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UNIFORM AND L_p APPROXIMATION FOR GENERALIZED INTEGRAL POLYNOMIALS

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Let X be a compact Hausdorff space and denote the space of all real valued continuous functions on X by $C(X, R)$. With pointwise operations this space becomes a linear space which we norm by defining $||f||=\sup_{x\in X}|f(x)|$ whenever $f\in C(X, R)$. A subset $\mathscr{F}$ of $C(X, R)$ is said to be point separating if for any two distinct points x and y in X there is an f in $\mathscr{F}$ with $f(x)\neq f(y)$. Let $Z[\mathscr{F}]$ denote the ring of all polynomials in elements of $\mathscr{F}$ which have integral coefficients. Thus an element q of $Z[\mathscr{F}]$ is an element of $C(X, R)$ with the special form

$$q=\sum_{j_1=0}^{r_1}\cdots\sum_{j_k=0}^{r_k}a_{j_1}\cdots{}_{j_k}f_k^{j_1}\cdots f_k^{j_k}$$

where the a's are integers, the f's belong to $\mathscr{F}$ and the r's are nonnegative integers. Such q are our integral polynomials. If X is a subset of n-dimensional Euclidean space and $\mathscr{F}$ is taken to be the set of n coordinate projections, then the elements of $Z[\mathscr{F}]$ are polynomials in the usual sense.

The well known Stone-Weierstrass theorem states that the set of all polynomials in elements of a point separating $\mathscr{F}$ are dense in $C(X, \boldsymbol{R})$. When the coefficients of the polynomials are constrained to be integers, the polynomials are no longer necessarily dense. The functions which can be approximated by elements of $\boldsymbol{Z}[\mathscr{F}]$ were characterized by Hewitt ond Zuckerman [3] as follows.

Let $B(X)$ denote the set of all q in $\boldsymbol{Z}[\mathscr{F}]$ with $||q||<1$ and set

$$J(X)=\{x\in X\colon q(x)=0 \quad \text{for all} \quad q\in B(X)\}\ .$$

Then their result reads as follows.

THEOREM 1. *An element $f\in C(X, \boldsymbol{R})$ is approximable by elements of $\boldsymbol{Z}[\mathscr{F}]$ if and only if*

$$(*)\qquad f(x)=q(x)\ ,\qquad x\in J(X)$$

for some q in $\boldsymbol{Z}[\mathscr{F}]$.

In this paper we establish two main results. The first is a version of Theorem 1 with approximation in L_p norms instead of the uniform norm. In the classical case, where X is a subset of the real line and $\mathscr{F}$ consists of the identity function, the "interpolatory" condition (*) no longer enters but a restriction on the "size" of X does. See the remarks following Theorem 2. The second result concerns the case where X is a product of subsets $\{X_\lambda\}_{\lambda \in \Lambda}$ of the real line and $\mathscr{F}$ is taken to be the set of coordinate projections. In this case

$$J(\prod_{\lambda \in \Lambda} X_\lambda) = \prod_{\lambda \in \Lambda} J(X_\lambda) .$$

See Theorem 6. Special cases of this result have been treated in Håstad [2] and Hewitt and Zuckerman [3].

It is possible to use Theorem 1 to obtain a result on approximation in L_p norms by polynomials with integral coefficients. Suppose that X is compact Hausdorff and μ is a finite regular positive Borel measure on X. We denote the space of real valued pth power integrable functions ($1 \leqq p < \infty$) on X by $L_p(\mu)$.

THEOREM 2. *With the above notation, if $J(X)$ is a μ-null set, then every f in $L_p(\mu)$ is approximable by elements of $\boldsymbol{Z}[\mathscr{F}]$.*

Before giving the proof we note that the hypothesis that $J(X)$ be μ-null is satisfied in many interesting cases. If $X \subset \boldsymbol{R}$, then we define its transfinite diameter $d(X)$ by

$$d(X) = \lim_{n \to \infty} \sup_{z_1, \cdots, z_n \in X} \Big(\prod_{1 \leqq i, j \leqq n} |z_i - z_j| \Big)^{2/n(n-1)} .$$

For a discussion of transfinite diameter and its properties, see Tsuji [5]. It can be shown (cf. Ferguson [1]) that if $X \subset \boldsymbol{R}$ and $d(X) < 1$, then $J(X)$ is finite, hence in the case where μ is Lebesgue measure and $d(X) < 1$, the hypothesis is satisfied. The transfinite diameter of an interval is 1/4 of its length, hence the hypothesis is satisfied whenever X is an interval length less than 4 and μ is Lebesgue measure.

Proof of Theorem 2. As a well known consequence of Lusin's theorem we know that $C(X, \boldsymbol{R})$ is dense in $L_p(\mu)$, hence, we assume without loss of generality that f is continuous. By the (outer) regularity of μ there are open sets

$$V_1 \supset V_2 \supset \cdots \supset J$$

such that $\mu(V_n) < 1/n$, all n, hence $\mu(\cap V_n) = 0$. For each n there is by Urysohn's lemma a ψ_n in $C(X, \boldsymbol{R})$ satisfying

$$0 \leqq \psi_n \leqq 1 ,$$

$$\operatorname{supp} \psi_n \subset V_n ,$$
$$\psi_n \equiv 1 \quad \text{on} \quad J .$$

Thus $\psi_n \to 0$ off $\cap V_n$, hence almost everywhere. Setting $\varphi_n = 1 - \psi_n$ we have $\varphi_n \to 1$ almost everywhere, hence $\varphi_n f \to f$ almost everywhere. By Theorem 1, for each n there exists a sequence $q_{n,m}$ in $Z[\mathscr{F}]$ such that

$$q_{n,m} \longrightarrow \varphi_n f \qquad \text{uniformly, } m \longrightarrow \infty .$$

It follows that $q_{n,n} \to f$ almost everywhere and that for sufficiently large n, $|q_{n,n}| \leqq |f| + 1$. By the Lebesgue dominated convergence theorem $q_{n,n} \to f$ in L_p-norm.

