PACIFIC JOURNAL OF MATHEMATICS
Vol. 82, No. 1, 1979

RIGHT SELF-INJECTIVE RINGS WHOSE ESSENTIAL RIGHT IDEALS ARE TWO-SIDED

KENNETH A. BYRD

A ring R of the kind described by the title is called a right q-ring and is characterized by the property that each of its right ideals is quasi-injective as a right R-module. The principal results of this paper are Theorem 6, which describes how an arbitrary right q-ring is constructed from division rings, local rings, and right q-rings with no primitive idempotent, and Theorem 5 which shows that a right q-ring cannot have an infinite set of orthogonal noncentral idempotents.

Ivanov described the structure of indecomposable, nonlocal right q-rings and conjectured that every right q-ring must be a direct sum of such rings together with a ring all of whose idempotents are central. Our results imply that though the structure of right q-rings is slightly more complicated than this (there are chain q-rings), one can still reduce the study of q-rings to ones which have only central idempotents. More precisely, the study of right q-rings is reduced to the study of right self-injective duo rings which are either local or have no primitive idempotent.

The work done here is an extension and generalization of Ivanov's investigations. We develop the finiteness conditions inherent in that work without the assumption of indecomposability and the structure of an arbitrary right q-ring is developed at the same time. Throughout the paper all rings have identity $1 \neq 0$ and all modules are unital.

Preliminaries. If one has a decomposition $A = A_1 \oplus A_2 \oplus \cdots \oplus A_n$ of a right R-module A as a finite direct sum of submodules then one has a representation of $\mathrm{End}_R A$, the ring of R-endomorphisms of A, as a ring of $n \times n$ "matrices" of the form (α_{ij}) where α_{ij} belongs to $\mathrm{Hom}_R(A_j, A_i)$. In particular, when one has a finite decomposition of the module R_R one also has a representation of the ring $R \cong \mathrm{End}_R R$ as a ring of matrices. A decomposition of $R_R = A \oplus B$ as a direct sum of two modules A and B which are unrelated in the sense that $\mathrm{Hom}_R(A, B)$ and $\mathrm{Hom}_R(B, A)$ are both the trivial group yields a representation of R as the product of the rings $\mathrm{End}_R A$ and $\mathrm{End}_R B$. For a direct sum decomposition of R_R, such unrelated summands may be achieved by summing over classes of related summands. When a module M is a direct sum of simple modules then a sum over a class of related summands is called an isotypic component of M, since two simple modules are related if and only if they are isomorphic.

If A is a right R-module then $E(A)$ denotes the injective hull of A. When R is right self-injective we will assume that $E(A)$ is a right ideal of R whenever A is a right ideal of R. The fact that the rings described by the title are the rings whose right ideals are quasi-injective is a consequence of the fact [6, 1.1 Theorem] that A is quasi-injective if and only if A is fully invariant in $E(A)$, that is $\mathrm{End}_R E(A) \cdot A \subseteqq A$.

Reduction to basic rings. A q-ring R will be called *basic* if each of the nonzero isotypic components of the socle of R_R is simple, i.e., R has no two distinct isomorphic minimal right ideals. We shall show that a right q-ring is the ring direct sum of a semisimple ring and a basic ring.

The following lemma of [3] is fundamental to our study.

LEMMA 1. *Let R be a right q-ring and A and B be independent right ideals of R. If f belongs to* $\mathrm{Hom}_R(A, B)$ *then $f(A)$ is semisimple.*

Proof. Recall that the socle of B is the intersection of the essential submodules of B. Let B_1 be an arbitrary essential submodule of B. It follows that $A \oplus B_1$ is essential in $A \oplus B$. Since R is a right q-ring, it follows that $A \oplus B_1$ is fully invariant in $A \oplus B$. Letting g be the endomorphism of $A \oplus B$ defined by $g(a + b) = f(a)$ for a in A and b in B, we see that $f(A) \subseteqq B_1$.

COROLLARY. *If A and B are independent isomorphic right ideals of R then each is injective and semisimple. An isotypic component of the socle of R which is not simple is injective.*

Proof. Assume that A and B are independent isomorphic right ideals. Since $E(A)$ and $E(B)$ are also independent and isomorphic then the above lemma implies that each is semisimple. It follows that $A = E(A)$ and $B = E(B)$.

Let H be an isotypic component which is not simple. If H is the direct sum of an infinite set of simple modules then $H = H_1 \oplus H_2$ where $H_1 \simeq H$ and $H_2 \simeq H$. Since H_1 and H_2 are injective then so is H. It follows that H must be a finite direct sum of at least two copies of a minimal right ideal S of R. Then S is injective and so is H.

PROPOSITION 1. *Let Γ be an independent set of right ideals of a right q-ring R. Suppose that for each member A of Γ there is a minimal right ideal $S(A)$ of R so that*

(1) *if $A \neq B$ then $S(A) \not\cong S(B)$,*

(2) *for each A, $\mathrm{Hom}_R(A, S(A)) \neq 0$,*

(3) $\Sigma\{A \mid A \in \Gamma\} \cap \Sigma\{S(A) \mid A \in \Gamma\} = 0$.

Then Γ is finite.

Proof. According to (2) there is for each A in Γ an epimorphism $\alpha_A: A \to S(A)$ and this induces on the direct sum, the epimorphism $\alpha: \Sigma_\Gamma A \to \Sigma_\Gamma S(A)$. Choose hulls in R and extend α to the mapping $\beta: E(\Sigma_\Gamma A) \to E(\Sigma_\Gamma S(A))$. From (3) and Lemma 1 we know that the image of β is $\Sigma_\Gamma S(A)$. On the other hand the image of β must be cyclic since $E(\Sigma A)$ is a direct summand of R. It follows that there are only finitely many nonisomorphic $S(A)$ for A in Γ, so (1) implies that Γ is finite.

THEOREM 1. *A right q-ring is isomorphic to the direct product of a semisimple ring and a basic right q-ring.*

Proof. The above proposition implies that Soc R has only a finite set $\{A_1, \cdots, A_k\}$ of isotypic components which are not simple. Since each of the A_i is injective we have a decomposition $R_R = (\Sigma A_i) \oplus B$. It follows easily that R is isomorphic to the product of the semisimple ring $\mathrm{End}_R(\Sigma A_i)$ and the ring $\mathrm{End}_R B$. If $B = eR$ where $e^2 = e$ then $\mathrm{End}_R B \simeq eRe$. Since $eR(1-e) \simeq \mathrm{Hom}_R(\Sigma A_i, B) = 0$ then $eRe = eR$ so that right ideals of eRe are the same as R-submodules of B. Then since B has no distinct pair of isomorphic simple submodules, it follows that eRe is basic.

DEFINITION. If A and B are right ideals of R then the notation $A \to B$ will indicate that $A \cap B = 0$ and $\mathrm{Hom}_R(A, B) \neq 0$. We shall write $A \to$ if $A \to B$ for some B, and we shall write $\to B$ if, for some A, $A \to B$.

The following finiteness condition is due to Ivanov [3, Lemma 3].

THEOREM 2. *Let R be a right q-ring. If Γ is an independent set of right ideals of R so that $A \to$ for each A in Γ then Γ is finite.*

Proof. By Theorem 1 we may assume that R is basic. By Lemma 1 we can find for each A in Γ an epimorphism α_A from A onto a minimal right ideal $S(A)$ such that $A \cap S(A) = 0$. Also, by taking injective hulls, we may assume that each A in Γ is a direct summand of R.

