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CONGRUENT SECTIONS OF A CONVEX BODY

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It is shown that if all the 3-dimensional sections of a convex body K, of dimension at least 4, through a fixed inner point are congruent, then K is a euclidean ball. A dual result concerning projections is also proved.

1. Introduction. W. Süss [8] showed that if all the plane sections of a 3-dimensional convex body passing through a fixed inner point are congruent, then the body is a euclidean ball. P. Mani [5] generalized this result to the case of congruent $2n$-dimensional sections of a $(2n+1)$-dimensional convex body. Both of these results are deduced immediately from topological proofs that a nonspherical $2n$-dimensional body cannot be completely turned in dimension $2n+1$, and the assumption that the sections fit together to form a convex body is only used to prove continuity. However, every centrally symmetric 3-dimensional body can be completely turned in 4-dimensional euclidean space E^4, so in this case a proof using properties of convex bodies is required; the present paper provides one. Our main results are:

THEOREM 1. *Let K be a convex body of dimension at least 4, let $\boldsymbol{p}$ be an inner point of K, and suppose that all 3-dimensional sections of K passing through $\boldsymbol{p}$ are congruent. Then K is a euclidean ball with center $\boldsymbol{p}$.*

THEOREM 2. *Let K be a convex body of dimension at least 4, and suppose all the 3-dimensional orthogonal projections of K are congruent. Then K is a euclidean ball.*

A result which follows directly from the work of Mani is the following:

THEOREM 3. *Let $n \geqq 1$, let K be a convex body of dimension at least $2n+1$ and let $\boldsymbol{p}$ be an inner point of K. Suppose all the $2n$-dimensional sections of K passing through $\boldsymbol{p}$ are affinely equivalent. Then K is an ellipsoid.*

2. Complete turnings of 3-dimensional bodies. When A is a d-dimensional convex body, a *field of bodies congruent to A* is a continuous function $A(\boldsymbol{u})$ defined for $\boldsymbol{u}$ in the unit sphere S^d, where $A(\boldsymbol{u})$ is a congruent copy of A lying in a hyperplane of E^{d+1} perpendicular to $\boldsymbol{u}$; here $A(\boldsymbol{u})$ is meant to be continuous in the Hausdorff metric. If additionally $A(\boldsymbol{u}) = A(-\boldsymbol{u})$ for each $\boldsymbol{u}$, we say $A(\boldsymbol{u})$ is a *complete turning of* A in E^{d+1}. Clearly if all the d-dimensional sections of a $(d+1)$-dimensional convex body through a fixed inner point are congruent, they give rise to a complete turning of some d-dimensional body in E^{d+1}. We make use of the methods of Mani [5] and H. Hadwiger [4] to determine which 3-dimensional convex bodies can be completely turned in E^4. When $\boldsymbol{v}$ is a fixed unit vector in E^4 and for $\boldsymbol{u}=(t_1, t_2, t_3, t_4) \in S^3$ we define $\boldsymbol{p}_1(\boldsymbol{u})=(-t_2, t_1, -t_4, t_3)$, $\boldsymbol{p}_2(\boldsymbol{u}) = (t_3, -t_4, -t_1, t_2)$, $\boldsymbol{p}_3(\boldsymbol{u}) = (-t_4, -t_3, t_2, t_1)$, then let Ψ_u be the orthogonal transformation such that $\Psi_u(\boldsymbol{v}) = \boldsymbol{u}$ and $\Psi_u(\boldsymbol{p}_i(\boldsymbol{v})) = \boldsymbol{p}_i(\boldsymbol{u})$ for $i = 1, 2, 3$. Notice that $\Psi_{-u} = -\Psi_u$.

LEMMA 2.1. *Let A be a 3-dimensional convex body whose symmetry group is finite, and suppose A can be completely turned in E^4. Then A is centrally symmetric.*

Proof. Let $A(\boldsymbol{u})$ be a complete turning of A in E^4. We may assume that each $A(\boldsymbol{u})$ has its centroid at the origin $\boldsymbol{o}$, and that $A = A(\boldsymbol{v})$ for some unit vector $\boldsymbol{v}$. Let Ψ_u be defined as above. Since $A(\boldsymbol{u})$ is a field of bodies congruent to A, the proof of Proposition 2 in [5] shows the existence of orthogonal transformations Φ_u depending continuously on $\boldsymbol{u}$ with $\Phi_u(A) = A(\boldsymbol{u})$. The restriction $\Phi_{-u}^{-1}\Phi_{u|A}$ is a continuously varying symmetry of A, and by connectedness it must be a constant Θ.

The map $\Psi_u^{-1}\Phi_u$ preserves the linear span of A, so consider $\Psi_u^{-1}\Phi_u(v)$ for a fixed $\boldsymbol{v} \in A$. The mapping $\boldsymbol{u} \mapsto \Psi_u^{-1}\Phi_u(v)$ maps S^3 continuously into a copy of E^3, so by the Borsuk-Ulam theorem (see [7], p. 266) it maps some pair of antipodal points into coincidence. Thus for some $\boldsymbol{u}$ we have

$$\Psi_{-u}^{-1}\Phi_{-u}(\boldsymbol{v}) = \Psi_u^{-1}\Phi_u(\boldsymbol{v})$$

and since $\Psi_{-u} = -\Psi_u$ this yields

$$-\Phi_{-u}(\boldsymbol{v}) = \Phi_u(\boldsymbol{v})$$

and so $-\boldsymbol{v} = \Phi_{-u}^{-1}\Phi_u(\boldsymbol{v}) = \Theta(\boldsymbol{v})$. It follows that Θ is a central reflection, and A is centrally symmetric.

LEMMA 2.2. *Let A be a 3-dimensional convex body whose symmetry group is infinite, and suppose A can be completely turned in E^4. Then A is centrally symmetric.*

Proof. Let $A(\boldsymbol{u})$ be a complete turning of A. We may assume that each $A(\boldsymbol{u})$ has its centroid at the origin, and that $A = A(\boldsymbol{v})$ where $\boldsymbol{v}$ is a unit vector. Let Ψ_u be the map defined above. Since

A has an infinite symmetry group, it has an axis of revolution; let such an axis be parallel to the unit vector $\boldsymbol{w}$.

Suppose that A is not centrally symmetric, so that A has only one axis of revolution, and for some $\lambda > 0$ the two sections

$$\{\boldsymbol{x} \in A: \boldsymbol{x} \cdot \boldsymbol{w} = \pm\lambda\}$$

are discs of different radii. Any symmetry of A maps the axis onto itself, and maps $\lambda\boldsymbol{w}$ onto $\lambda\boldsymbol{w}$ also.

