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# ON THE ORDER OF DIRICHLET $L$-FUNCTIONS

G. KOLESNIK

**1. Introduction.** Let $L(s, \chi)$ be a Dirichlet $L$-function, where $\chi$ is a nonprincipal character $(\bmod q)$ and $s = \sigma + it$. A standard estimate for $L(s, \chi)$ based on bounds for $\zeta(s, w)$, is

$$(1) \qquad |L(s, \chi)| \leqq C_1(\varepsilon)\cdot\tau^{c(1-\sigma)+\varepsilon}q^{1-\sigma}, \quad \frac{1}{2} \leqq \sigma \leqq 1 ,$$

where $\tau = |t| + 2$, $c = 1/6$ (see, for example, Prachar [5, (4.12)]), and in fact, $c$ can be replaced by a constant $< 1/6$. An immediate application of Richert's work [6] gives

$$(2) \qquad |L(s, \chi)| \leqq C_1\tau^{100(1-\sigma)^{3/2}}q^{1-\sigma}\log^{2/3}\tau, \quad \frac{1}{2} \leqq \sigma \leqq 1 ,$$

which is better than (1) if $\sigma$ is near 1.

Another estimate can easily be obtained from $|L(1 + it, \chi)| \leqq C_2 \log \tau q$ and the functional equation of $L(s, \chi)$ as follows. First,

$$\begin{aligned} |L(it, \chi)| &= 2\cdot|(2\pi)^{it-1}q^{1/2-it} \\ &\quad \times \cos\frac{1}{2}\pi\Big(1 - it + \frac{1}{2} - \frac{1}{2}\,\bar{\chi}(-1)\Big)\Gamma(1 - it)L(1 - it, \bar{\chi})| \\ &\leqq C_3\sqrt{\tau q}\log \tau q . \end{aligned}$$

Now the convexity principle yields for

$$(3) \qquad \begin{aligned} |L(s, \chi)| &\leqq (C_3\sqrt{\tau q}\log \tau q)^{1-\sigma}\cdot(C_2\log \tau q)^{\sigma} \leqq C_4(\tau q)^{1/2(1-\sigma)} \\ &\quad \times \log \tau q,\ 0 \leqq \sigma \leqq 1 . \end{aligned}$$

Neglecting dependence on $\tau$, Davenport [2], improved (3):

$$(4) \qquad |L(s, \chi)| \leqq C_2(\tau)q^{1/2(1-\sigma)}, \quad 0 \leqq \sigma \leqq 1 .$$

Also, Burgess [1] improved (4) by establishing

$$|L(s, \chi)| \leqq C_1(\varepsilon, \tau)q^{3/8(1-\sigma)+\varepsilon} , \quad \frac{1}{2} \leqq \sigma \leqq 1 .$$

By examining Burgess' proof, it can be seen that the constant $C(\varepsilon, \tau)$ can be taken to be $C_2(\varepsilon)\pi^{2(1-\sigma)}$ and his result can be further sharpened to yield

$$(5) \qquad |L(s, \chi)| \leqq C_6\tau^{2(1-\sigma)}q^{3/8(1-\sigma)}C^{\omega}\log \tau, \quad \frac{1}{2} \leqq \sigma \leqq 1 ,$$

where $\omega = \log q/\log\log q$. The estimates (3), (4), and (5) are better than (1) if $q$ is large compared to $\tau$.

For $\sigma = 1/2$, the previous estimates were improved by Fujii, Gallagher and Montgomery, [3], who showed that if $P$ is a fixed set of primes and $q$ is composed only of primes in $P$, then

$$\left|L\left(\frac{1}{2} + it, \chi\right)\right| \leqq C(\varepsilon, P)(\tau q)^{1/6+\varepsilon} . \tag{6}$$

In this paper we prove two more estimates which imply (1), (4), and (5) and which are better than (2), (3), and (6) in some range of $\sigma, \tau$, and $q$. We prove:

THEOREM 1. *Let $\chi$ be a nonprincipal character* (mod $q$). *Let $1/2 \leqq \sigma \leqq 1, \tau = |t| + 2$ and $\omega = \log q/\mathrm{loglog}\, q$. Then*

$$|L(s, \chi)| \ll \tau^{-\sigma} q^{3/8(1-\sigma)} C^{\omega} \log \tau , \tag{7}$$

*where $C$ is some absolute constant.*

THEOREM 2. *Let $\chi$ be a character* (mod $q$). *Let $1/2 \leqq \sigma \leqq 1$ and $\tau = |t| + 2$. Then*

$$|L(s, \chi)| \ll \tau^{35/108(1-\sigma)} q^{1-\sigma} \log^3 \tau q . \tag{8}$$

*In particular,* (7) *and* (8) *imply*

$$\left|L\left(\frac{1}{2} + it, \chi\right)\right| \ll \sqrt{\tau}\, q^{3/16} C^{\omega} \log \tau$$

*and*

$$\left|L\left(\frac{1}{2} + it, \chi\right)\right| \ll \tau^{35/216} \sqrt{q} \log^3 \tau q .$$

The estimates of $L(s, \chi)$ for $\sigma \in [0, 1/2]$ can be obtained by using (7) or (8) and the functional equation of $L(s, \chi)$.

