colon f \in AP(T, Y; B)\}$ and each $f(Y)$ is relatively compact in B (because (T, Y) has a dense orbit), Z is compact in L. Define $u(t) \in L(AP(T, Y; B), AP(T, Y; B))$ by $(u(t)f)(y) = f(ty)$ $(t \in T, y \in Y)$, and let U denote the strong operator closure of $u(T)$. Then an argument similar to the one for Z shows that U is compact in that topology [5, Theorem 3.2]. Now let $g \in C(Z)$ and let (t_n) be any sequence in T. There exists a subnet (q_i) of (t_n) and $v \in U$ such that $u(q_i) \to v$ in the strong operator topology. This implies that $g \circ e(q_i y) = g(e(y)u(q_i))$ converges uniformly to $g(e(y)v)$ in $y \in Y$. Therefore $g \circ e \in AP(T, Y)$ for every $g \in C(Z)$.

Given $\varepsilon > 0$ and $f \in AP(T, Y; B)$, let $Z_1, \cdots, Z_n$ be an open covering of Z such that $\|zf - wf\| < \varepsilon$ whenever $z, w \in Z_j$ $(j = 1, \cdots, n)$. Let $g_1, \cdots, g_n \in C(Z)$ such that support $(g_j) \subset Z_j$ and $\sum_{j=1}^n g_j = 1$.

[7, p. 170.] Choose $z_j \in Z_j$ and set $b_j = z_j f$, $h_j = q_j \circ e$. Then $\|f - \sum_{j=1}^{n} h_j \otimes b_j\| < \varepsilon$, hence $f \in AP(T, Y) \otimes B$.

The following theorem is now immediate:

THEOREM 2. *Let (S, X) and (T, Y) be transformation semigroups such that S has an identity and (T, Y) has a dense orbit. Then*

$$AP((S \circledT T, X \times_\sigma Y)) = AP(T, Y) \otimes AP(S \circledT T, X)$$

if and only if conditions (A) *and* (B) *hold.*

COROLLARY 1. *Let (S, X) and (T, Y) be as in Theorem 2. Then*

$$AP(S \times T, X \times Y) = AP(S, X) \otimes AP(T, Y) .$$

COROLLARY 2. *Let (S, X) and (T, Y) be as in Theorem 2, and suppose that T contains a dense subgroup and (S, X) has a dense orbit. If either* (a) *(T, Y) is almost periodic, or* (b) *X is compact and (T, X, σ) is distal, then*

$$AP((S \circledT T, X \times_\sigma Y)) = AP(T, Y) \otimes AP(S \circledT T, X) .$$

Proof. By Lemma 1, it suffices to show that condition (B) holds. Let $f \in F$ and suppose that (a) holds. For each $x \in X$, then, $f(x, \cdot) \in AP(T, Y)$, and since $f(X, \cdot)$ is relatively compact in $C(Y)$ (because (S, X) has a dense orbit), Lemma 2 implies that $f \in AP(T, X \times Y, \beta)$.

Now suppose (b) holds. If (t_i) is a net in T, there exists a subnet (q_j) and $\xi \in E(T, X, \sigma)$ such that $\sigma(q_j, x) \to \xi x$ for all $x \in X$. Fix $x_0 \in X$ and let $x' = \xi^{-1} x_0$. We may assume that $f(\sigma(q_j, x), q_j y) \rightrightarrows g(x, y)$ for some $g \in C(X \times Y)$. Since (S, X) has a dense orbit, $x \to f(x, \cdot): X \to C(Y)$ is continuous, hence $f(\sigma(q_j, x), q_j y) - f(x_0, q_j y) \rightrightarrows 0$. Therefore $f(x_0, q_j y) \rightrightarrows g(x', y)$, so $f(x_0, \cdot) \in AP(T, Y)$. Now proceed as in first paragraph.

4. Almost periodic compactification of $S \circledT T$. Let T be a semitopological semigroup. An *almost periodic compactification* of T is a pair (T', Ψ), where T' is a compact topological semigroup, and $\Psi: T \to T'$ is a continuous homomorphism with dense image such that $\Psi^* C(T') = AP(T)$ (where $\Psi^*: C(T') \to C(T)$ is the adjoint mapping $f \to f \circ \Psi$). Almost periodic compactifications always exist and are unique up to isomorphism [1, 2, 5].