It follows that for each $\boldsymbol{u} \in S^3$, there is a unit vector $\boldsymbol{w}(\boldsymbol{u})$ in the linear span of $A(\boldsymbol{u})$ such that $\Phi(\boldsymbol{w}) = \boldsymbol{w}(\boldsymbol{u})$ for every orthogonal transformation Φ with $\Phi(A) = A(\boldsymbol{u})$. Hence $\boldsymbol{w}(\boldsymbol{u})$ is a continuous function of $\boldsymbol{u}$ and $\boldsymbol{w}(-\boldsymbol{u}) = \boldsymbol{w}(\boldsymbol{u})$. The mapping $\boldsymbol{u} \mapsto \Psi_u^{-1}(\boldsymbol{w}(\boldsymbol{u}))$ is a continuous map of S^3 into a copy of E^3, so by the Borsuk-Ulam theorem, for some $\boldsymbol{u}$ we have

$$\Psi_u^{-1}(\boldsymbol{w}(\boldsymbol{u})) = \Psi_{-u}^{-1}(\boldsymbol{w}(-\boldsymbol{u})) = -\Psi_u^{-1}(\boldsymbol{w}(\boldsymbol{u}))$$

so that $\boldsymbol{w}(\boldsymbol{u}) = -\boldsymbol{w}(\boldsymbol{u})$ which is impossible. We conclude that A is centrally symmetric.

REMARKS. Lemmas 2.1 and 2.2 show that any 3-dimensional convex body which can be completely turned in E^4 is centrally symmetric. Conversely, the map Ψ_u allows every 3-dimensional centrally symmetric convex body to be completely turned in E^4.

3. Congruent central sections of a convex body. Throughout this section K will be a fixed 4-dimensional convex body in E^4 having the origin as center of symmetry, and such that all the 3-dimensional central sections of K are congruent. We assume K is not a euclidean ball, and seek a contradiction. For nonzero $\boldsymbol{u}$ and $\boldsymbol{v}$ the hyperplane $\{\boldsymbol{x} \in E^4: \boldsymbol{x} \cdot \boldsymbol{u} = 0\}$ is denoted $H(\boldsymbol{u})$, the orthogonal projection on $H(\boldsymbol{u})$ is denoted π_u and $\Phi_{u,v}$ is some orthogonal transformation which maps $H(\boldsymbol{u}) \cap K$ onto $H(\boldsymbol{v}) \cap K$; clearly the choice of $\Phi_{u,v}$ may not be unique.

LEMMA 3.1. *Let $\boldsymbol{v} \in S^3$. Then the section $H(\boldsymbol{v}) \cap K$ is not a body of revolution.*

Proof. Suppose the lemma is false. Then since $H(\boldsymbol{v}) \cap K$ is not a euclidean ball, it has just one axis of rotation l. Consider a plane Λ with $l \subset \Lambda \subset H(\boldsymbol{v})$. For any $\boldsymbol{u}^* \in X = S^3 \cap \Lambda^\perp$, there is a neighborhood of $\boldsymbol{u}^*$ in X in which $\Phi_{u,v}$ can be chosen as a continuous function of $\boldsymbol{u}$. Let X_0 be a compact simple arc of X containing $\boldsymbol{v}$ in its interior. By compactness X_0 can be dissected into a finite collection of interior-disjoint arcs, on each of which $\Phi_{u,v}$ is chosen continuously; if this gives rise to two choices $\Phi'_{u,v}$ and $\Phi''_{u,v}$ of $\Phi_{u,v}$ at a common end $\boldsymbol{u}$ of two such arcs, then $\Phi''_{u,v}\Phi'^{-1}_{u,v}$ preserves $H(\boldsymbol{v}) \cap K$, so by composing $\Phi'_{u,v}$ with a suitable orthogonal transformation we can suppose $\Phi''_{u,v} = \Phi'_{u,v}$. Hence we can choose $\Phi_{u,v}$ continuously for $\boldsymbol{u} \in X_0$.

We claim $\Phi_{u,v}(\Lambda)$ contains l for every $\boldsymbol{u} \in X_0$. Suppose this is false, and let $\boldsymbol{x} \in l \cap bdK$. Then as $\boldsymbol{u}$ varies on X_0, a nontrivial arc on a sphere is described by $\Phi_{u,v}(\boldsymbol{x})$, so $H(\boldsymbol{v}) \cap bdK$ contains a maximal spherical cap A with pole $\boldsymbol{x}$ and at constant distance from $\boldsymbol{o}$. Let $\boldsymbol{y}$ and z be the points of Λ on the perimeter of A. Then for each $\boldsymbol{u} \in X_0$, the points $\Phi_{u,v}(\boldsymbol{y})$ and $\Phi_{u,v}(z)$ lie within clA and $||\Phi_{u,v}(\boldsymbol{y}) - \Phi_{u,v}(z)|| = ||\boldsymbol{y} - z||$, so l, $\Phi_{u,v}(\boldsymbol{y})$ and $\Phi_{u,v}(z)$ are coplanar. This contradiction shows that $\Phi_{u,v}(\Lambda)$ contains l for each $\boldsymbol{u} \in X_0$.

By composing $\Phi_{u,v}$ with a suitable continuously varying orthogonal transformation that acts as a symmetry on $H(\boldsymbol{v}) \cap K$ we can suppose $\Phi_{u,v}(\Lambda) = \Lambda$ for each $\boldsymbol{u} \in X_0$ and $\Phi_{v,v}$ is the identity map, so $\Phi_{u,v}(\boldsymbol{u}) = \boldsymbol{v}$. Since the symmetry group of $\Lambda \cap K$ is finite, $\Phi_{u,v|\Lambda}$ is the identity for all $\boldsymbol{u} \in X_0$. Thus l is the axis of $H(\boldsymbol{u}) \cap K$ for all $\boldsymbol{u} \in X_0$, and hence (by letting X_0 tend to X) for all $\boldsymbol{u} \in X$. Then for any $\boldsymbol{s} \in l^{\perp} \cap bdK$, the length $||\boldsymbol{s}||$ is equal to the radius of the central section of $H(\boldsymbol{v}) \cap K$ perpendicular to l. It follows that $l^{\perp} \cap K$ is a euclidean ball and so K is a euclidean ball contrary to hypothesis. This proves the lemma.

REMARKS. From Lemma 3.1 it follows that each $H(\boldsymbol{u}) \cap K$ has only a finite symmetry group. It follows from the proof of Proposition 2 in [5] that for fixed $\boldsymbol{v} \in S^3$ we can choose $\Phi_{u,v}$ as a continuous function of $\boldsymbol{u} \in S^3$. We can further suppose $\Phi_{v,v}$ is the identity so $\Phi_{u,v}(\boldsymbol{u}) = \boldsymbol{v}$. When $\boldsymbol{u}$ and $\boldsymbol{v}$ are not unit vectors, we define $\Phi_{u,v} = \Phi_{u',v'}$ where $\boldsymbol{u}' = ||\boldsymbol{u}||^{-1}\boldsymbol{u}$, $\boldsymbol{v}' = ||\boldsymbol{v}||^{-1}\boldsymbol{v}$.

LEMMA 3.2. *K is smooth.*

Proof. Let K^* be the polar reciprocal of K relative to the origin. Then $\Phi_{u,v}(\pi_u K^*) = \pi_v K^*$ for each $\boldsymbol{u}, \boldsymbol{v} \in S^3$. To prove K is smooth, it will suffice to show K^* is strictly convex. In the ensuing argument, faces are meant to be exposed faces.