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## 2. Notation.

$$e(f(x)) = \exp(2\pi i f(x)) .$$
$$\omega = \log q/\log\log q .$$
$$s = \sigma + it,\ \frac{1}{2} \leqq \sigma \leqq 1 .$$
$$\tau = |t| + 2 .$$

$C$ denotes some appropriate absolute constant, not always the same.

**3. Application of the estimate of Burgess.** In this section we will show that

$$|L(s, \chi)| \ll \pi^{1-\sigma} q^{3/8(1-\sigma)} C^{\omega} \log^3 \tau .$$

We need the following result of E. Bombieri:

LEMMA. *Let $N$ and $m$ be nonnegative integers. Let $\alpha_j, \beta_j$ be numbers such that $|\alpha_j - \beta_j| \leqq (2\pi m N^j)^{-1}$ for $1 \leqq j \leqq m$, and let $f(x) = \alpha_1 x + \cdots + \alpha_m x^m$, $g(x) = \beta_1 x + \cdots + \beta_m x^m$. Let $c_1, c_2, \cdots$ be complex, and let*

$$S(\bar{\alpha}, N) = \max_{1 \leqq N_1 < N} \Big| \sum_{1 \leqq n \leqq N_1} c_n e(f(n)) \Big| ,$$

*where $\bar{\alpha} = (\alpha_1, \cdots, \alpha_m)$. Then $S(\bar{\beta}, N) \leqq 6S(\bar{\alpha}, N)$.*

*Proof.* For every $N_1 \in [1, N]$ we have:

$$\begin{aligned} \sum_{1 \leqq n \leqq N_1} c_n e(g(n)) &= \sum_{1 \leqq n \leqq N_1} c_n e(f(n)) \prod_{j=1}^{m} e((\beta_j - \alpha_j) n^j) \\ &= \sum_{k_1, \cdots, k_m = 0}^{\infty} \left( \prod_{j=1}^{m} \frac{\{2\pi i(\beta_j - \alpha_j)\}^{k_j}}{k_j!} \right) \sum_{1 \leqq n \leqq N_1} c_n n^{mk_m + \cdots + k_1} e(f(n)) . \end{aligned}$$

Using Abel's summation formula, we obtain:

$$\begin{aligned} S(\bar{\beta}, N) &\leqq \sum_{k_1, \ldots, k_m = 0}^{\infty} \prod_{j=1}^{m} \frac{|2\pi(\beta_j - \alpha_j)|^{k_j}}{k_j!} \cdot N^{mk_m + \cdots + k_1} \cdot 2S(\bar{\alpha}, N) \\ &\leqq 2S(\bar{\alpha}, N) \cdot \sum_{k_1, \ldots, k_m = 0}^{\infty} \prod_{j=1}^{m} \frac{|(2\pi(\beta_j - \alpha_j) N^j|^{k_j}}{k_j!} \\ &\leqq 2S(\bar{\alpha}, N) \left( \sum_{k=0}^{\infty} m^{-k}/k! \right)^m \leqq 6S(\bar{\alpha}, N) . \end{aligned}$$

LEMMA 2. *Let $q \geqq 2$ and let $M, N$ be integers. Let $\chi$ be a primitive character* (mod $q$). *Then*

$$\Big| \sum_{1 \leqq n \leqq N} \chi(n + M) \Big| \leqq \sqrt{N} q^{3/16} C^{\omega} .$$

*This lemma can be proven similarly to Theorem 2,* [1].

LEMMA 3. *Let $q$ and $N$ be integers such that $q \geqq 2$ and $N \leqq \tau q$. Let $\chi$ be a primitive character* (mod $q$). *Then*

$$|S = \max_{N \leqq N_1 \leqq 2N} \Big| \sum_{N+1 \leqq n \leqq N_1} \chi(n) n^{-it} \Big| \ll \sqrt{N\tau \log \tau} \cdot q^{3/16} C^{\omega} .$$

*Proof.* We can obviously suppose that $\tau \log \tau q \leqq N$ since otherwise the estimate is trivial. Taking $H = [N(\tau \log \tau q)^{-1}]$ and $m = [\log \tau q]$, and dividing the sum in $S$ into $\leqq 2NH^{-1}$ subsums, we obtain:

$$|S| \leqq 2NH^{-1} \max_{N \leqq M \leqq 2N} \max_{1 \leqq H_1 \leqq H} \Big| \sum_{M+1 \leqq n \leqq M+H_1} \chi(n) n^{-it} \Big| .$$

