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EXISTENCE OF A STRONG LIFTING COMMUTING WITH A COMPACT GROUP OF TRANSFORMATIONS II

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Let G be a locally compact group with left Haar measure γ. The well-known "Theorem LCG" of A. and C. Ionescu-Tulcea states that there is a strong lifting of $M^\infty(G,\gamma)$ commuting with left translations. Our purpose here is to prove a generalization of this theorem in case G is compact. Thus let (G, X) be a free left transformation group with X and G compact. Let ν_0 be a Radon measure on $Y=X/G$, and let μ be the Haar lift of ν_0. Let ρ_0 be a strong lifting of $M^\infty(Y,\nu_0)$. We will show that $M^\infty(X,\mu)$ admits a strong lifting ρ which extends ρ_0 and commutes with G.

In [6], the result just stated was proved when G and X satisfied certain restrictions. The following theorem, which may be of independent interest, enables us to remove the conditions imposed in [6]: Let H be a closed normal Lie subgroup of a compact group G: then there is a D' sequence (see 1.2 and [1] in H, consisting of compact neighborhoods $V_n(n \geqq 1)$ of the identity, such that $g^{-1}V_ng = V_n$ for all $g \in G$.

1.

NOTATION 1.1. Let G be a compact topological group, H a closed, normal, real Lie subgroup. Let γ be normalized Haar measure on G, and let λ be normalized Haar measure on H. For each $g \in G$, define $\alpha_g: H \rightarrow H: h \rightarrow g^{-1}hg$. Let $\mathfrak{H}$ be the Lie algebra of H; let exp: $\mathfrak{H} \rightarrow H$ be the exponential map.

DEFINITION 1.2. ([1]). A *D'-sequence in H* is a sequence $(W_n)_{n=1}^\infty$ of λ-measurable subsets of H such that (i) $W_n \supset W_{n+1}(n \geqq 1)$; (ii) $0 < \lambda(W_nW_n^{-1}) < C\cdot\lambda(W_n)$ for some $C > 0$ and all n; (iii) every neighborhood of idy ($\equiv$identity) $\in H$ contains some W_n.

PROPOSITION 1.3. *There is a D'-sequence $(V_n)_{n=1}^\infty$ in H, consisting of compact neighborhoods of* idy, *such that $g^{-1}V_ng = V_n(n \geqq 1$, $g \in G)$.*

Proof. Let W be a neighborhood of 0 in $\mathfrak{H}$ such that $\exp|_W$ is a diffeomorphism onto $\exp(W) \subset H_0$, the identity component of H. Define log to be the inverse of $\exp|_W$. There is a neighborhood $N \subset \exp(W)$ of idy such that $g^{-1}Ng \subset W(g \in G)$. Let $\varphi_g(x) = \log\circ\alpha_g\circ$

$\exp(x) = \log(g^{-1}\cdot\exp(x)\cdot g)$ for all $x \in W_1 = \log(N)$. Then $\varphi_g\colon W_1 \to W$, and $\varphi_g(0) = 0 (g \in G)$.

Each map α_g is a continuous isomorphism of H, hence is analytic ([9], Theorem 5.22). Let $\mathrm{Ad}_g\colon \mathfrak{H} \to \mathfrak{H}$ be the derivative at idy $\in H$ of α_g. Then $\mathrm{Ad}_g(x) = D\varphi_g(0)\cdot x (x \in \mathfrak{H})$. The map $g \to \mathrm{Ad}_g$ is a homomorphism of G into $GL(\mathfrak{H})$. We show that it is continuous. Let $G_0 = \{g \in G \mid g^{-1}hg = h$ for all $h \in H_0\}$. Then G_0 is a closed normal subgroup of G. The group G/G_0 acts effectively on H_0 via the map $\eta\colon G/G_0 \times H_0 \to H_0\colon (gG_0, h) \to g^{-1}hg$. Therefore G/G_0 is a Lie group, and the map η is analytic ([8], pp. 208, 212, 213). It follows that $g \to \mathrm{Ad}_g$ is continuous.

Let $\langle\ ,\ \rangle_1$ be an inner product on $\mathfrak{H}$. Define an inner product $\langle\ ,\ \rangle$, invariant under each Ad_g, by

$$\langle x, y\rangle = \int_G \langle \mathrm{Ad}_g(x), \mathrm{Ad}_g(y)\rangle_1\, d\gamma(g) (x, y \in \mathfrak{H})\ .$$

Observe that, if $B_r = \{x \in \mathfrak{H} \mid ||x|| \leqq r$, where $||x||^2 = \langle x, x\rangle\}$, then $\mathrm{Ad}_g(B_r) = B_r (g \in G)$. Also observe that, if m is a Lebesgue measure on $\mathfrak{H}$, then there is a constant β such that $m(B_r) = \beta r^k$, where $k = \dim H$.