Suppose first that K^* has a 2-face F, and let F be the face of K^* in the direction of $\boldsymbol{w} \in S^3$. Fix a unit vector $\boldsymbol{v}$ perpendicular to $\boldsymbol{w}$ and the affine hull $aff\ F$. Then $\pi_u F$ is a 2-face of $\pi_u K^*$ for every $\boldsymbol{u}$ perpendicular to $\boldsymbol{w}$ and close to $\boldsymbol{v}$, and by continuity $\Phi_{u,v}(\pi_u F) = \pi_v F$. However, if $\boldsymbol{u}$ is chosen perpendicular to $\boldsymbol{w}$ but not perpendicular to $aff\ F$, then $\pi_u F$ has smaller area than $\pi_v F$. This contradiction shows that K^* has no 2-faces.

Next suppose that K^* has 3-faces, and consider any 3-face G, having an outer unit normal $\boldsymbol{m}$ say at its centroid. If $\boldsymbol{u}$ is any unit vector perpendicular to $\boldsymbol{m}$ then $\pi_u G$ is a 2-face of $\pi_u K^*$. Conversely, suppose J is a 2-face of a projection $\pi_w K^*$. Then there is a face G' of K^* such that $\pi_w G' = J$. We necessarily have $dim\, G' \geqq dim\, J$, and since K^* has no 2-faces, G' must be a 3-face. Hence $\boldsymbol{w}$ is perpendicular to the normal of K^* at the centroid of G'. Since the facets of K^* form a countable set, $\pi_w(K^*)$ can only have a 2-face when $\boldsymbol{w}$ lies in a certain countable union of hyperplanes. This is impossible since all the 3-dimensional orthogonal projections of K^* are congruent. We conclude that K^* has no 3-faces.

Finally suppose K^* has an edge L, with ends $\boldsymbol{x}$ and $\boldsymbol{x} + \lambda \boldsymbol{t}$ where $\lambda > 0$ and $\boldsymbol{t}$ is a unit vector. Let L be the face of K^* in the direction of the unit vector $\boldsymbol{p}$, let Θ be the plane through $\boldsymbol{o}$ orthogonal to $\boldsymbol{p}$ and $\boldsymbol{t}$, and let $\boldsymbol{v}$ be a unit vector in Θ. For each $\boldsymbol{u} \in \Theta \cap S^3$ the line segment $L(\boldsymbol{u}) = \Phi_{u,v}(\pi_u L)$ is an edge of $\pi_v K^*$ and has length λ; we claim that $L(\boldsymbol{u})$ is the same edge for every $\boldsymbol{u} \in \Theta \cap S^3$. Suppose this is false; then by continuity the region $\cup\{L(\boldsymbol{u}): \boldsymbol{u} \in \Theta \cap S^3\}$ contains on open neighborhood N in the relative boundary of $\pi_v K^*$. Choose $\boldsymbol{u} \in \Theta \cap S^3$ such that $L(\boldsymbol{u})$ intersects N. For every unit vector $\boldsymbol{w}$ orthogonal to $\boldsymbol{p}$ and close to $\boldsymbol{u}$, the segment $L(\boldsymbol{w}) = \Phi_{w,v}(\pi_w L)$ is an edge of $\pi_v K^*$ that intersects N, so $L(\boldsymbol{w}) = L(\boldsymbol{u}')$ for some $\boldsymbol{u}' \in \Theta \cap S^3$. Hence $L(\boldsymbol{w})$ has length λ. But we can choose $\boldsymbol{w}$ not to be orthogonal to $\boldsymbol{t}$, in which case $L(\boldsymbol{w})$ is shorter than L. This contradiction shows that $L(\boldsymbol{u})$ is the same edge for all $\boldsymbol{u} \in \Theta \cap S^3$.

It follows that $\Phi_{u,v}(\pi_u \boldsymbol{x}) = \pi_v(\boldsymbol{x})$ and $\Phi_{u,v}(\pi_u(\boldsymbol{x} + \lambda \boldsymbol{t})) = \pi_v(\boldsymbol{x} + \lambda \boldsymbol{t})$ for all $\boldsymbol{u} \in \Theta \cap S^3$, and since π_u and π_v fix $\boldsymbol{t}$ we find that $\Phi_{u,v}(\boldsymbol{t}) = \boldsymbol{t}$. Further $\pi_u(\boldsymbol{p}) = \pi_v(\boldsymbol{p}) = \boldsymbol{p}$ so $\Phi_{u,v}(\boldsymbol{p}) = \boldsymbol{p}$, and it follows that $\Phi_{u,v}$ fixes all points of $\Theta^\perp$ for $\boldsymbol{u} \in \Theta \cap S^3$. Hence all sections of K parallel to Θ are circular and have centers on $\Theta^\perp$. It follows that K has 3-dimensional central sections which are bodies of revolution, contrary to Lemma 3.1. We conclude that K^* is strictly convex, so K is smooth.

DEFINITION. An open neighborhood A on the relative boundary of a section $H(\boldsymbol{v}) \cap K$ is said to be *contoured* if the intersection of A with every sphere with center $\boldsymbol{o}$ is empty or a circular arc.

LEMMA 3.3. *Let $\boldsymbol{x}$ be a boundary point of K at which the unit outward normal $\boldsymbol{n}$ is not a multiple of $\boldsymbol{x}$, let $\boldsymbol{v}$ be a unit vector perpendicular to $\boldsymbol{x}$, and suppose relbd $H(\boldsymbol{v}) \cap K$ contains no contoured neighborhoods. Then $\Phi_{u,v}$ is a differentiable function of $\boldsymbol{u}$ for $\boldsymbol{u}$ close to $\boldsymbol{v}$.*

Proof. Choose a neighborhood A of $\boldsymbol{x}$ in the boundary of K such that at no point of A is the normal direction to K parallel to the radius vector. We show A contains a neighborhood $B \subset \mathit{relbd}\, H(\boldsymbol{v}) \cap K$ so that at no point of B is the outward normal to $H(\boldsymbol{v}) \cap K$ parallel to the radius vector. Suppose this is false so by continuity of the normal directions, the normal to $H(\boldsymbol{v}) \cap K$ at each point of $H(\boldsymbol{v}) \cap A$ is parallel to the radius vector. Hence $H(\boldsymbol{v}) \cap A$ is a subset of a 3-sphere S with center $\boldsymbol{o}$. For $\boldsymbol{u} \in S^3$ we have $\Phi_{u,v}^{-1}(H(\boldsymbol{v}) \cap A) \subset S$, and the regions $\Phi_{u,v}^{-1}(H(\boldsymbol{v}) \cap A)$ cover a neighborhood of $\boldsymbol{x}$ in bdK. Thus $\boldsymbol{x}$ is parallel to $\boldsymbol{n}$ contrary to hypothesis. We deduce the existence of B as required.