For every $M$ and $H_1$ in the above range , we get

$$(6) \qquad \Big|\sum_{M+1 \leqq n \leqq M+H_1} \chi(n) n^{-it}\Big| = \Big| \sum_{1 \leqq n \leqq H_1} X(n+M) \left(\frac{n+M}{M}\right)^{-it} \Big|$$
$$\leqq \Big| \sum_{1 \leqq n \leqq H_1} \chi(n+M) e\left(-\frac{t}{2\pi}\left\{\frac{n}{M} - \frac{n^2}{2M^2} + \cdots + \frac{(-1)^m \cdot n^m}{mM^m}\right\}\right)\Big|$$
$$+ \frac{|t| H^{m+2}}{M^{m+1}} .$$

Let $\beta_j = 0$ and $\alpha_j = (-1)^j t / 2\pi j M^j$. Then for $1 \leqq j \leqq m \, |\alpha_j - \beta_j| = |t| \cdot (2\pi j M^j)^{-1} \leqq (2\pi m H^j)^{-1}$. Applying Lemmas 1 and 2, we obtain:

$$|S| \leqq |2NH^{-1} \max_{N \leqq M \leqq 2N} \max_{1 \leqq H_1 \leqq H} \Big| \sum_{1 \leqq n \leqq H} \chi(n+M) \Big| + 2 \frac{\tau H^{m+1}}{N^m}$$
$$\ll NH^{-1} \sqrt{H} q^{3/16} C^{\omega} + N\tau(\tau \log \tau q)^{-\log \tau q}$$
$$\ll \sqrt{N \cdot \tau \log \tau q}\, q^{3/16} C^{\omega} .$$

From this, the result is easily obtained.

Now we can prove Theorem 1. First, we suppose that $\chi$ is primitive. Let $N = [\tau q]$, $M = [\tau q^{3/8}]$, $L = \log(N/M)/\log 2$, $N_l = M2^l (l = 0, \cdots, L)$. Using Abel's formula, the Polya-Vinogradov estimate for character sums and Lemma 3, we get:

$$|L(s, \chi)| \leqq \sum_{n < M} n^{-\sigma} + \Big| \sum_{M \leqq n \leqq N} \chi(n) n^{-\sigma - it} \Big| + \Big| \sum_{n > N} \chi(n) n^{-s} \Big|$$
$$\ll M^{1-\sigma} \log M + \sum_{l=0}^{L} \max_{N_l \leqq N_l^1 \leqq 2N_l} \Big| \sum_{N_l \leqq n \leqq N_l^1} \chi(n) n^{-\sigma - it} \Big|$$
$$+ \sum_{n > N} \tau n^{-\sigma - 1} \Big| \sum_{N \leqq x \leqq n} \chi(n) \Big|$$
$$\ll M^{1-\sigma} \log M + \sum_{l=0}^{L} N_l^{-\sigma} \max_{N_l \leqq N_l^1 \leqq 2N_l} \Big| \sum_{N_l \leqq n \leqq N_l^1} \chi(n)^{-it} \Big|$$
$$+ \tau \sqrt{q}\, N^{-\sigma} \log q$$
$$\ll M^{1-\sigma} \log M + \sum_{l=0}^{L} N_l^{1/2-\sigma} \sqrt{\tau}\, q^{3/16} C^{\omega} \sqrt{\log \tau} + \tau \sqrt{q}\, N^{-\sigma} \log q$$
$$\ll M^{1-\sigma} \log M + L M^{1/2-\sigma} \sqrt{\tau}\, q^{3/16} C^{\omega} \sqrt{\log \tau} + \tau \sqrt{q}\, N^{-\sigma} \log q$$
$$\ll \tau^{1-\sigma} q^{3/8(1-\sigma)} C^{\omega} \log \tau .$$

If $X$ is not primitive, then there is a $q_1 | q$ and a primitive character $\chi_1 \pmod{q_1}$, associated with $\chi$, such that we can write (see, for example, [5, (6.12)]):

$$|L(s, \chi)| = |L(s, \chi_1)| \prod_{p|q} \left|1 - \frac{\chi_1(p)}{p^s}\right| \leqq |L(s, \chi_1)| \cdot \prod_{p|q} 2 \leqq |L(s, \chi_1)| \cdot 2^{\omega} ,$$

and the theorem follows.