Suppose that $S(A) = S(B)$ for some B, B in Γ where $A \neq B$. From the projectivity of A there is a mapping $\beta: A \to B$ so that $\alpha_B \beta = \alpha_A$. It follows from Lemma 1 that Im β contains a copy of $S(B)$, so that $S(B) \subseteqq B$ since R is basic. This contradiction implies that if $A, B \in \Gamma$ and $A \neq B$ then $S(A) \not\cong S(B)$.

Let Γ_1 be the set of all A_1 in Γ so that $S(A_1) \subseteqq \Sigma\{A \mid A \in \Gamma\}$. Since R is basic and the sum is direct then for each member A_1 of Γ_1 there is a unique member $\gamma(A_1)$ of Γ so that $S(A_1) \subseteqq \gamma(A_1)$. We use the mapping $\gamma: \Gamma_1 \to \Gamma$ to form the partition $\{\gamma^{-1}(A) \mid A \in \operatorname{Im} \gamma\}$ of Γ_1. Since $A \notin \gamma^{-1}(A)$ for each A in Im γ, it follows from Proposition 1 that each member of this partition is a finite set.

Assume that Γ_1 is infinite and let ϕ be a function which chooses a member from each nonempty subset of Γ_1. If X is a finite subset of Γ_1 then $X \cup \gamma(X) \cup \gamma^{-1}(X)$ is also finite where $\gamma^{-1}(X) = \cup \{\gamma^{-1}(B) \mid B \in X\}$ and $\gamma^{-1}(B) = \varnothing$ if $B \notin \operatorname{Im} \gamma$. Denote by X' the set complement of $X \cup \gamma(X) \cup \gamma^{-1}(X)$ in Γ_1. We note $X' \neq \varnothing$ for all finite subsets X of Γ_1. Define the sequence $\{A_i\}_{i=1}^{\infty}$ in Γ_1 by setting $A_1 = \phi(\Gamma_1)$ and if $A_1, \cdots, A_n$ are already chosen then $A_{n+1} = \phi(\{A_1, \cdots, A_n\}')$. Suppose that $(\Sigma A_i) \cap (\Sigma S(A_i)) \neq 0$. Since R is basic this means that for some j, k one has $\gamma(A_j) = A_k$ and this cannot happen by the construction of the sequence. The existence of such a sequence contradicts Proposition 1 so we conclude that Γ_1 is finite.

Since $\Gamma - \Gamma_1$ is clearly finite by Proposition 1 then Γ is finite.

Injective hulls of minimal right ideals. Let $\mathscr{S}$ be the set of minimal right ideals of a basic right q-ring R and let $E(\mathscr{S}) = \{E(S) \mid S \in \mathscr{S}\}$ be a chosen set of injective hulls in R for the members of $\mathscr{S}$. For each S in $\mathscr{S}$ there is a primitive idempotent e_S of R such that $e_S R = E(S)$. According to Lemma 1 if e is a primitive idempotent of R and $\to eR$ then eR is isomorphic to a member of $E(\mathscr{S})$. In fact if eR is not isomorphic to a member of $E(\mathscr{S})$ then e is central as the next proposition shows.

PROPOSITION 2. *Let e be a primitive idempotent of a basic right q-ring R. If $eR\to$, then $\to eR$.*

Proof. Suppose the proposition is false so that $(1 - e)\,\mathrm{Re} \simeq \operatorname{Hom}_R(eR, (1 - e)R) \neq 0$ but $eR(1 - e) = 0$. Since e is primitive $eR = eRe$ is a local ring and since $eR(1 - e) = 0$ then the right ideals of eRe are precisely the R-submodules of eR. If J is the Jacobson radical of R then eJ is the unique maximal right ideal of eRe. If $eJ = 0$ then eR is simple and since R is basic it follows that

$(1-e)Re = 0$ contrary to the assumptions. So $eJ \neq 0$ and it follows that eJ contains a nonzero cyclic submodule L for which there is a eRe-epimorphism $\beta\colon L \to eR/eJ$ which is also an R-epimorphism. The assumption that $(1-e)Re \neq 0$ together with Lemma 1 implies that the simple image eR/eJ of eR embeds in $(1-e)R$. Since $(1-e)R$ is injective there is an R-homomorphism $\alpha\colon eR \to (1-e)R$ so that $\alpha|_L = \beta$. Since Im α is semisimple it follows that $\alpha(eJ) = 0$ so that $\beta = 0$ which is a contradiction.

PROPOSITION 3. *If e is a primitive idempotent of a basic right q-ring R and $\to eR$, then* (1) $S = eR(1-e)$ *is a minimal right ideal of R,* (2) $eRe \simeq \mathrm{End}_R S$, *and* (3) *$S$ is the only proper nonzero submodule of eR.*

Proof. (1) Since $\to eR$ then $eR(1-e)$ is nonzero and it is contained in the socle of eR. Since e is primitive it follows that $eR = E(S)$ for some minimal right ideal S containing $eR(1-e)$. If $\mathrm{Hom}_R(eR, S) \neq 0$ then there is a copy of S in $(1-e)R$ contradicting the fact that R is basic. It follows that $se = 0$ for every $s \in S$, that is $S \subseteq eR(1-e)$. Thus $S = eR(1-e)$.

(2) If J is the Jacobson radical of R then eRe has radical $eJe = \{x \in eRe \mid xS = 0\}$. Since $eR = S \oplus eRe$ as abelian groups one has

$$(eJe)R = (eJe)(eR) \subseteq (eJe)S + eJe = eJe$$

so that eJe is a right R-submodule of eR. Since $S \cap eJe = 0$ then $eJe = 0$ so that eRe is a division ring. Restriction to S is an isomorphism from $\mathrm{End}_R eR$ onto $\mathrm{End}_R S$.

(3) If K is a nonzero submodule of eR then $S \subseteq K$ and $K = Ke \oplus K(1-e)$. It follows that $S = K(1-e)$. Since Ke is a right ideal of eRe then either $Ke = 0$ or $Ke = eRe$. Thus $K = S$ or $K = eR$.

Let $\mathscr{A}(R) = \{E(S) \in E(\mathscr{S}) \mid \to E(S)\}$. We consider the restriction of the $\to$-relation to $\mathscr{A}(R)$. Note that $E(S_1) \to E(S_2)$ for $E(S_1)$, $E(S_2)$ members of $\mathscr{A}$ means that the top, $E(S_1)/S_1$, of $E(S_1)$ is isomorphic to the bottom, S_2, of $E(S_2)$.

Let D be the domain and T be the range of the restriction of $\to$ to the set $\mathscr{A}$. It is easy to show that $\to$ is a one-to-one function from D onto T. Define $a\colon \mathscr{A} \to \mathscr{A}$ by $a(E_1) = E_2$ if $E_1 \in D$ and $E_1 \to E_2$ and $a(E_1) = E_1$ if $E_1 \notin D$. Similarly $a^{-1}\colon \mathscr{A} \to \mathscr{A}$ is defined by $a^{-1}(E_2) = E_1$ if $E_2 \in R$ and $E_1 \to E_2$ and $a^{-1}(E_2) = E_2$ if $E_2 \notin T$. Then for each $E \in \mathscr{A}$ let $\vec{E} = \{a^k(E) \mid k \in \mathbf{Z}\}$. It is easy to see that (1) $E \in \vec{E}$ since a^0 is the identity mapping, and (2) if $F \in \vec{E}$ then $\vec{F} =$

$\vec{E}$ so that the set $\vec{\mathscr{A}}$ of $\rightarrow$—classes $\vec{E}$ for $E \in \mathscr{A}$ is a partition of $\mathscr{A}$. In fact the associated equivalence relation on $\mathscr{A}$ is just the smallest equivalence relation on $\mathscr{A}$ which contains the restriction of $\rightarrow$ to $\mathscr{A}$.