THEOREM 3. *Let S, T be semitopological semigroups with identities, and $S \circledT T$ a semidirect product. The following are equivalent:*

(a) *There exist compact topological semigroups S', T'; continuous homomorphisms $\phi: S \to S'$, $\Psi: T \to T'$ with dense images; and a jointly continuous multiplication on the compact topological space $S' \times T'$ such that $(S' \times T', \phi \times \Psi)$ is an a.p. compactification of $(S \circledS T)$ (where $(\phi \times \Psi)(s, t) = (\phi(s), \Psi(t))$).*

(b) $AP(S \circledS T) = AP(S \circledS T, S) \otimes AP(T)$.

(c) *Every member of $AP(S \circledS T)$ is a.p. with respect to both of the actions*

$$(S \circledS T) \times (S \times T) \longrightarrow S \times T: ((s, t), (x, y)) \longrightarrow (s\tau(t, x), y)$$

$$T \times (S \times T) \longrightarrow S \times T: (t, (x, y)) \longrightarrow (x, ty)\ .$$

If (a) *holds then (T', Ψ) is an a.p. compactification of T, and $S' \times T'$ is a semidirect product $S' \circledP T'$, where $\rho(\Psi(t), \phi(s)) = \phi(\tau(t, s))$.*

Proof. Statements (b) and (c) are equivalent by Theorem 2.

(b) implies (a): Let $((S \circledS T)', \theta)$ denote an a.p. compactification of $S \circledS T$, S' the spectrum of the C^*-algebra $AP(S \circledS T, S)$, $\phi: S \to S'$ the evaluation mapping $(\phi(s): f \to f(s))$, and (T', Ψ) an a.p. compactification of T. By hypothesis there exists an isometric isomorphism $V: C(S') \otimes C(T') \to C((S \circledS T)')$ such that $V(\hat{g} \otimes \hat{h}) = (g \otimes h)\hat{}$, $(g \in AP(S \circledS T, S), h \in AP(T))$, where $\phi^*(\hat{g}) = g$, $\Psi^*(\hat{h}) = h$ and $\theta^*((g \otimes h)\hat{}) = g \otimes h$. Since $C(S') \otimes C(T') = C(S' \times T')$ there exists a homeomorphism $\eta: (S \circledS T)' \to S' \times T'$ such that $\eta^* = V$. If $s \in S$, $t \in T$, $g \in AP(S \circledS T, S)$ and $h \in AP(T)$, then $(\hat{g} \otimes \hat{h})(\eta(\theta(s, t))) = V(\hat{g} \otimes \hat{h})(\theta(s, t)) = (g \otimes h)\hat{}(\theta(s, t)) = g(s)h(t) = (\hat{g} \otimes \hat{h})(\phi(s), \Psi(t))$. It follows that $\phi \times \Psi = \eta \circ \theta$. Let $S' \times T'$ have the unique multiplication which makes η a semigroup isomorphism. If $s, s' \in S$ then $(\phi(s), \Psi(1))(\phi(s'), \Psi(1)) = \eta \circ \theta(ss', 1) = (\phi(ss'), \Psi(1))$, hence $\phi(S) \times \Psi(1)$, and therefore also $S' \times \Psi(1)$, is a subsemigroup of $S' \times T'$. Thus we may define multiplication in S' so that ϕ and S' have the required properties.

(a) implies (b): If (a) holds then in particular $AP(S \circledS T) = (\phi \times \Psi)^* C(S' \times T') = (\phi \times \Psi)^* C(S') \otimes C(T')$. We shall show that $\phi^* C(S') = AP(S \circledS T, S)$. Let $g \in \phi^* C(S')$ and $((s_i, t_i))$ be any net in $S \circledS T$. Choose a subnet (s'_j, t'_j) such that $(\phi(s'_j), \Psi(t'_j))$ converges to some $(x, y) \in S' \times T'$. Then since $S' \times T'$ is a compact topological semigroup, the first coordinate of $(\phi(s'_j\tau(t'_j, s)), \Psi(t'_j)) = (\phi(s'_j), \Psi(t'_j)) \times (\phi(s), \Psi(1))$ converges uniformly in $s \in S$ to the first coordinate of $(x, y)(\phi(s), \Psi(1))$, and it follows that $g \in AP(S \circledS T, S)$. Conversely, let $g \in AP(S \circledS T, S)$. Then $g \otimes 1 \in AP(S \circledS T)$ so there exists $h \in C(S' \times T')$ such that $g \otimes 1 = h \circ (\phi \times \Psi)$. If $k(x) = h(x, \Psi(1))$ then $k \in C(S')$ and $g = k \circ \phi$. Therefore $\phi^* C(S') = AP(S \circledS T, S)$. A similar argument shows that $\Psi^* C(T') = AP(T)$. Thus $AP(S \circledS T) = AP(S \circledS T, S) \otimes AP(T)$, and (T', Ψ) is an a.p. compactification of T.