Consider the measure $\lambda|_{\exp W}$. By ([7], Corollary 2, p. 106), there is a Lebesgue measure m on $\mathfrak{H}$ and an analytic function $\rho\colon W \to \boldsymbol{R}$, satisfying $\rho(0) = 1$, such that $\lambda(\exp B) = \int_B \rho(x)dm(x)$ for each Borel set $B \subset W$. Let W_2 be a neighborhood of $0 \in \mathfrak{H}$ such that $1/2 \leqq \rho(x) \leqq 2 (x \in W_2)$.

Now let $0 < \varepsilon < 1$ satisfy $(1-\varepsilon)^k > 1/2 (k = \dim H)$. Recall that $\varphi_g(0) = 0$ for all $g \in G$, that $\mathrm{Ad}_g(x) = D\varphi_g(0)\cdot x$, that G is compact, and that $(gG_0, x) \to \varphi_g(x)\colon G/G_0 \times W_2 \to W$ is analytic. We can therefore find $r' > 0$ such that

(*) $||\varphi_g(x) - \mathrm{Ad}_g(x)|| < \varepsilon ||x||$ for all $g \in G$ if $||x|| \leqq r'$ (recall $||x||^2 = \langle x, x\rangle$). Choose $r_0 \leqq r'$ such that $B_{3r} \subset W_2$ and $\exp(B_r)\cdot\exp(B_r) \subset \exp B_{3r}$ if $r \leqq r_0$. Let $r_n = r_0/n$. Define $C_n = \bigcap_{g\in G}\varphi_g(B_{r_n})$, and let $V_n = \exp(C_n)$. By (*), $B_{(1-\varepsilon)r_n} \subset C_n$ for each n. Hence V_n is a compact neighborhood of idy for each $n (n \geqq 1)$.

We show that $(V_n)_{n=1}^{\infty}$ is the desired D'-sequence in H. First note that $g^{-1}V_n g = \alpha_g\circ\exp(C_n) = \exp\circ\varphi_g(C_n) = \exp C_n = V_n$ for all $g \in G$. Next, observe that $V_n V_n^{-1} = \exp(C_n)\cdot\exp(-C_n) \subset \exp(B_{r_n})\cdot\exp(B_{r_n}) \subset \exp B_{3r_n}$. So $\exp(B_{(1-\varepsilon)r_n}) \subset V_n \subset V_n V_n^{-1} \subset \exp B_{3r_n}$. So, on the one hand, $\lambda(V_n V_n^{-1}) \leqq \lambda(\exp B_{3r_n}) = \int_{B_{3r_n}} \rho(x)dm(x) \leqq 2\cdot\beta\cdot 3^k\cdot(r_n)^k$, while on the other hand,

$$\lambda(V_n) \geqq \int_{B_{(1-\varepsilon)r_n}} \rho(x)dm(x) \geqq 1/2\beta(1-\varepsilon)^k(r_n)^k > 1/4\beta(r_n)^k.$$

Hence $\lambda(V_n V_n^{-1}) \leqq 8\cdot 3^k\lambda(V_n)$, so (ii) of 1.2 is satisfied with $C = 8\cdot 3^k$.

It is easy to see that $(V_n)_{n=1}^{\infty}$ satisfies (i) and (iii) of 1.2. This completes the proof of 1.3.

REMARK 1.4. The sequence $(V_n)_{n=1}^{\infty}$ is also a D''-sequence ([1]); that is, each V_n contains a subset U_n such that $U_n \cup U_n U_n^{-1} \subset V_n$, and $\lambda(V_n) < C'\lambda(U_n)$ for some constant $C'(n \geqq 1)$. To see this, let $s_n = (1 - \varepsilon)r_n/3$, and let $U_n = \exp B_{s_n}$. Then $U_n \cdot U_n^{-1} \subset \exp B_{(1-\varepsilon)r_n} \subset V_n$, and it is easy to see that we may choose $C' = 8 \cdot 3^k$.

2. The reader is warned that much of the terminology of this section was discussed in ([6]); that discussion will not be repeated in all detail.

NOTATION 2.1. Let X be a compact Hausdorff space, and let G be a compact Hausdorff topological group. Suppose (G, X) is a (left) transformation group (thus there is a continuous map $\Phi: G \times X \rightarrow X: (g, x) \rightarrow g \cdot x$ satisfying (i) $\mathrm{idy} \cdot x = x$; (ii) $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x (x \in X; g, g_1, g_2 \in G)$). Suppose also that G acts *freely* (thus $g \cdot x = x \Rightarrow g = \mathrm{idy}\,(g \in G, x \in X)$). Let $Y = X/G$ be the space of G-orbits, with the quotient topology; let $\pi_0: X \rightarrow Y$ be the canonical projection. Let γ be normalized Haar measure on G, and fix a Radon measure ν_0 on Y. Let $M^{\infty}(Y, \nu_0)$ be the algebra of all bounded ν_0-measurable complex functions on Y, and let $L^{\infty}(Y, \nu_0)$ be the (usual) space of equivalence classes in $M^{\infty}(Y, \nu_0)$.

DEFINITION 2.2. The *Haar lift* μ of ν_0 is defined as follows: $\mu(f) = \int_Y \left(\int_G f(g \cdot x) d\gamma(g) \right) d\nu_0(y)$ for each $f \in C(X)$.