It is immediate from Theorem 2 that the set of classes $\vec{E}$ with more than one member is a finite set and also that each class $\vec{E}$ is itself finite. It is straightforward to show that these classes $\vec{E}$ are of two kinds namely;

(1) Chain: $\rightarrow E_1 \rightarrow E_2 \rightarrow \cdots E_l$ where $E_1 \notin T$ and $E_l \notin D$.

(2) Loop: $E_1 \rightarrow E_2 \rightarrow \cdots \rightarrow E_l \rightarrow E_1$.

In each case the cardinality l of $\vec{E}$ will be called the *length* of $\vec{E}$.

LEMMA 2. *Suppose that e is a primitive idempotent of a basic right q-ring R and $\rightarrow eR$ so that $S = eR(1-e) \neq 0$.*

(1) *The right annihilator $S^r = \{x \in R \mid Sx = 0\}$ is a maximal right ideal.*

(2) *If f is also a primitive idempotent of R and $eR \rightarrow fR$ then $eRe \simeq fRf$.*

Proof. (1) If s is a nonzero element of S then $s^r = M$ is a maximal right ideal. The right ideal M must be essential since otherwise $S = eR$ and since R is basic this contradicts the assumption $eR(1-e) \neq 0$. It follows that M is a two-sided ideal of R. For any nonzero element s_1 of S one has $s_1 = sr$ for some r in R so $s_1M = srM \subseteq sM = 0$. Thus $M = S^r$.

(2) Let T be the simple submodule of fR. Since $T = fRe$ is a simple right R-module, it is a 1-dimensional eRe-space on the right. Jacobson's density theorem [5, p. 28] and (1) imply that T is also a 1-dimensional fRf-space on the left. Choose a nonzero element t of T. The correspondence $a \leftrightarrow b$ if and only if $at = tb$ is an isomorphism between fRf and eRe.

The $\rightarrow$-classes $\vec{E}$ of R are determined "up to isomorphism" by our choice of a representative set of injective hulls of minimal right ideals of R. However, the sum of an $\rightarrow$-class is independent of this choice. This is a consequence of the following proposition.

PROPOSITION 4. *Let e be an idempotent of the basic right q-ring R. There is a one-to-one correspondence between the set $(1-e)Re$ and the set of copies of eR in R such that to the element z of $(1-e)Re$ corresponds the module $(1+z)eR$.*

Proof. If z belongs to $(1-e)Re$ then $f = (1+z)e$ is idempotent and since $ef = e$ and $fe = f$ it follows that $fR \simeq eR$. If $(1 + z_1)eR =$

$(1 + z_2)eR$ for z_1, z_2 in $(1 - e)Re$ then for some r in R, $(1 + z_1)e = (1 + z_2)er$. It follows that $e = er$ and $z_1 = z_1e = z_2e = z_2$. Thus the correspondence is one-to-one.

Let E be a copy of eR in R. Since $(1 - e)R$ contains no non-zero copy of a submodule of eR then the kernel, $E \cap (1 - e)R$, of the projection $x \mapsto ex$ of E into eR is zero. It follows that $eR = eE \oplus A$ for some submodule A of eR. But A must be zero since there is a copy of A in eE. Thus the projection of E into eR is an isomorphism onto eR. Choose a in E so that $ea = e$ and let $z = (1 - e)ae$. If $x \in E$ then $e(aex - x) = 0$ and it follows that $aex = x$ for all x in E. Then for x in E one has

$$x = ex + (1 - e)x = ex + (1 - e)aex = ex + zx = (1 + z)ex .$$

Thus one has $E = (1 + z)eR$.

In particular if $eR \to fR$ with eR and fR members of $\mathscr{A}$ then every copy of eR in R is contained in $eR \oplus fR$ because $(1 - e)Re = fRe$. Thus the sum of an $\to$-class is independent of the choice of the injective hulls.

DEFINITION. A basic right q-ring R is called a *loop q-ring* if R has only one $\to$-class, that class is a loop, and R is the sum of its loop.

NOTATION. Let D be a division ring. We denote by D_0 the $D - D$ bimodule D equipped with the zero multiplication.

THEOREM 3. *If R is a loop q-ring of length l then there is a division ring D so that R is isomorphic to the ring $H(l, D)$ of $l \times l$ matrices with elements on the diagonal from D and elements in the positions $(2, 1), (3, 2), \cdots, (l, l - 1), (1, l)$ from D_0 and zero entries elsewhere. Conversely every ring $H(l, D)$ is a loop q-ring.*

Proof. The first statement is an immediate consequence of the matrix representation of $R = \mathrm{End}_R R$ where $R_R = \sum_{i=1}^{l} E_i$ and $E_1 \to E_2 \to \cdots \to E_l \to E_1$. One may take $D = \mathrm{End}_R E_1$ and use Lemma 2 (2). The converse is proved in [3, Theorem 3].

The following theorem may be proved by a straightforward induction on the number of loops of R.

THEOREM 4. *Let R be a basic right q-ring. There is a set $\{l_1, l_2, \cdots, l_k\}$ of integers $\geqq 2$ and a set of division rings $\{D_1,$*

$D_2, \cdots, D_k\}$ so that

$$R \cong \prod_{i=1}^{k} H(l_i, D_i) \times R_1$$

where R_1 is a basic right q-ring which has no loops.

Chain q-rings. Assume that R is a basic right q-ring with no loops. Suppose that $\mathscr{C} = \{\vec{E}_i | 1 \leqq i \leqq m\}$ is a finite set of chains of R where $\vec{E}_i$ is $\to E_{i1} \to E_{i2} \to \cdots \to E_{il_i}$ with $E_{ij} = e_{ij}R$ for a primitive idempotent e_{ij} of R. Let $f = 1 - \Sigma e_{ij}$. Then for each i one has $fR \to E_{ij}$ exactly when $j = 1$. Also since $fR(1 - f) = 0$ then $fR = fRf$ is a ring with identity f.

PROPOSITION 5. *With the notation above, the ring fR is a basic right q-ring. The set of arrow classes of R is the disjoint union of the set of arrow classes of fR with the set $\mathscr{C}$. For each i, the fR-module $e_{i1}Rf$ is simple, injective and is not embeddable in fR.*

Proof. The first two statements are straightforward consequences of the facts that the right ideals of fR coincide with the R-submodules of fR and $\mathrm{Hom}_{fR}(K, L) = \mathrm{Hom}_R(K, L)$ for any right ideals K and L of R on which f acts as a right identity. Since $e_{i1}Rf$ is a simple R-module it is a simple fR-module and as R is basic it cannot be isomorphic to a right ideal of fR. The fR-injectivity of $e_{i1}Rf$ follows from Baer's criterion and Lemma 1.