It now follows from the Implicit Function theorem that each set $C(\alpha) = \{\boldsymbol{y} \in B: \|\boldsymbol{y}\| = \alpha\}$ is a union of simple continuously differentiable arcs if it is nonempty. We may suppose B is chosen so that each $C(\alpha)$ is connected. Consider two curves $C(\alpha)$ and $C(\beta)$ with $\alpha \neq \beta$, and let $\boldsymbol{a}_0 \in C(\alpha)$ and $\boldsymbol{b}_0 \in C(\beta)$ be two points for which $\boldsymbol{a}_0 - \boldsymbol{b}_0$ is not perpendicular to the tangent line of $C(\beta)$ at $\boldsymbol{b}_0$. We can continuously differentiably select $\boldsymbol{f}_{\alpha,\beta}(\lambda, \boldsymbol{a}) \in C(\beta)$ with $\|\boldsymbol{f}_{\alpha,\beta}(\lambda, \boldsymbol{a}) - \boldsymbol{a}\| = \lambda$ for $\boldsymbol{a} \in C(\alpha)$ close to $\boldsymbol{a}_0$ and λ close to $\|\boldsymbol{a}_0 - \boldsymbol{b}_0\|$, such that $\boldsymbol{f}_{\alpha,\beta}(\|\boldsymbol{a}_0 - \boldsymbol{b}_0\|, \boldsymbol{a}_0) = \boldsymbol{b}_0$.

Let us suppose there exist open neighborhoods M, N in B such that for each $\alpha \neq \beta$, each $\boldsymbol{a}_0 \in C(\alpha) \cap M$ and each $\boldsymbol{b}_0 \in C(\beta) \cap N$ with $\boldsymbol{a}_0 - \boldsymbol{b}_0$ not perpendicular to the tangent line of $C(\beta)$ at $\boldsymbol{b}_0$, we have

$$(*) \qquad D_2\|\boldsymbol{f}_{\alpha,\beta}(\lambda, \boldsymbol{a}) - \boldsymbol{f}_{\alpha,\beta}(\mu, \boldsymbol{a})\| = 0$$

for all λ and μ close to $\|\boldsymbol{a}_0 - \boldsymbol{b}_0\|$ and $\boldsymbol{a}$ on $C(\alpha)$ close to $\boldsymbol{a}_0$. Additionally we may suppose that each $C(\alpha)$ intersects M and N in (connected, but possibly empty) arcs.

Consider $\boldsymbol{a}_0 \in M$ with $\|\boldsymbol{a}_0\| = \alpha$. Suppose N contains a neighborhood P such that each $\boldsymbol{b} \in P$ satisfies $\|\boldsymbol{b}\| \neq \alpha$ and $\boldsymbol{b} - \boldsymbol{a}_0$ is perpendicular to the tangent line of $C(\|\boldsymbol{b}\|)$ at $\boldsymbol{b}$. We can suppose the intersection of P with each $C(\beta)$ is connected, so that each $C(\beta)$ which intersects P is at constant distance from $\boldsymbol{a}_0$; thus each such $C(\beta)$ is a circular arc, being in the intersection of two spheres. Hence P is a contoured neighborhood contrary to hypothesis. Thus for the given $\boldsymbol{a}_0$, for a dense set of $\boldsymbol{b}_0$ in N we have $\boldsymbol{a}_0 - \boldsymbol{b}_0$ not perpendicular to the tangent line of $C(\beta)$ at $\boldsymbol{b}_0$ and $D_2\|\boldsymbol{f}_{\alpha,\beta}(\lambda, \boldsymbol{a}) - \boldsymbol{f}_{\alpha,\beta}(\mu, \boldsymbol{a})\| = 0$ for all $\boldsymbol{a}$ on $C(\alpha)$ close to $\boldsymbol{a}_0$ and λ, μ close to $\|\boldsymbol{a}_0 - \boldsymbol{b}_0\|$ where $\beta = \|\boldsymbol{b}_0\|$. Consider such a $\boldsymbol{b}_0$, which we can suppose chosen so that $\boldsymbol{a}_0 - \boldsymbol{b}_0$ is not perpendicular to the tangent line of $C(\alpha)$ at $\boldsymbol{a}_0$, let $\lambda_0 = \|\boldsymbol{a}_0 - \boldsymbol{b}_0\|$, and suppose $D_2\|\boldsymbol{f}_{\alpha,\beta}(\lambda, \boldsymbol{a}) - \boldsymbol{f}_{\alpha,\beta}(\mu, \boldsymbol{a})\| = 0$ for all λ and μ in an interval J with center λ_0 and all $\boldsymbol{a}$ in an arc F of $C(\alpha)$ surrounding $\boldsymbol{a}_0$.

Then $\|\boldsymbol{f}_{\alpha,\beta}(\lambda, \boldsymbol{a}) - \boldsymbol{f}_{\alpha,\beta}(\mu, \boldsymbol{a})\|$ is a function only of λ and μ for $\lambda, \mu \in J$, $\boldsymbol{a} \in F$. For fixed $\lambda, \mu \in J$, the triangles $\{\boldsymbol{a}, \boldsymbol{f}_{\alpha,\beta}(\lambda, \boldsymbol{a}), \boldsymbol{f}_{\alpha,\beta}(\mu, \boldsymbol{a})\}$ are then all congruent for $\boldsymbol{a} \in F$. Letting μ tend to λ, the angle between the tangent line to $C(\beta)$ at $\boldsymbol{f}_{\alpha,\beta}(\lambda,\boldsymbol{a})$ and the vector $\boldsymbol{f}_{\alpha,\beta}(\lambda,\boldsymbol{a})-\boldsymbol{a}$ is a function of λ only, say $\rho(\lambda)$, for $\lambda\in J$ and $\boldsymbol{a}\in F$. We can suppose F and J are so short that $\boldsymbol{f}_{\beta,\alpha}(\mu, \boldsymbol{f}_{\alpha,\beta}(\lambda, \boldsymbol{a}))$ is defined for $\lambda, \mu\in J$, $\boldsymbol{a}\in F$.