It remains to prove that if (a) holds, then $S' \times T'$ is a semidirect product. We may take S' to be the spectrum of $AP(S \circledτ T, S)$ and $\phi: S \to S'$ the evaluation map. For each $g \in AP(S \circledτ T, S)$ and $t \in T$ define $v(t)g \in AP(S \circledτ T, S)$ by $(v(t)g)(s) = g((1, t)s) = g(\tau(t, s))$. Then $\delta(t, x) = v(t)^*x$ defines an action δ of on S' such that

$$\delta(t, \phi(s)) = \phi(\tau(t, s)) \ . \tag{6}$$

Since this action is equicontinuous, $E = E(T, S', \delta)$ is a compact topological semigroup, and (6) shows that $E \subset \mathrm{Hom}\,(S)$. Since $t \to \delta(t, \cdot): T \to E$ is a continuous homomorphism there exists a continuous homomorphism $y \to \rho(y, \cdot): T' \to E$ such that $\rho(\Psi(t), \cdot) = \delta(t, \cdot)$ [5]. In particular, $\rho(\Psi(t), \phi(s)) = \phi(\tau(t, s))$. Since ϕ, Ψ, and $\phi \times \Psi$ are homomorphisms, $(\phi(s), \Psi(t))(\phi(s'), \Psi(t')) = (\phi(s\tau(t, s')), \Psi(t)\Psi(t')) = (\phi(s)\rho(\Psi(t), \phi(s')), \Psi(t)\Psi(t'))$ $(s, s' \in S; t, t' \in T)$, and taking nets we see that $(x, y)(x', y') = (x\rho(y, x'), yy')$ $(x, x' \in S'; y, y' \in T')$. Therefore $S' \times T' = S' \circledρ T'$.

The following corollary is an extension of the main results of [10].

COROLLARY 1. *Let S and T be semitopological semigroups with identities, $S \circledτ T$ a semidirect product, and suppose that T contains a dense subgroup G. Then in the notation of Theorem 3, $(S' \circledρ T', \phi \times \psi)$ exists and is an a.p. compactification of $S \circledτ T$, and $AP(S \circledτ T) = AP(S \circledτ T, S) \otimes AP(T)$.*

Proof. For each $t \in T$ define $U(t): AP(S \circledτ T) \to C(S \circledτ T)$ by $U(t)f(s', t') = f(\tau(t, s'), t')$. By Lemma 1 of §3, it suffices to show that $U(T)f$ is relatively norm compact for each $f \in AP(S \circledτ T)$. Since $t \to U(t)f$ is continuous in the topology of pointwise convergence on $C(S \times T)$, it is enough to show that $U(G)f$ is relatively norm compact. For $x \in S \circledτ T$ let $L(x)$ and $R(x)$ denote respectively the left and right translation operators on $AP(S \circledτ T)$. If $t \in G$ then

$$U(t)f(s', t') = L(1, t)R(1, t^{-1})R(1, t')f(s', 1)$$

$(s' \in S, t' \in T)$, hence by Lemma 2 applied to $K = R(1, T)f$ and $A = \{L(1, t)R(1, t^{-1}): t \in G\}$, any sequence (t_n) of G has a subsequence (r_n) such that $U(r_n)f$ converges in norm.

COROLLARY 2. [6, 11]. *Let S, T be semitopological semigroups with identities and a.p. compactifications (S', ϕ), (T', ψ). Then $(S' \times T', \phi \times \psi)$ is an a.p. compactification of the direct product $S \times T$.*

REMARKS. J. Berglund and P. Milnes have given an example of two left zero semigroups S and T for which the conclusion of

Corollary 2 is false [3]. Also, the corresponding result for the weakly almost periodic compactification can fail even if S and T are locally compact abelian topological groups. (See, for example, [11], p. 663.)

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Received July 22, 1977.

THE GEORGE WASHINGTON UNIVERSITY
WASHINGTON, D. C. 20052