Suppose that $fR = gR + hR$ where g and h are orthogonal idempotents of fR. For each $\vec{E}_i$ in $\mathscr{C}$ exactly one of $gR \to E_{i1}$ or $hR \to E_{i1}$ is true because if both gR and hR mapped onto the simple submodule of E_{i1} then projectivity of gR would imply that hR contained a copy of that simple module thus violating the agreement that R is basic. If, say, $gR \to E_{i1}$ we say *the chain $\vec{E}_i$ is associated with gR.* In this way each decomposition of f as a sum of orthogonal idempotents induces a corresponding partition of the set of chains $\mathscr{C}$. The proof of the next proposition describes a procedure for decomposing f in such a way that each component summand of fR has associated with it exactly one chain from $\mathscr{C}$.

PROPOSITION 6. *Let Λ be an independent set of right ideals of a right q-ring R. If there is a right ideal A of R such that* (1) *$A \to B$ for every $B \in \Lambda$ and* (2) *$A \cap (\Sigma_{\Lambda} B) = 0$, then Λ is finite.*

Proof. We may assume that R is basic, that the members of

$\varLambda$ are minimal right ideals and that $A = eR$ for some idempotent e of R.

Suppose that B_1 and B_2 belong to $\varLambda$ and $B_1 \neq B_2$. Since R is basic $B_1 \not\cong B_2$ and it follows from Lemma 2(1) that $B_1^r \neq B_2^r$. From $eR \to B$ for each $B \in \varLambda$ it follows that $Be \neq 0$. In particular $e \notin B_i^r$ for $i = 1, 2$ and since R/B_i^r is a division ring it follows that $1-e \in B_1^r \cap B_2^r$. The modular law implies that

$$B_i^r = (1 - e)R + (B_i^r \cap eR)$$

and

$$\begin{aligned} B_i^r \cap eR &= B_i^r \cap (eRe + eR(1 - e)) \\ &= eR(1 - e) + (B_i^r \cap eRe) . \end{aligned}$$

Thus if $B_1 \neq B_2$ then $eRe \cap B_1^r \neq eRe \cap B_2^r$.

Choose $x \in (eRe \cap B_1^r) - B_2^r$. Let J denote the Jacobson radical of R. Since by [1, Theorem 3.1] eRe/eJe is a regular ring there is an element y of eRe such that $x - xyx$ belongs to eJe. Since idempotents of eRe lift modulo eJe by [1, Theorem 4.1] then there is an idempotent g of eRe such that $xy - g$ belongs to eJe. We note that $g \in B_1^r - B_2^r$. Thus one has the decomposition $A = (e - g)R \oplus gR$ where $gR \to B_2$ and $(e - g)R \to B_1$.

Assume that $\varLambda$ is infinite. Choose one of gR and $(e - g)R$ which has infinitely many members of $\varLambda$ as homomorphic images and call it A_1' and call the other A_1 so that $A = A_1 \oplus A_1'$. Replace A by A_1' and repeat the above process so that $A_1' = A_2 \oplus A_2'$ where A_2' has infinitely many homomorphic images in $\varLambda$. In this way we construct an infinite sequence $\{A_i\}_{i=1}^{\infty}$ which satisfies the three conditions of Proposition 1 and this is a contradiction.

DEFINITION. A basic right q-ring R is called a *chain q-ring* if $R = fR \oplus E_1 \oplus \cdots \oplus E_l$ where $\to E_1 \to E_2 \to \cdots \to E_l$ is the only $\to$-class of R and $fR \to E_1$. We call fR the corner of R in this case.

Note that in a chain q-ring fR is a basic right q-ring all of whose idempotents are central since fR has no $\to$-classes. Also fR is not a right cogenerator since the simple module $E_1 f$ does not embed in fR. For instance fR might be an infinite product of division rings.

PROPOSITION 7. *If R is a basic loopless right q-ring then R is isomorphic to the product of a finite set of chain q-rings each of which has as a corner an infinite product of division rings together with a basic loopless q-ring which has no projective minimal right* *ideal.*

Proof. In the basic loopless q-ring R let $\{e_iR \mid i \in I\}$ be the set of projective minimal right ideals. For each i in I one has $e_iR(1-e_i) = 0$ so that $e_iR = e_iRe_i$ is a division ring. Consider the usual embedding α of the direct sum Σe_iR into Πe_iR where α maps e_j to $(\delta_{ij}e_i)_{i\in I}$. Since $e_iR = e_iRe_i$ and $e_iRe_j = 0$ for $i \neq j$ then α is an *essential* embedding. It follows that there is an R-monomorphism $\phi\colon \Pi e_iR \to R$ so that $\phi\cdot\alpha$ is the inclusion of Σe_iR in R. Let ψ be the splitting map for ϕ so $\psi\phi = 1$. One may show that $\psi(1) = (e_i)_{i\in I}$. If $g = \phi\psi(1)$ then the image of ϕ is gR and $\psi(g) = (e_i)_{i\in I}$. Since R is basic, $gR(1-g) = 0$ so $gR = gRg$. One has for r, s in R

$$\psi(gr\cdot gs) = \psi(grs) = \psi(g)rs = (e_i)_{i\in I}rs = (e_irs)_{i\in I} = (e_ir)_{i\in I}\cdot(e_is)_{i\in I}$$

where the last multiplication is componentwise. Thus ϕ is a *ring* isomorphism from Πe_iR onto gR.

Proposition 6 implies that the set of chains $\mathscr{C}(gR)$ of R associated with gR is finite, and that there is a decomposition $g = g_1 + g_2 + \cdots + g_k$ so that the g_i are orthogonal idempotents associated one-to-one with the chains of $\mathscr{C}(gR)$, i.e., each $\mathscr{C}(g_iR)$ is a singleton. Let $\bar{g}_i$ be an idempotent such that $\bar{g}_iR = g_iR \oplus \Sigma\mathscr{C}(g_iR)$. One checks that for each $i = 1, \cdots, k$, $\bar{g}_i$ is central. For instance $\bar{g}_iR(1-\bar{g}_i) = 0$ since otherwise $(1-\bar{g}_i)R$ has a simple image in $\bar{g}_iR$ and by projectivity must contain a copy of that simple module, thus contradicting the fact that R is basic. Thus $R = \bar{g}R \oplus (1-\bar{g})R$ where $\bar{g} = \sum_1^k \bar{g}_i$ and each $\bar{g}_i$ is central. For each i, $\bar{g}_iR$ is a chain q-ring with corner, g_iR, a product of division rings. Also $(1-\bar{g})R$ is a basic loopless q-ring which has no projective minimal right ideal since any such must be contained in $\bar{g}R$ by construction.