DEFINITION 2.3. Let ρ_0 be a fixed strong lifting ([6], 1.4; see the references given there) of $M^{\infty}(Y, \nu_0)$. Let ρ be a linear lifting of $M^{\infty}(X, \mu)$. Note that $M^{\infty}(Y, \nu_0)$ may be embedded in $M^{\infty}(X, \mu)$ via $f \rightarrow f \circ \pi$. Say ρ *extends* ρ_0 if $\rho|_{M^{\infty}(Y, \nu_0)} = \rho_0$. Say ρ *commutes with* G if

$$\rho(f \cdot g)(x) = \rho(f)(g \cdot x)(g \in G, x \in X, f \in M^{\infty}(X, \mu)) ;$$

here $(f \cdot g)(x) \equiv f(g \cdot x)$.

The following theorem was proved in ([6]) subject to various additional assumptions. We prove it here in full generality.

THEOREM 2.4. *Suppose (G, X) is a free left transformation group. Let ρ_0 be a strong lifting of $M^{\infty}(Y, \nu_0)$. Then there exists a strong lifting ρ of $M^{\infty}(X, \mu)$ which extends ρ_0 and commutes with*

G, where μ is the Haar lift of ν_0.

More notation is necessary before we can discuss the proof of 2.4.

NOTATION 2.5. Let H be a closed, normal, real Lie subgroup of G. Let $Z = X/H$, and let $\pi: X \to Z$ be the projection. Note $(G/H, Z)$ is a free left transformation group. Write $g \cdot z$ for $(gH) \cdot z (g \in G, z \in Z)$. Define a Radon measure ν on Z by $\nu = \pi(\mu)$. Let λ be normalized Haar measure on H. For each $z \in Z$, let λ_z be the Radon measure on X defined by $\lambda_z(f) = \int_H f(h \cdot x) d\lambda(h)$ for one (hence all) $x \in \pi^{-1}(z)$. Then $\mu(f) = \int_Z \lambda_z(f) d\nu(z)$ for all $f \in C(X)$.

It can be shown that 2.4 follows from 2.6 below. See the paragraphs under "Proof of 2.2, using 2.7" in ([6]), and the reference given there. See also the proofs of Theorems 2 and 3 in ([5], Chpt. IV).

THEOREM 2.6. *Let H, Z, ν, π be as in 2.5, and suppose there is a strong lifting δ of $M^\infty(Z, \nu)$ which commutes with G/H. Then there is a strong lifting ρ of $M^\infty(X, \mu)$ which extends δ and commutes with G.*

To prove 2.6, we need only revise the proof of Proposition 3.11 in ([6]). For each $z_0 \in Z$ and $f \in M^\infty(X, \mu)$, define $R^f(z_0)$ as in ([6], 3.3–3.5). Thus $R^f(z_0)$ is an element of $L^\infty(X, \lambda_{z_0})$. Abusing notation, we think of $R^f(z_0)$ as a function on $\pi^{-1}(z_0)$. We repeat Proposition 3.9 of ([6]):

PROPOSITION 2.7. $R^{f \cdot g}(z_0)(h \cdot x_0) = R^f(g \cdot z_0)(ghg^{-1} \cdot gz_0)(x_0 \in X, z_0 = \pi(x_0), h \in H, g \in G)$.

DEFINITION 2.8. Let $(V_n)_{n=1}^\infty$ be the D'-sequence of § 1. Let $x_0 \in X$, $z_0 = \pi(x_0)$. As in ([6], 3.10, Case I), define

$$T_n^f(x_0) = \frac{1}{\lambda(V_n)} \int_X R^f(z_0)(\bar{x}) \psi_{V_n \cdot x_0}(\bar{x}) d\lambda_{z_0}(\bar{x})$$
$$= \frac{1}{\lambda(V_n)} \int_H R^f(z_0)(hx_0) \psi_{V_n}(h) d\lambda(h)$$

(here ψ denotes characteristic function).

PROPOSITION 2.9. $T_n^{f \cdot g}(x_0) = T_n^f(g \cdot x_0)(g \in G, x_0 \in X)$.

Proof.

$$\begin{aligned}
T_n^{f\cdot g}(x_0) &= \frac{1}{\lambda(V_n)}\int_H R^{f\cdot g}(z_0)(h\cdot x_0)\psi_{V_n}(h)d\lambda(h) \\
&= (\text{by 2.7 above})\ \frac{1}{\lambda(V_n)}\int_H R^f(g\cdot z_0)(ghg^{-1}\cdot gx_0)\psi_{V_n}(h)d\lambda(h) \\
&= (\text{by ([2], 28.72e))}\frac{1}{\lambda(V_n)}\int_H R^f(g\cdot z_0)(h\cdot gx_0)\psi_{gV_ng^{-1}}(h)d\lambda(h) \\
&= T_n^f(g\cdot x_0).
\end{aligned}$$

2.10. *Proof of* 2.6. Combine the following: (i) the just-proved 2.9; (ii) the reasoning of the Case I portions of ([6], 3.12, 3.13, and 3.14); (iii) ([6], 3.15).

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