Consider $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ in the interior of F, let $\boldsymbol{b}_i = \boldsymbol{f}_{\alpha,\beta}(\lambda_0, \boldsymbol{a}_i)$ and let $\boldsymbol{g}_i(\lambda) = \boldsymbol{f}_{\beta,\alpha}(\lambda, \boldsymbol{b}_i)\in C(\alpha)$ for $i = 1, 2$. We can choose an open interval J' with $\lambda_0\in J'\subset J$ which is so short that $\boldsymbol{g}_i(\lambda)\in F$ for all $\lambda\in J'$, $i = 1, 2$. Then $\boldsymbol{f}_{\alpha,\beta}(\lambda, \boldsymbol{g}_i(\lambda)) = \boldsymbol{b}_i$; choose unit vectors $\boldsymbol{t}_i$ parallel to the tangent lines of $C(\beta)$ at $\boldsymbol{b}_i$ so that $(\boldsymbol{g}_i(\lambda) - \boldsymbol{b}_i)\cdot \boldsymbol{t}_i = \lambda\cos\rho(\lambda)$. There is an orthogonal transformation Ψ in $H(\boldsymbol{v})$ with $\Psi(\boldsymbol{b}_1) = \boldsymbol{b}_2$, $\Psi(\boldsymbol{t}_1) = \boldsymbol{t}_2$ and $\Psi(\boldsymbol{a}_1) = \boldsymbol{a}_2$. The continuously varying points $\boldsymbol{g}_i(\lambda)$ satisfy:

$$\begin{aligned}
&||\boldsymbol{g}_2(\lambda)|| = ||\Psi\boldsymbol{g}_1(\lambda)|| = \alpha\\
&||\boldsymbol{g}_2(\lambda) - \boldsymbol{b}_2|| = ||\Psi\boldsymbol{g}_1(\lambda) - \boldsymbol{b}_2|| = \lambda\\
&(\boldsymbol{g}_2(\lambda) - \boldsymbol{b}_2)\cdot \boldsymbol{t}_2 = (\Psi\boldsymbol{g}_1(\lambda) - \boldsymbol{b}_2)\cdot \boldsymbol{t}_2 = \lambda\cos\rho(\lambda)
\end{aligned}$$

and these conditions ensure $\Psi\boldsymbol{g}_1(\lambda) = \boldsymbol{g}_2(\lambda)$ for all $\lambda\in J'$. Thus Ψ maps $\boldsymbol{a}_1$ onto $\boldsymbol{a}_2$ and maps a neighborhood of $\boldsymbol{a}_1$ in $C(\alpha)$ onto a neighborhood of $\boldsymbol{a}_2$ in $C(\alpha)$. If F contains in its interior a point of 2-fold differentiability of $C(\alpha)$, then F has constant curvature, and since it lies on a sphere it must be an arc of a circle.

Since *relbd* $H(\boldsymbol{v})\cap K$ is twice differentiable almost everywhere, $C(\alpha)\cap M$ has a point of two-fold differentiability for a dense set of α. If $C(\alpha)\cap M$ is twice differentiable somewhere, the above arguments show it contains a circular arc; choose a maximal such arc C. Then the above arguments apply taking a_0 as an end of C, and this contradicts the maximality of C unless $C = C(\alpha)\cap M$. We conclude that $C(\alpha)\cap M$ is a circular arc for a dense set of α; by taking limits M is contoured contrary to hypothesis.

It follows that our supposition (*) is false. Thus for a dense of $(\boldsymbol{a}_0, \boldsymbol{b}_0)$ in $B\times B$, for $\alpha = ||\boldsymbol{a}_0||$ and $\beta = ||\boldsymbol{b}_0||$ we find that the tangent line of $C(\beta)$ at $\boldsymbol{b}_0$ is not perpendicular to $\boldsymbol{a}_0 - \boldsymbol{b}_0$ and $D_2||\boldsymbol{f}_{\alpha,\beta}(\lambda, \boldsymbol{a}) - \boldsymbol{f}_{\alpha,\beta}(\mu, \boldsymbol{a})|| \neq 0$ for $(\lambda, \mu, \boldsymbol{a})$ arbitrarily close to $(\lambda_0, \lambda_0, \boldsymbol{a}_0)$ in $\boldsymbol{R}\times\boldsymbol{R}\times C(\alpha)$ where $\lambda_0 = ||\boldsymbol{a}_0 - \boldsymbol{b}_0||$. We can therefore choose λ, μ, ν, $\boldsymbol{a}_0$, $\boldsymbol{b}_0$, $\boldsymbol{c}_0$ with $||\boldsymbol{a}_0|| = \alpha$, $||\boldsymbol{b}_0|| = \beta$, $\boldsymbol{b}_0 = \boldsymbol{f}_{\alpha,\beta}(\lambda, \boldsymbol{a}_0)$, $\boldsymbol{c}_0 = \boldsymbol{f}_{\alpha,\beta}(\mu, \boldsymbol{a}_0)$, $\nu = ||\boldsymbol{b}_0 - \boldsymbol{c}_0||$, such that the tangent lines of $C(\alpha)$ at $\boldsymbol{b}_0$ and $\boldsymbol{c}_0$ are not prependicular to $\boldsymbol{b}_0 - \boldsymbol{a}_0$ and $\boldsymbol{c}_0 - \boldsymbol{a}_0$ respectively, $D_2||\boldsymbol{f}_{\alpha,\beta}(\lambda, \boldsymbol{a}_0) - \boldsymbol{f}_{\alpha,\beta}(\mu, \boldsymbol{a}_0)|| \neq 0$, and by choosing λ, μ and ν small with $\boldsymbol{b}_0 - \boldsymbol{a}_0$ not too nearly parallel to the tangent line of $C(\beta)$ at $\boldsymbol{b}_0$ we can also ensure that $\{\boldsymbol{a}_0, \boldsymbol{b}_0, \boldsymbol{c}_0\}$ is linearly independent.

We can write $K = \{\boldsymbol{y}: h(\boldsymbol{y}) \leqq 1\}$ where h is a positive-homogeneous continuously differentiable convex function. Regarding points of E^4 as column matrices, for points $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{u} \neq \boldsymbol{o}$ define

$$F\begin{bmatrix} a \\ b \\ c \\ u \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\|a\|^2 \\ h(a) \\ \frac{1}{2}\|b\|^2 \\ h(b) \\ \frac{1}{2}\|c\|^2 \\ h(c) \\ \frac{1}{2}\|a-b\|^2 \\ \frac{1}{2}\|b-c\|^2 \\ \frac{1}{2}\|c-a\|^2 \\ u\cdot a \\ u\cdot b \\ u\cdot c \end{bmatrix} \text{ so that } DF\begin{bmatrix} a \\ b \\ c \\ u \end{bmatrix} = \left[\begin{array}{ccc|c} a^T & & & \\ \nabla h(a) & & & \\ & b^T & & \\ & \nabla h(b) & & \\ & & c^T & \\ & & \nabla h(c) & \\ a^T-b^T & b^T-a^T & & \\ & b^T-c^T & c^T-b^T & \\ a^T-c^T & & c^T-a^T & \\ \hline u^T & & & a^T \\ & u^T & & b^T \\ & & u^T & c^T \end{array}\right]$$

where ∇h is the gradient of h; notice that if y is a boundary point of K then $\nabla h(y)$ is a nonzero multiple of the unit normal to K at y. We will show that

$$\text{(1)} \qquad rank\, D_{abc}F\begin{bmatrix} a_0 \\ b_0 \\ c_0 \\ v \end{bmatrix} = 12\,.$$

To this end define $m(x)$ to be the orthogonal projection of $\nabla h(x)^T$ on $H(v)$, and let

$$Q' = \begin{bmatrix} a_0^T & & \\ m^T(a_0) & & \\ & b_0^T & \\ & m^T(b_0) & \\ & & c_0^T \\ & & m^T(c_0) \\ a_0^T - b_0^T & b_0^T - a_0^T & \\ & b_0^T - c_0^T & c_0^T - b_0^T \\ a_0^T - c_0^T & & c_0^T - a_0^T \end{bmatrix}.$$

We first prove $rank\, Q' = 9$.