**4. The proof of Theorem 2.** To prove Theorem 2, we need two lemmas.

LEMMA 4. *Let $t \geqq 0$, $0 \leqq a \leqq 1$, and let $X$ and $X_1$ be integers such that $0 < X \leqq X_1 \leqq 2X \leqq \tau^{143/108}$. Then*

$$S_1 \equiv \sum_{X \leqq x \leqq X_1} e(t \log (x + a)) \ll \sqrt{X}\tau^{35/216} \log^2 \tau .$$

*Proof.* If $X \leqq \sqrt{\tau}$, then the result can be proven similarly to Corollary 2, [4]. The same method yields

$$\sum_{X \leqq x \leqq X_1} e(t \log x - ax) \ll \sqrt{X}\tau^{35/216} \log^2 \tau , \tag{9}$$

for $X \leqq \sqrt{\pi}$. If $\sqrt{\tau} \leqq X \leqq \tau^{143/108}$, then, by Lemma 3 of [4]

$$|S_1| \leqq \sum_{t/(X_1+a) \leqq n \leqq t/(X+a)} \frac{\sqrt{t}}{n} e(t \log n - an)| + 0(X\tau^{-1/2}) .$$

Here $t/(X + a) \leqq \sqrt{\tau}$. With the use of Abel's inequality, (9) yields the result for $\sqrt{\tau} \leqq X \leqq \tau^{143/108}$.

LEMMA 5. *Let $1/2 \leqq \sigma \leqq 1$, $t \geqq 1$ and $0 \leqq a \leqq 1$. Then*

$$\zeta(s, a) \equiv \sum_{n=0}^{\infty} (n + a)^{-s} \ll a^{-\sigma} + \tau^{35(1-\sigma)/108} \log^3 \tau .$$

*Proof.* Let $N = \tau^{143/108}$. Using the Euler-Maclaurin formula [see, for example, [5], (1.7), p. 372]), we obtain similarly to [5], (5.8), p. 114:

$$\begin{aligned}\zeta(s, a) - \sum_{n=0}^{N-1} (n + a)^{-s} &= \frac{(N + a)^{1-s}}{1 - s} - s\int_N^{\infty} \frac{x - [x]}{(x + a)^{s+1}} dx \\ &= \frac{(N + a)^{1-s}}{1 - s} - \frac{1}{2}\, s\, \frac{(x - [x])^2}{(x + a)^{s+1}}\Big]_N^{\infty} + \frac{1}{2}\, s(s + 1) \int_N^{\infty} \frac{(x - [x])^2}{(x + a)^{s+2}} dx \\ &\ll 1 + \tau^2 \int_N^{\infty} u^{-\sigma-2} du \leqq 1 + \tau^2 \cdot N^{-\sigma-1} \ll \tau^{35(1-\sigma)/108} .\end{aligned}$$

If we denote $M = [\tau^{35/108}]$, $L = [\log (N/M)/\log 2]$, $N_l = M \cdot 2^l$ for $l = 0, \cdots, L$ and $N_{L+1} = N$, then we have

$$S \equiv \sum_{n=0}^{N-1} (n+a)^{-s} \ll \sum_{0<n<M} (n+a)^{-\sigma} + \sum_{0 \leqq l \leqq L} \left| \sum_{N_l \leqq n < N_{l+1}} (n+a)^{-s} \right| .$$

Using Abel's formula and Lemma 4, we obtain:

$$S \ll a^{-\sigma} + M^{1-\sigma} \log M + \sum_{0 \leqq l \leqq L} N_l^{-\sigma} \max_{N_l \leqq N_l' \leqq N_{l+1}} \left| \sum_{N_l \leqq n \leqq N_l'} (n+a)^{-it} \right|$$

$$\ll a^{-\sigma} + M^{1-\sigma} \log M + \sum_{0 \leqq l \leqq L} N_l^{1/2-\sigma} \cdot \tau^{35/216} \log^2 \tau \ll a^{-\sigma}$$

$$+ \tau^{35(1-\sigma)/108} \log^3 \tau .$$

This proves the lemma.

To prove Theorem 2, we can obviously suppose $t \geqq 1$, otherwise the result follows from (1). Using Lemma 5, we obtain:

$$|L(s, \chi)| = \left| q^{-s} \sum_{m=1}^{q} \chi(m) \zeta(s, m/q) \right|$$

$$< q^{-\sigma} \sum_{m=1}^{q} ((q/m)^{\sigma} + \tau^{35(1-\sigma)/108} \log^3 \tau) \ll \tau^{35(1-\sigma)/108} q^{1-\sigma} \log^3 \tau q .$$

*Note Added in Proof.* We would like to draw attention to a recent paper by D. R. Heath-Brown, "Hybrid bounds for Dirichlet $L$-function," Inventiones Mathematicae, 44 (1978), 149–170, which contains a better result than our Theorem 7.

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CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CA 91125
AND
STATE UNIVERSITY OF NEW YORK AT BUFFALO
BUFFALO, NY 14214