Matrix representation of chain q-rings. A chain q-ring R is a q-ring with orthogonal idempotents $f, e_1, e_2, \cdots, e_l$ such that the e_i, $1 \leqq i \leqq l$, are primitive, $fR \to e_1R \to e_2R \to \cdots \to e_lR$, $fR = fRf$, and $R = fR \oplus e_1R \oplus \cdots \oplus e_lR$. Since the $\to$-relations shown are the only ones which exist between the modules $fR, e_1R, \cdots, e_lR$ one has the matrix representation

$$R \simeq \begin{bmatrix} fR & 0 & 0 & \cdots & 0 \\ e_1Rf & e_1Re_1 & 0 & \cdots & 0 \\ 0 & e_2Re_1 & e_2Re_2 & \cdots & 0 \\ 0 & 0 & \ddots & & \vdots \\ \vdots & \vdots & \ddots & e_{l-1}Re_{l-1} & 0 \\ 0 & 0 & \cdots & e_lRe_{l-1} & e_lRe_l \end{bmatrix}.$$

Since e_1Rf, is 1-dimensional as a left e_1Re_1-space if we select $x_1 \in e_1Rf$, $x_1 \neq 0$ and if $M = x_1^r$ then $d_1x_1 = x_1\bar{d}_1$ implements a ring isomorphism $d_1 \mapsto \bar{d}_1$ from e_1Re_1 onto fR/M and at the same time $e_1Rf \simeq fR/M$ as an fR-module. If we use these isomorphisms to identify e_1Re_1 with fR/M and e_1Rf with the fR-module fR/M then the left action of e_1Re_1 on e_1Rf corresponds to the natural left fR/M-module structure of fR/M. Similarly for $i \geqq 1$ each $e_{i+1}Re_i$ is 1-dimensional on each side so that selecting $x_{i+1} \in e_{i+1}Re_i$, $x_{i+1} \neq 0$ we have isomorphisms $d_{i+1} \mapsto \overline{d_{i+1}}$ from $e_{i+1}Re_{i+1}$ onto e_iRe_i given by $d_{i+1}x_{i+1} = x_{i+1}\overline{d_{i+1}}$. If we denote by $(fR/M)_0$ the abelian group of fR/M with its usual left and right module structures over the rings fR and fR/M and with the zero multiplication then it is easy to see that

$$R \simeq \begin{bmatrix} fR & 0 & 0 & \cdots & 0 \\ (fR/M)_0 & fR/M & 0 & \cdots & 0 \\ 0 & (fR/M)_0 & fR/M & & \vdots \\ 0 & 0 & \ddots & \ddots & \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & (fR/M)_0 & fR/M \end{bmatrix}.$$

The following proposition shows that, conversely, every ring of this form is a right q-ring.

DEFINITION. Let A be a right q-ring with an essential maximal right ideal M such that A/M is injective and does not embed in A. We denote by $C(A, M, l)$ the ring of $(l+1) \times (l+1)$ matrices with entries in the $(1, 1)$ position from A, entries in the other main diagonal positions from A/M, entries on the sub-diagonal from $(A/M)_0$, and zero entries elsewhere. (It is convenient to allow l to be any integer $\geqq 0$.)

PROPOSITION 8. *For any $l \geqq 0$, the ring $C(A, M, l)$ as defined above is a right q-ring.*

Proof. Let A and M be as described above. For each $l \geqq 1$ let $A_l = C(A, M, l)$ and let M_l denote the ideal of A_l whose members are those matrices with zero entry in the $(l+1, l+1)$ position. We wish to show by induction that for every $l \geqq 1$ the ring A_l is a right q-ring with the essential maximal right ideal M_l such that A_l/M_l is A_l-injective and does not embed in A_l.

If $l \geqq 1$ there is an obvious ring isomorphism between A_{l+1} and $C(A_l, M_l, 1)$. Using this, the proof by induction is reduced to proving that the statement holds when $l = 1$.

Let e_i, $i = 1, 2$ be the idempotent matrix of A_1 with zero entries except at the (i, i) position where the entry is 1. Since $e_1A_1e_2 = 0$ then a minimal right ideal of A_1 is either a minimal right ideal of e_1A_1 (i.e., a minimal right ideal of A) or it is a simple submodule of e_2A_1. The kernel of the ring homomorphism from A_1 onto A/M which sends a matrix X to Xe_2 is $M_1 = A_1e_1$. Since A/M is a division ring, the ideal M_1 is a maximal right ideal of A_1. It is easy to check that $S = e_2A_1e_1$ is an essential submodule of e_2A_1 so that M_1 is an essential right ideal of A_1. Since $e_2A_1/S \simeq A_1/M_1$ it follows that S is the only proper nonzero submodule of e_2A_1.

Suppose that K is an essential right ideal of A_1. If $K \supseteqq e_2A_1$ then $K = e_2A_1 \oplus (K \cap e_1A_1)$. Otherwise $K \cap e_2A_1 = S$ so $Ke_2 = 0$ and $K \subseteqq M_1 = e_1A_1 \oplus S$. It follows that $K = S \oplus (K \cap e_1A_1)$. Since A is a right q-ring it is easy to see that $K \cap e_1\mathrm{A}_1$ is a two-sided ideal of e_1A_1 and it follows easily that in either of the above cases, K is a two-sided ideal of A_1.

To see that A_1 is right self-injective it suffices to apply Baer's criterion as follows. Let $\phi: K \to A_1$ be an A_1-homomorphism where K is an essential right ideal of A_1. Since K is an ideal $K = e_1K \oplus e_2K$. Let ϕ_1 be the restriction of ϕ to e_1K. Since $e_1A_1 = e_1A_1e_1$ then $\operatorname{Im}\phi_1 \subseteqq A_1e_1 = e_1A_1 \oplus S$. The injectivity of A/M as an A-module implies that S is an injective e_1A_1-module and by assumption $e_1A_1 \simeq A$ is right self-injective. Since ϕ_1 is an e_1A_1-homomorphism it follows that there is an element a of A and an element s of A/M so that for every X in e_1K one has $\phi_1(X) = \begin{pmatrix} a & 0 \\ s & 0 \end{pmatrix} \cdot X$. Let ϕ_2 be the restriction of ϕ to e_2K. Since no submodule of e_2A_1 has a nonzero image in e_1A_1 then the image of ϕ_2 must be contained in e_2A_1. It is then easy to see that there is an element d of A/M so that for each Y in e_2K one has $\phi_2(Y) = \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} \cdot Y$. It follows that for all Z in K, $\phi(Z) = \begin{pmatrix} a & 0 \\ s & d \end{pmatrix} \cdot Z$. So A_1 is right self-injective.

If A_1/M_1 embeds in A_1 then either A_1/M_1 embeds in e_1A_1 or $A_1/M_1 \simeq S$. Since $A_1/M_1 \simeq e_2A_1/S$ and $e_1A_1e_2 = 0$ then A_1/M_1 does not embed in e_1A_1. If $A_1/M_1 \simeq S$ then there is an epimorphism from e_2A_1 onto S. But since $Se_2 = 0$ there is no such mapping. Thus A_1/M_1 does not embed in A_1. To see that A_1/M_1 is A_1-injective suppose that $\phi: K \to A_1/M_1$ is an epimorphism where K is an essential right ideal of A_1. Since K is an ideal then $K = e_1K \oplus e_2K$. Because $e_1A_1e_2 = 0$ and A_1/M_1 is an image of e_2A_1 it follows that $\phi(e_1K) = 0$. If $e_2K = e_2K_1$ then ϕ extends immediately to A_1. Otherwise, $e_2K = S$ so that ϕ is an isomorphism between S and A_1/M_1 which we have just shown to be impossible.

The finiteness condition. The finiteness results Propositions 1 and 6 and Theorem 2 will be subsumed in the following theorem whose proof will be given as a sequence of lemmas.