Let $\boldsymbol{s}^*$, $\boldsymbol{t}^*$, $\boldsymbol{w}^*$ be unit vectors parallel to the tangent lines of $C(\alpha)$ at $\boldsymbol{a}_0$, of $C(\beta)$ at $\boldsymbol{b}_0$ and of $C(\beta)$ at $\boldsymbol{c}_0$ respectively.

Suppose that there are points $\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{w} \in H(\boldsymbol{v})$ such that

$$Q'\begin{bmatrix} \boldsymbol{s} \\ \boldsymbol{t} \\ \boldsymbol{w} \end{bmatrix} = \boldsymbol{o} .$$

Then $\boldsymbol{a}_0 \cdot \boldsymbol{s} = 0$ and $\boldsymbol{m}(\boldsymbol{a}_0) \cdot \boldsymbol{s} = 0$ which ensures that $\boldsymbol{s}$ is a multiple of $\boldsymbol{s}^*$. Similarly $\boldsymbol{t}$ and $\boldsymbol{w}$ are multiples of $\boldsymbol{t}^*$ and $\boldsymbol{w}^*$ respectively. By choice of $\boldsymbol{a}_0$, $\boldsymbol{b}_0$, and $\boldsymbol{c}_0$ we have

$$(\boldsymbol{a}_0 - \boldsymbol{b}_0) \cdot \boldsymbol{t}^* \neq 0 , \quad (\boldsymbol{a}_0 - \boldsymbol{c}_0) \cdot \boldsymbol{w}^* \neq 0$$

and this ensures that the equations

$$(2) \qquad (\boldsymbol{a}_0 - \boldsymbol{b}_0) \cdot (\sigma \boldsymbol{s}^* - \tau \boldsymbol{t}^*) = 0$$

$$(3) \qquad (\boldsymbol{a}_0 - \boldsymbol{c}_0) \cdot (\sigma \boldsymbol{s}^* - \omega \boldsymbol{w}^*) = 0$$

have a one-dimensional space of solutions (σ, τ, ω). We can choose numbers τ^* and ω^* such that

$$\begin{aligned} \tau^* \boldsymbol{t}^* &= D_2 \boldsymbol{f}_{\alpha,\beta}(\lambda, \boldsymbol{a}_0) \\ \omega^* \boldsymbol{w}^* &= D_2 \boldsymbol{f}_{\alpha,\beta}(\mu, \boldsymbol{a}_0) ; \end{aligned}$$

if we take $\sigma^* = 1$ then $(\sigma^*, \tau^*, \omega^*)$ is a solution of (2) and (3) since $\|\boldsymbol{f}_{\alpha,\beta}(\lambda, \boldsymbol{a}) - \boldsymbol{a}\| = \lambda$ and $\|\boldsymbol{f}_{\alpha,\beta}(\mu, \boldsymbol{a}) - \boldsymbol{a}\| = \mu$ for $\boldsymbol{a}$ on $C(\alpha)$ close to $\boldsymbol{a}_0$. Also if χ is the projection on the 8th coordinate of $\boldsymbol{R}^{12}$ we have

$$\chi Q'\begin{bmatrix} \sigma^* \boldsymbol{s}^* \\ \tau^* \boldsymbol{t}^* \\ \omega^* \boldsymbol{w}^* \end{bmatrix} = \frac{1}{2} D_2 \|\boldsymbol{f}_{\alpha,\beta}(\lambda, \boldsymbol{a}_0) - \boldsymbol{f}_{\alpha,\beta}(\mu, \boldsymbol{a}_0)\|^2 \neq 0 .$$

Thus

$$Q'\begin{bmatrix} \sigma \boldsymbol{s}^* \\ \tau \boldsymbol{t}^* \\ \omega \boldsymbol{w}^* \end{bmatrix} = \boldsymbol{o} \text{ implies } \sigma = \tau = \omega = 0 ,$$

which shows that $rank\, Q' = 9$.

Suppose $\boldsymbol{p}$, $\boldsymbol{q}$, and $\boldsymbol{r}$ are vectors in E^4 for which

$$(4) \qquad D_{abc} F \begin{bmatrix} \boldsymbol{a}_0 \\ \boldsymbol{b}_0 \\ \boldsymbol{c}_0 \\ \boldsymbol{v} \end{bmatrix} \begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{q} \\ \boldsymbol{r} \end{bmatrix} = \boldsymbol{o} .$$

By considering the last 3 components in (4) we find that $\boldsymbol{v} \cdot \boldsymbol{p} = \boldsymbol{v} \cdot \boldsymbol{q} = \boldsymbol{v} \cdot \boldsymbol{r} = 0$, so if coordinates are chosen such that $\boldsymbol{v}$ is on the x_4 axis, we have $\boldsymbol{p} = (\boldsymbol{p}', 0)$, $\boldsymbol{q} = (\boldsymbol{q}', 0)$, $\boldsymbol{r} = (\boldsymbol{r}', 0)$. Also the 4th, 8th and 12th columns of Q' are zero, (4) show that

$$Q'\begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{q} \\ \boldsymbol{r} \end{bmatrix} = \boldsymbol{o}$$

and since $rank\, Q' = 9$ it follows that $\boldsymbol{p}' = \boldsymbol{q}' = \boldsymbol{r}' = \boldsymbol{o}$. Hence $\boldsymbol{p} = \boldsymbol{q} = \boldsymbol{r} = \boldsymbol{o}$ which proves (1). Now it follows from the Implicit Function theorem that in a certain neighborhood of $(\boldsymbol{a}_0, \boldsymbol{b}_0, \boldsymbol{c}_0)$, for each $\boldsymbol{u}$ close to $\boldsymbol{v}$ the equation

$$F\begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \\ \boldsymbol{c} \\ \boldsymbol{u} \end{bmatrix} = F\begin{bmatrix} \boldsymbol{a}_0 \\ \boldsymbol{b}_0 \\ \boldsymbol{c}_0 \\ \boldsymbol{v} \end{bmatrix}$$

has a unique solution $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$, and $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ are differentiable functions of $\boldsymbol{u}$. Roughly, we can say that no tetrahedron close to $(\boldsymbol{o}, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ with $\boldsymbol{o}$ as a vertex and 3 vertices on $H(\boldsymbol{u}) \cap bdK$ is congruent to $(\boldsymbol{o}, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$. It follows that $\Phi_{u,v}(\boldsymbol{a}) = \boldsymbol{a}_0$, $\Phi_{u,v}(\boldsymbol{b}) = \boldsymbol{b}_0$ and $\Phi_{u,v}(\boldsymbol{c}) = \boldsymbol{c}_0$. Thus $\Phi_{u,v}$ is a differentiable function of $\boldsymbol{u}$ near $\boldsymbol{v}$.

LEMMA 3.4. *Some 3-dimensional central section of K has a contoured neighborhood on its relative boundary.*

Proof. Suppose the lemma is false. Since K is assumed not to be a euclidean ball, there is a point $\boldsymbol{x}$ on the boundary of K at which the unit outward normal vector $\boldsymbol{n}$ is not parallel to $\boldsymbol{x}$. Let $\boldsymbol{v}$ be the unit vector perpendicular to $\boldsymbol{x}$ and which is coplanar with $\boldsymbol{n}$ and $\boldsymbol{x}$ having $\boldsymbol{n} \cdot \boldsymbol{v} > 0$. Then $\Phi_{u,v}$ is a differentiable function of $\boldsymbol{u}$ by Lemma 3.3 for $\boldsymbol{u}$ close to $\boldsymbol{v}$. For real θ let $\boldsymbol{u} = \boldsymbol{u}(\theta) = -\theta \boldsymbol{x} + \boldsymbol{v}$, let $\boldsymbol{y} = \boldsymbol{y}(\theta) = \Phi_{u,v}^{-1}(\boldsymbol{x})$ and let $\boldsymbol{f} = \boldsymbol{y}'(0)$. We have $\boldsymbol{y}(0) = \boldsymbol{x}$ and $\boldsymbol{y}(\theta) \cdot \boldsymbol{u}(\theta) = 0$. Since $||\boldsymbol{y}(\theta)||$ is constant we have $\boldsymbol{y} \cdot \boldsymbol{y}' = 0$ so $\boldsymbol{x} \cdot \boldsymbol{f} = 0$. Thus

$$(\boldsymbol{x} + \boldsymbol{f}\theta + o(\theta)) \cdot (-\theta \boldsymbol{x} + \boldsymbol{v}) = 0$$

whence

$$-\theta \boldsymbol{x} \cdot \boldsymbol{x} + \boldsymbol{x} \cdot \boldsymbol{v} - \theta^2 \boldsymbol{f} \cdot \boldsymbol{x} + \theta \boldsymbol{f} \cdot \boldsymbol{v} = o(\theta)$$

so that

$$-\boldsymbol{x}\cdot\boldsymbol{x} + \boldsymbol{f}\cdot\boldsymbol{v} = o(1)$$

as $\theta \to 0$. It follows that $\boldsymbol{f}\cdot\boldsymbol{v} = \boldsymbol{x}\cdot\boldsymbol{x} > 0$.

We can write $\boldsymbol{n} = \alpha\boldsymbol{x} + \beta\boldsymbol{v}$ where $\beta = \boldsymbol{n}\cdot\boldsymbol{v} > 0$, and then

$$\begin{aligned}\boldsymbol{n}\cdot\boldsymbol{y} - \boldsymbol{n}\cdot\boldsymbol{x} = \boldsymbol{n}\cdot(\boldsymbol{y}-\boldsymbol{x}) &= (\alpha\boldsymbol{x} + \beta\boldsymbol{v})\cdot(\theta\boldsymbol{f} + o(\theta))\\ &= \theta\alpha\boldsymbol{x}\cdot\boldsymbol{f} + \theta\beta\boldsymbol{v}\cdot\boldsymbol{f} + o(\theta)\\ &= \theta\beta\boldsymbol{v}\cdot\boldsymbol{f} + o(\theta)\end{aligned}$$

which is positive for small positive θ. This is impossible since $\boldsymbol{n}\cdot\boldsymbol{x} \geqq \boldsymbol{n}\cdot z$ for all $z \in K$. We conclude that some $H(\boldsymbol{w}) \cap K$ has a contoured neighborhood on its relative boundary.

LEMMA 3.5. *No 3-dimensional central section of K has a contoured neighborhood on its relative boundary. Our assumption that K is not a euclidean ball is therefore untenable.*

Proof. Suppose $\boldsymbol{v}$ is a unit vector and that *relbd* $H(\boldsymbol{v}) \cap K$ contains a contoured neighborhood A. Define $C(\alpha) = \{\boldsymbol{x} \in A \colon \|\boldsymbol{x}\| = \alpha\}$. First consider the possibility that all of the circular arcs $C(\alpha)$ are parallel to a certain plane Λ through $\boldsymbol{o}$ in $H(\boldsymbol{v})$. Let Θ be a plane through $\boldsymbol{o}$ in $H(\boldsymbol{v})$ which intersects A and which makes a positive angle γ with Λ. Then $\Theta \cap K$ is not circular, for then A would contain a spherical region which is impossible since A is contoured. The symmetry group of $\Theta \cap K$ is therefore finite.

Suppose that $\Phi_{u,v}(\Theta) = \Theta$ for every $\boldsymbol{u} \in \Theta^\perp \cap S^3$; then $\Phi_{u,v|\Theta}$ would be a continuously varying symmetry of $\Theta \cap K$, and since $\Phi_{v,v}$ is the identity we find $\Phi_{u,v|\Theta}$ is the identity for all $\boldsymbol{u} \in \Theta^\perp \cap S^3$. It follows that every section of K parallel to $\Theta^\perp$ is circular with center on Θ. Hence some 3-dimensional central sections of K are bodies of revolution, contrary to Lemma 3.1.

Therefore there exists some $\boldsymbol{u}$ such that $\Phi_{u,v}(\Theta) \neq \Theta$. Choose distinct numbers α and β such that $C(\alpha)$ and $C(\beta)$ both intersect Θ. There is arc Γ of $\Theta^\perp \cap S^3$ which has $\boldsymbol{v}$ as one end, such that $\Phi_{u,v}(\Theta)$ intersects $C(\alpha)$ and $C(\beta)$ for every $\boldsymbol{u} \in \Gamma$ but $\Phi_{u,v}(\Theta) \neq \Theta$ for some $\boldsymbol{u} \in \Gamma$. For all $\boldsymbol{u} \in \Gamma$ we have $\Phi_{u,v}(C(\alpha) \cap \Theta) = C(\alpha) \cap \Phi_{u,v}(\Theta)$ and $\Phi_{u,v}(C(\beta) \cap \Theta) = C(\beta) \cap \Phi_{u,v}(\Theta)$, so $\Phi_{u,v}(\Theta)$ makes an angle γ with Λ. Hence for every $\boldsymbol{x}$ in $\Theta \cap bdK$, the arc $\{\Phi_{u,v}(\boldsymbol{x}) \colon \boldsymbol{u} \in \Gamma\}$ is a compact circular arc in $H(\boldsymbol{v}) \cap bdK$, is parallel to Λ and has its center on the the line l in $H(\boldsymbol{v})$ through $\boldsymbol{o}$ perpendicular to Λ. By taking various values of γ, it follows that for any plane Λ' in $H(\boldsymbol{v})$ parallel to Λ but distinct from Λ, the closed curve $\Lambda' \cap bdK$ is a union of compact circular arcs centerd on l. We can express $\Lambda' \cap bdK$ as the union of a countable collection $\mathscr{F}$ of interior-disjoint maximal compact circular arcs with centers on l. The end-points of the arcs in $\mathscr{F}$ form a compact countable set $\mathscr{E}$. If $\mathscr{E}$ is nonempty, it follows from the Baire Category theorem that some point of $\mathscr{E}$ is isolated; such an isolated point is a common end-point of two members of $\mathscr{F}$, which cannot exist. We conclude that $\mathscr{E}$ is empty so that $\Lambda' \cap bdK$ is a circle with its center on l. It follows that $H(v) \cap K$ is a body of revolution contrary to Lemma 3.1.