THEOREM 5. *A right q-ring has no infinite set of orthogonal, noncentral idempotents.*

It suffices to prove the result for basic rings where from Lemma 1 and Proposition 2, the theorem is equivalent to the assertion that $\mathscr{A}(R)$ is a finite set. From Theorem 4 and Proposition 7 we may assume R has no loops and no projective minimal right ideals. We now reduce the problem to the case where R has no chain of length $l > 1$. Let $\{\vec{E}_i \mid 1 \leqq i \leqq m\}$ be the set of chains of R of length $l>1$, where $\vec{E}_i$ is $\rightarrow E_{i1} \rightarrow E_{i2} \rightarrow \cdots \rightarrow E_{il_i}$ with $E_{ij} = e_{ij}R$ for a primitive idempotent e_{ij} of R. Let $f = 1 - \Sigma e_{ij}$ so that $fR \rightarrow E_{i1}$ for each i and $fR = fRf$. It follows that fR is a right q-ring which is basic, loopless, without projective minimal right ideals and whose $\rightarrow$-classes are exactly the chains of R of length 1.

LEMMA 3. *Let R be a basic right q-ring whose only $\rightarrow$-classes are chains of length* 1 *and let $\mathscr{S}$ be the set of minimal right ideals of R. To each subset A of $\mathscr{S}$ we associate an idempotent e_A so that e_AR is an injective hull of $\Sigma\{S \mid S \in A\}$. If $A \subseteqq \mathscr{S}$ then there is a subset A_1 of $\mathscr{S}$ so that $A \Delta A_1$ is finite and e_{A_1} is central. (Here Δ denotes symmetric difference of sets.)*

Proof. Let $A \subseteqq \mathscr{S}$ and e_AR be a hull of $\Sigma\{S \mid S \in A\}$. Let $B \subseteqq \mathscr{S}$ be the finite set of simple images of e_AR in $(1 - e_A)R$ and let $C \subseteqq \mathscr{S}$ be the finite set of simple images of $(1-e_A)R$ in e_AR. One has $e_AR = eR + e_CR$ and $(1 - e_A)R = fR + e_BR$ where e, f, e_C and e_B are pairwise orthogonal idempotents. If $A_1 = (A - C) \cup B$ then $A \Delta A_1$ is finite and $(e + e_B)R$ is a hull of $\Sigma\{S \mid S \in A_1\}$. It is routine to check that $e + e_B$ is central. For instance, to see that $\mathrm{Hom}_R((1-e-e_B)R, (e + e_B)R) = \mathrm{Hom}_R(fR + e_CR, eR + e_BR) = 0$ one argues as follows: Any simple submodule of e_BR is an image of e_AR. It cannot also be an image of $(1 - e_A)R$ so it cannot be an image of fR and thus $\mathrm{Hom}_R(fR, e_BR) = 0$. Since the $\rightarrow$-classes are all chains of length one then $\mathrm{Hom}_R(e_CR, eR) = \mathrm{Hom}_R(e_CR, e_BR) = 0$ and $\mathrm{Hom}_R(fR, eR) = 0$ by the definition of C. The argument that $\mathrm{Hom}_R((e + e_B)R, (1 - e - e_B)R) = 0$ is similar.

Proof of Theorem 5. The proof is by contradiction. Assume the theorem is false. Then there is a right q-ring R such that

$\mathscr{S} = \{S_i \mid i \in I\}$, the set of minimal right ideals of R, is infinite, $S_i \not\cong S_j$ for $i \neq j$, and if for each i we let e_jR be a hull of S_i then $(1 - e_i)Re_i = 0$ and $e_iR(1 - e_i) \neq 0$. If we let A be a countably infinite subset of $\mathscr{S}$ then by the Lemma 3 there is a countably infinite subset A_1 so that the hull of $\Sigma\{S \mid S \in A_1\}$ is generated by a central idempotent e_{A_1}. It follows that we can assume that $\mathscr{S}$ is countable (so we take I to be the set of positive integers) and that R_R is the hull of $\sum_{i=1}^{\infty} S_i$, i.e., the socle of R_R is essential in R_R. The proof is given as a sequence of eleven assertions proved individually.

(1) If J is the Jacobson radical of R then

$$J = \{r \in R \mid re_i = 0, \forall i\} = \bigcap_{i=1}^{\infty} R(1 - e_i) .$$

Proof of (1). Since $(1 - e_i)Re_i = 0$ then $Re_i = e_iRe_i$. Since e_iRe_i is a division ring then Re_i is a minimal left ideal of R so that $R(1 - e_i)$ is a maximal left ideal. Thus $J \subseteq \cap R(1 - e_i)$. For the other containment, if $r \in \cap R(1 - e_i)$ then $rS_i = 0$ for all i so $r(\mathrm{Soc}\ R) = 0$. Then since $\mathrm{Soc}\ R$ is essential in R_R it follows that $r \in J$ by [1, Theorem 3.1]. Thus (1) is proved.

(2) The mapping $\alpha: R_R \to \Pi e_iR$ defined by $\alpha(r) = (e_ir)_{i \in I}$ is an R-monomorphism.

Proof of (2). If $r \in R$ and $r \neq 0$ then since Soc R is essential in R there is $r_1 \in R$ so that $0 \neq rr_1 \in \mathrm{Soc}\ R$. For some $j \in I$ one has $e_jrr_1 \neq 0$ so $e_jr \neq 0$ and $\alpha(r) \neq 0$. Thus (2) is proved.

We will identify the module ΠS_i with its image in Πe_iR under the mapping induced by the inclusions $S_i \hookrightarrow e_iR$. Since $\alpha(R)$ is injective then $\Pi e_iR = \alpha(R) \oplus L$ for some submodule L_R.

(3) $L \subseteq \Pi S_i$.

Proof of (3). If $L \not\subseteq \Pi S_i$ then there is $(x_i)_{i=1}^{\infty} \in L$ such that $x_j \notin S_j$ for some j. Since $S_j = e_jR(1 - e_j)$ and $x_je_j = e_jx_je_j$ it follows that $x_je_j \neq 0$. Then since $x_ie_j = \delta_{ij}x_je_j$ we have $0 \neq (x_i)_{i=1}^{\infty} e_j = \alpha(x_je_j) \in L \cap \alpha(R)$ which is a contradiction. Thus (3) is proved.

From (3) and the modular law one has

$$\Pi S_i = \Pi S_i \cap (\alpha(R) \oplus L) = (\Pi S_i \cap \alpha(R)) \oplus L .$$

(4) $\Pi S_i \cap \alpha(R) = \alpha(J)$.

Proof of (4). Since $J = \cap R(1 - e_i)$ from (1), it is clear that $\alpha(J) \subseteq \Pi S_i \cap \alpha(R)$. For the other containment suppose that $r \in R$ and $\alpha(r) \in \Pi S_i$. Then for each i one has $e_ir = e_ir(1 - e_i)$ so that $re_i = e_ire_i = 0$. It follows from (1) that $r \in J$. Thus (4) is proved.

We have $\Pi S_i = \alpha(J) \oplus L$ and since $(\Pi S_i)J = 0$, it follows from

(2) that $J^2 = 0$.

(5) The rings R/J and $\Pi e_i Re_i$ are isomorphic.

Proof of (5). For each i, $e_i R = S_i \oplus e_i Re_i$ as abelian groups. The projections onto the second summands induce an abelian group epimorphism $\pi: \Pi e_i R \to \Pi e_i Re_i$. Let $\beta = \pi\alpha$ map R into $\Pi e_i Re_i$. Using $Re_i = e_i Re_i$ one can see that β is a ring homomorphism. Since by (3) $L \subseteqq \Pi S_i$ then $\pi(L) = 0$ and since $\Pi e_i R = \alpha(R) \oplus L$ it follows that $\operatorname{Im} \pi = \pi\alpha(R) = \operatorname{Im} \beta$ so β is an epimorphism. We note that $\operatorname{Ker} \beta = \alpha^{-1}(\operatorname{Ker} \pi) = \alpha^{-1}(\Pi S_i) = J$ by (4). Thus (5) is proved.

From Lemma 2, each S_i is a 1-dimensional left vector space over $e_i Re_i$. The componentwise multiplication $(e_i r_i e_i)_{i=1}^{\infty}(s_i)_{i=1}^{\infty} = (e_i r_i e_i s_i)_{i=1}^{\infty}$ makes ΠS_i a left $\Pi e_i Re_i$-module. Since each S_i is a left ideal of R and $JS_i = 0$ then ΠS_i is naturally a left R/J-module where the multiplication is given by $(r + J)(s_i)_{i=1}^{\infty} = (rs_i)_{i=1}^{\infty}$. We denote by D_i the ring $e_i Re_i$.

(6) As left R/J-modules, ΠS_i is isomorphic to R/J.

Proof. In each S_i select a nonzero element x_i. This produces a map $\delta: \Pi S_i \to \Pi D_i$ where $\delta(s_i)_{i=1}^{\infty} = (d_i)_{i=1}$ when $s_i = d_i x_i$. The mapping δ is clearly a ΠD_i-isomorphism. The mapping β of (5) induces a ring isomorphism $\bar{\beta}: R/J \to \Pi D_i$. One checks that if $\bar{r} = r + J$ for $r \in R$ and $s \in \Pi S_i$ then $\delta(\bar{r}s) = \beta(r)\delta(s)$ so that if we identify R/J and ΠD_i via $\bar{\beta}$ then δ yields the desired isomorphism. Thus (6) is proved.

Since $J^2 = 0$ then J is a left R/J-module.

(7) The restriction of α to J is an R/J-monomorphism from ${}_{R/J}J$ into ΠS_i.

Proof of (7). If $s \in \Pi S_i$ and $\bar{r} = r + J$ for $r \in R$ then one always has $\bar{r}s = \bar{\beta}(\bar{r})\cdot s$ where $\cdot$ denotes componentwise multiplication, since $\bar{\beta}(\bar{r})\cdot s = \beta(r)\cdot s = (e_i re_i)\cdot(s_i) = (e_i re_i s_i) = (re_i s_i) = (rs_i) = r(s_i) = \bar{r}s$. Let j belong to J. Then $\alpha(\bar{r}j) = \alpha(rj) = (e_i r_i)_{i=1}^{\infty} j = (e_i re_i j + e_i r(1-e_i)j)_{i=1}^{\infty} = (e_i re_i j)_{i=1}^{\infty} = (e_i re_i)_{i=1}^{\infty}.\ (e_i j)_{i=1}^{\infty} = \bar{\beta}(r)\cdot\alpha(j) = \bar{r}\alpha(j)$. Thus (7) is proved.

(8) The mapping $\bar{\beta}^{-1}\delta\alpha$ is an *essential* embedding of J into R/J as left R/J-modules.

Proof of (8). From (6) and (7), $\bar{\beta}^{-1}\delta\alpha$ is an R/J-embedding of J into R/J. To show that $\bar{\beta}^{-1}\delta\alpha(J)$ is essential in R/J is equivalent to showing that $\delta\alpha(J)$ is an essential left ideal of ΠD_i. It suffices to show that for each j the idempotent E_j where $E_j = (\delta_{ij})_{i=1}^{\infty}$ belongs to $\delta\alpha(J)$. If $x_i \in S_i$ then for each k one has $x_i e_k = 0$ so that $\Sigma S_i \subseteqq J$. Then with the elements $x_i \in S_i$ chosen in (6) one has $\delta\alpha(x_j) \in \delta\alpha(J)$ and $\delta\alpha(x_j) = \delta(e_i x_j)_{i=1}^{\infty} = \delta(\delta_{ij}x_j) = {}_{i=1}^{\infty}E_j$. Thus (8) is proved.

Consider the bimodule ${}_{R/J}J_{R/J}$. The right hand action of elements of R/J on J produces a ring homomorphism $\gamma\colon R/J \to \mathrm{End}\,({}_{R/J}J)$ whose kernel is $\{r + J \mid Jr = 0\}$.

(9) If r belongs to R then $Jr = 0$ if and only if Supp $r = \{i \in I \mid re_i \neq 0\}$ is finite.

Proof of (9). Suppose that Supp r is finite. Since $r_1 = r - \sum_{i \in \mathrm{Supp}\, r} re_i$ left annihilates all the e_i then by (1) r_1 belongs to J and hence $Jr_1 = 0$ since $J^2 = 0$. But clearly $J \sum_{i \in \mathrm{Supp}\, r} re_i = J \sum_{i \in \mathrm{Supp}\, r} e_i re_i = 0$. Thus $Jr = 0$.

Suppose that $Jr = 0$ and that Supp r is infinite. From Lemma 3 there is a central idempotent f of R so that if $I_1 = \{i \in I \mid e_i \in fR\}$ then Supp $r \Delta I_1$ is finite. Replacing R by fR we can assume that Supp r is cofinite in I. Let $r_2 = r + \sum_{i \notin \mathrm{Supp}\, r} e_i$. Since $J(\Sigma e_i) = 0$ then $Jr_2 = 0$. But for all i in I, $r_2 e_i \neq 0$ so $\beta(r_2)$ is a unit of ΠD_i. Then there is an element t of R so that $1 - r_2 t \in J$ and hence $J = J(1 - r_2 t) \subseteqq J^2 = 0$, a contradiction. Thus (9) is proved.

It follows from (9) that $\mathrm{Ker}\,\gamma = \{\bar{r} \mid Jr = 0\} = \{\bar{r} \mid re_i = e_i re_i = 0$ a.e.$\} = \{\bar{r} \mid \bar{\beta}(r) \in \mathrm{Soc}\,\Pi D_i\} = \mathrm{Soc}\,(R/J)$. Let $D = \Pi D_i = \Pi e_i R e_i$. Then γ induces a ring monomorphism from $D/\mathrm{Soc}\,D$ into $\mathrm{End}\,({}_{R/J}J)$. Since as a left R/J-module J is isomorphic to an essential left ideal of R/J by (8), then $\mathrm{End}\,({}_{R/J}J) \simeq R/J$ because R/J is a left self-injective regular ring (in fact, a product of division rings). It follows that γ induces a ring monomorphism from $D/\mathrm{Soc}\,D$ into D. We then arrive at a contradiction from the following two facts.

(10) If G is a set of nonzero orthogonal idempotents of $D = \Pi D_i$ then $|G| \leqq \aleph_0$.

(11) The ring $D/\mathrm{Soc}\,D$ has a set of orthogonal idempotents of cardinality c.

Proof of (10). For each $i = 1, 2, \cdots$ let ε_i be the sequence of D with ith slot e_i and zero elsewhere. If g_1 and g_2 belong to G and $\varepsilon_i g_1 = \varepsilon_i$ and $\varepsilon_i g_2 = \varepsilon_i$ then $g_1 = g_2$ since otherwise we have $\varepsilon_i^2 = 0$. It follows that if we let $E = \{\varepsilon_i \mid \varepsilon_i g = \varepsilon_i$ for some g in $G\}$ then the mapping from E to G which maps ε_i to g if $\varepsilon_i g = \varepsilon_i$ is well-defined and it is clearly a surjection. It follows that $|G| \leqq |E| \leqq \aleph_0$.