We may therefore assume that not all of the arcs $C(\alpha)$ are parallel to one plane. We can then chose distinct numbers α and β and a plane Λ through $\boldsymbol{o}$ in $H(\boldsymbol{v})$ such that Λ intersects each of $C(\alpha)$ and $C(\beta)$ in two points, and $C(\alpha)$ is not in a plane parallel to the plane of $C(\beta)$. For no plane Λ' through $\boldsymbol{o}$ in $H(v)$ close to Λ are the configurations $(\boldsymbol{o}, \Lambda \cap C(\alpha), \Lambda \cap C(\beta))$ and $(\boldsymbol{o}, \Lambda' \cap C(\alpha), \Lambda' \cap C(\beta))$ congruent, so it follows that $\Phi_{u,v}(\Lambda) = \Lambda$ for all $\boldsymbol{u} \in \Lambda^{\perp} \cap S^3$. Further, $\Lambda \cap K$ is not circular so $\Phi_{u,v|\Lambda}$ is the identity for all $\boldsymbol{u} \in \Lambda^{\perp} \cap S^3$. It follows as in the case considered above that K has 3-dimensional central sections which are bodies of revolution contrary to Lemma 3.1.

Lemma 3.5 contradicts Lemma 3.4, so we conclude that K is a euclidean ball.

We have now proved:

PROPOSITION. *If K is a centrally symmetric 4-dimensional convex body and all the 3-dimensional central sections of K are congruent, then K is a euclidean ball.*

4. Proof of the theorems.

Proof of Theorem 1. Let d denote the dimension of K, and consider first the case when $d = 4$. For $\boldsymbol{u} \in S^3$ let $A(\boldsymbol{u})$ be the section of K through $\boldsymbol{p}$ which is perpendicular to the direction $\boldsymbol{u}$. Then $A(\boldsymbol{u})$ is a complete turning of some 3-dimensional body A in E^4, so by Lemmas 2.1 and 2.2, A is centrally symmetric. Hence $A(\boldsymbol{u})$ is centrally symmetric for each $\boldsymbol{u} \in S^3$. Consider an orthogonal projection K_0 of K on a 3-flat through $\boldsymbol{p}$. Then every 2-dimensional section of K_0 through $\boldsymbol{p}$ is a projection of a 3-dimensional section of K through $\boldsymbol{p}$. Thus all 2-dimensional sections of K_0 through $\boldsymbol{p}$ are centrally symmetric, and it follows from a result of Rogers [6] that K_0 is centrally symmetric. Every 2-dimensional orthogonal projection of K is a projection of some 3-dimensional projection, and so is centrally symmetric. It follows from another result of Rogers [6] that K is centrally symmetric.

If $\boldsymbol{p}$ is the center of K, it follows immediately from the Proposition above that K is a euclidean ball with center $\boldsymbol{p}$. Suppose therefore that the center of K is $\boldsymbol{a} \neq \boldsymbol{p}$, and consider a 3-dimensional orthogonal projection π with $\pi(\boldsymbol{a}) \neq \pi(\boldsymbol{p})$. As we have seen above, every 2-dimensional section of $\pi(K)$ through $\pi(\boldsymbol{p})$ is centrally symmetric, but $\pi(\boldsymbol{a})$ is the center of $\pi(K)$. It follows from the False Center theorem of Aitchison, Petty and Rogers [1] that $\pi(K)$ is an ellipsoid. Since $\pi(\boldsymbol{a}) \neq \pi(\boldsymbol{p})$ for almost all projections π, by taking limits we find that every 3-dimensional projection of K is an ellipsoid, so K is an ellipsoid by the dual of a result of Busemann [2, p. 91]. The 3-dimensional central sections of K are all similar, and it is easily shown that K must therefore be a euclidean ball. Since the 3-dimensional sections of K through $\boldsymbol{p}$ are all congruent, $\boldsymbol{p}$ must be the center of K.

In the case $d > 4$, it follows from the 4-dimensional case considered above that every 4-dimensional section of K through $\boldsymbol{p}$ is a euclidean ball with center $\boldsymbol{p}$, so K is a euclidean ball with center $\boldsymbol{p}$.

Proof of Theorem 2. We may assume that the centroid of K is $\boldsymbol{o}$. Consider an orthogonal projection K_0 of K on a 4-flat through $\boldsymbol{o}$. The 3-dimensional orthogonal projections of K_0 are all orthogonal projections of K and are therefore congruent. So the 3-dimensional orthogonal projections of K_0 give rise to a complete turning of some 3-dimensional convex body in 4 dimensions, and by Lemmas 2.1 and 2.2 they are all centrally symmetric. Hence K_0 is centrally symmetric. It follows that K is centrally symmetric with center $\boldsymbol{o}$, using a result of Rogers. Let K^* be the polar reciprocal of K about $\boldsymbol{o}$. Then all the central 3-dimensional sections of K^* are congruent so by Theorem 1, K^* is a euclidean ball with center $\boldsymbol{o}$. Hence K is a euclidean ball.

Proof of Theorem 3. First consider the case when the dimension of K is $2n+1$. For each unit vector $\boldsymbol{u}$ let $K(\boldsymbol{u})$ be the $2n$-dimensional section of K through $\boldsymbol{p}$ perpendicular to $\boldsymbol{u}$, and let $F(\boldsymbol{u})$ be the $2n$-dimensional ellipsoid of least volume containing $K(\boldsymbol{u})$; the uniqueness of $F(\boldsymbol{u})$ was proved by Danzer, Laugwitz, and Lenz [3]. The affine transformation Φ_u which maps $F(\boldsymbol{u})$ onto a $2n$-dimensional euclidean unit ball $B(\boldsymbol{u})$ in the hyperplane of $F(\boldsymbol{u})$ by dilating its principal axes is a continuous function of $\boldsymbol{u}$. Then all $\Phi_u K(\boldsymbol{u})$ for $\boldsymbol{u} \in S^3$ are congruent, so $\Phi_u K(\boldsymbol{u})$ is a field of congruent $2n$-dimensional bodies in E^{2n+1}. A result of Mani [5] shows that each $\Phi_u K(\boldsymbol{u})$ is a euclidean ball, so $K(\boldsymbol{u})$ is an ellipsoid. It follows from a theorem of Busemann [2, p. 91] that K is an ellipsoid.

Now suppose the dimension of K is greater than $2n+1$. From the case already considered it follows that every $(2n+1)$-dimensional section of K through $\boldsymbol{p}$ is an ellipsoid, and Busemann's result then shows that K is an ellipsoid.

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