Proof of (11). The set N of natural numbers has a set $\mathscr{A}$ of c subsets of N, each of cardinality $\aleph_0$, any two of which have finite intersection. (Match N with the set of rational numbers and choose for each real number a strictly increasing sequence of rational numbers converging to it.) For each subset X of N let e_X be the idempotent of D such that $e_X(i) = e_i$ if i belongs to X and

$e_X(i) = 0$ otherwise. The set $\bar{\mathscr{A}} = \{e_X + \operatorname{Soc} D \mid X \in \mathscr{A}\}$ is a set of pairwise orthogonal idempotents of $D/\operatorname{Soc} D$. Since $X \Delta Y$ is infinite when X and Y are distinct members of $\mathscr{A}$ then $e_X + \operatorname{Soc} D \neq e_Y + \operatorname{Soc} D$. It follows that $\bar{\mathscr{A}}$ has cardinality c.

Thus (10) and (11) hold and Theorem 5 is proved.

COROLLARY. *Let R be a basic right q-ring which has no projective minimal right ideals and has no loops. Then R is a finite product of chain q-rings whose corners are right q-rings with no noncentral idempotents.*

Proof. It follows from Theorem 5 that $\mathscr{A}(R)$ is a finite set all of whose members are chains. If $\mathscr{C}(R)$ is the union of the sets in $\mathscr{A}(R)$ then $\Sigma\mathscr{C}(R)$ is injective so there is an idempotent g of R such that $R_R = gR + \Sigma\mathscr{C}(R)$. If the chains are denoted $\vec{E}_i$ or $\rightarrow E_{i1} \rightarrow E_{i2} \rightarrow \cdots \rightarrow E_{il_i}$ for $1 \leqq i \leqq m$ then $gR \rightarrow E_{i1}$ for each i. As in Proposition 6 we can find orthogonal idempotents g_i, $1 \leqq i \leqq m$ so that $g = \sum_{i=1}^{m} g_i$ and $g_i R \rightarrow E_{j1}$ if and only if $i = j$. If $\bar{g}_i$ is an idempotent such that $\bar{g}_i R = g_i R \oplus \sum_{j=1}^{l_i} E_{ij}$ then $\bar{g}_i$ is central in R. As a ring $g_i R$ is a chain right q-ring such that the corner $g_i R = g_i R g_i$ is a right q-ring with $\mathscr{A}(g_i R) = \varnothing$. It follows from Lemma 1 that each idempotent of $g_i R$ is central.

PROPOSITION 9. *If R is a right q-ring with no projective minimal right ideals all of whose idempotents are central then $R \simeq Z \times L$ where Z is a right q-ring with no primitive idempotent and L is a product of local right q-rings none of which is a division ring.*

Proof. Let $\{e_i \mid i \in I\}$ be the set of primitive idempotents of R. As in the proof of Proposition 7 there is an idempotent g of R so that gR is ring-isomorphic to the product of local rings $L = \Pi e_i R$, in such a way that $e_i R \subseteqq gR$ corresponds to its usual image in $\Pi e_i R$. Clearly, $(1 - g)R$ has no primitive idempotent.

We note that local rings in the product L which are division rings would correspond to projective minimal right ideals of R.

PROPOSITION 10. *Let R be a chain right q-ring without projective minimal right ideals and with corner gR a ring with all idempotents central. Then $R \simeq R_1 \times L$ where R_1 is a chain right q-ring with corner Z a ring with no primitive idempotents and L is a product of local right q-rings none of which is a division ring.*

Proof. By Proposition 9, $gR = g_1 R \oplus g_2 R$ where $Z = g_1 R$ has no primitive idempotent and $g_2R \simeq L$ is a product of local right q-rings none of which is a division ring. The chain of R is associated with Z and not with L. This follows from Proposition 5 and the fact that if L is a product of local rings which are not division rings then there is no simple, injective right L-module which is not embeddable in L. For suppose that $L = \Pi L_i$ where each L_i is a local right q-ring so that $J_i \neq 0$ where J_i is the Jacobson radical of L_i. Suppose L/M is simple, injective L-module and is not embeddable in L. Then the maximal right ideal M of L is essential and therefore M is an ideal of L. Choose $u \in L$ so that $u = (u_i)$ where for each i, $u_i \in J_i$ and $u_i \neq 0$. The right annihilator u^r of u in L is contained in the radical ΠJ_i of L so that in particular $u^r \subseteqq M$. Thus the mapping $ua \mapsto a + M$ from uL to L/M is a well-defined epimorphism. Since L/M is injective there is an element $x \in L/M$ so that for each $a \in L$, $a + M = x(ua)$. But $u \in \Pi J_i \subseteqq M$ so that $xu = 0$ and we have a contradiction.

We can summarize all of the structure theorems of the paper in the following way.

THEOREM 6. *A right q-ring is isomorphic to a finite product of rings of the following kinds*:

(1) *Semisimple artinian ring.*

(2) *Loop q-ring*: $H(l, D)$.

(3) *ΠD_i-chain q-ring*: $C(\Pi D_i, M, l)$ *where the corner* ΠD_i *is an infinite product of division rings.*

(4) *Z-chain q-ring*: $C(Z, M, l)$ *where the corner Z is a right q-ring with no primitive idempotent.*

(5) *A product of local right q-rings none of which is a division ring.*

Final remarks. The further study of q-rings would examine the structure of the local ones and the ones which have no primitive idempotent. The latter clearly have zero right socle and for both kinds, all idempotents are central so that one would expect the investigation of them to require methods very different from those of the present paper.

With regard to the symmetry question for the q-ring condition, it is easy to see that a chain right q-ring (of length $\geqq 1$) is not left self-injective so that a right q-ring need not be also a left q-ring. For consider $R = C(A, M, 1)$ and let E_1 and E_2 be the idempotent matrices with zero entries except for entries of 1 in the (1, 1) and (2, 2) positions respectively. It is easy to see that the obvious correspondence between $S = E_2RE_1$ and RE_2 where $\begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix}$

corresponds to $\begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}$ is an isomorphism of left R-modules. If R were left self-injective then by Baer's criterion the isomorphism from S to RE_2 could be realized as a right multiplication by some element of RE_2, but $SRE_2 = 0$. One might rephrase the question thus: Is every right q-ring with no chain of length $\geqq 0$ also a left q-ring? [Cf. 2, Remark 2.14.] With regard to this symmetry question one would like to know whether there is a local, right self-injective duo ring which is not left self-injective.

REFERENCES

1. C. Faith and Y. Utumi, *Quasi-injective modules and their endomorphism rings*, Arch. Math., **15** (1964), 166–174.

2. D. A. Hill, *Semiperfect q-Rings*, Math. Ann., **200** (1973), 113–121.

3. G. Ivanov, *Non-local rings whose ideals are all quasi-injective*, Bull. Austral. Math. Soc., **6** (1972), 45–52.

4. ———, *Non-local rings whose ideals are all quasi-injective*: Addendum, Bull. Austral. Math. Soc., **12** (1975), 159–160.

5. N. Jacobson, *Structure of Rings*, AMS Colloquium Publ., Vol. 37, Providence, R. I., 1964.

6. R. E. Johnson and E. T. Wong, *Quasi-injective modules and irreducible rings*, J. London Math. Soc., **36** (1961), 260–268.

Received August 2, 1977 and in revised form July 31, 1978.

THE UNIVERSITY OF NORTH CAROLINA
GREENSBORO, NC 27412