In Theorems 1 and 2 we see the crucial role which the set $J(X)$ plays in this approximation problem. In the special case where X is a product of subsets of the real line, we can write $J(X)$ as the product of the J's of the corresponding factors. Since the proof is long, we break it up into several stages of increasing generality.

The following result is a strengthening of Hewitt and Zuckerman [3, Thm. 6.7] and Håstad [2, Thm. 5] who prove it when the factor spaces are intervals.

When X is a product of subsets of $\boldsymbol{R}$, we take $\mathscr{F}$ to be the family of coordinate projections. Thus if

$$X = \prod_{i=1}^{n} X_i$$

then

$$\mathscr{F} = \{\pi_1, \cdots, \pi_n\}$$

where π_i, $1 \leqq i \leqq n$, is given by

$$\pi_i(x_1, \cdots, x_n) = x_i .$$

For an arbitrary infinite product $X = \prod_{\lambda \in \Lambda} X_\lambda$, we let $\mathscr{F} = \{\pi_\lambda\}_{\lambda \in \Lambda}$ where π_λ sends each x in X into its λth coordinate. Thus, at least in the finite case, the elements of $Z[\mathscr{F}]$ are the usual polynomials in n variables with integral coefficients.

THEOREM 3. *Let n be a positive integer and $X_1, \cdots, X_n$ be compact subsets of $\boldsymbol{R}$, each with transfinite diameter $d(X_i) < 1$. We let $X = \prod_{i=1}^{n} X_i$ and define $\mathscr{F}$ as above. Then*

$$J(\prod_{i=1}^{n} X_i) = \prod_{i=1}^{n} J(X_i) ,$$

where each $J(X_i)$ *is defined by taking the corresponding family* $\mathscr{F}$ *to consist simply of the identity function.*

Proof. We first need to know that for each i $(1 \leqq i \leqq n)$ there is a polynomial $q^{(i)}$ in a single variable with integral coefficients (i.e., $q^{(i)} \in \boldsymbol{Z}[t]$) such that $\|q^{(i)}\|_{X_i} = \sup_{t \in X_i} |q^{(i)}(t)| < 1$ and for t in X_i, $q^{(i)}(t) = 0$ if and only if $t \in J(X_i)$. We can see this as follows. If $B(X) = \{0\}$ (actually impossible here) then take $q^{(i)} \equiv 0$. Otherwise let $0 \not\equiv q_1 \in B(X)$. Then q_1 has at most finitely many zeros and for each of the elements $t_1, \cdots, t_k$ of $(Z_{q_1} \cap X_i) \backslash J(X_i)$ (where Z_{q_1} denotes the set of zeros of q_1), if such there be, let $q_2, \cdots, q_{k+1}$ be elements of $B(X)$ with $q_{i+1}(t_i) \neq 0$ $(1 \leqq i \leqq k)$. Then for a large enough positive even integer m we can take $q^{(i)}$ to be the polynomial

$$q_1^m + \cdots + q_{k+1}^m .$$

We remark that $q^{(i)} \geqq 0$ although we will not need this fact.

Since each $(q^{(i)} \cdot \pi_i) \in B(X)$ we have

$$J(X) \subset \bigcap_{i=1}^{n} \{x \in X \colon (q^{(i)} \cdot \pi_i)(x) = 0\} = \prod_{i=1}^{n} J(X_i) . \tag{1}$$

If some $J(X_i)$ is void, then by (1) so is $J(X)$ and then theorem holds. We therefore suppose that all $J(X_i)$ are nonvoid.

To establish the reverse inclusion in (1) we shall prove that if $f \in C(X, \boldsymbol{R})$ and f is approximable on X by elements of $\boldsymbol{Z}[\mathscr{F}]$, then f can be interpolated on $\prod_{i=1}^{n} J(X_i)$ by an element of $\boldsymbol{Z}[\mathscr{F}]$. The reverse inclusion in (1) is then a consequence of Theorem 1 as follows. We suppose that

$$y \in \left[\prod_{i=1}^{n} J(X_i)\right] \backslash J(X)$$

and derive a contradiction. Let γ be any transcendental real number. Clearly there exists a φ in $C(X, \boldsymbol{R})$ with $\varphi(J(X)) \subset \{0\}$ and $\varphi(y) = \gamma$. Then by Theorem 1, φ is approximable on X; hence, by what we are going to prove φ is interpolable on $\prod_{i=1}^{n} J(X_i)$. This is a contradiction since $y \in \prod_{i=1}^{n} J(X_i)$ and $q(y)$ is therefore algebraic for all q in $\boldsymbol{Z}[\mathscr{F}]$.

Now let f be an element of $C(X, \boldsymbol{R})$ which is approximable on X. We will be done once we show that f is interpolated on $\prod_{i=1}^{n} J(X_i)$ by an element of $\boldsymbol{Z}[\mathscr{F}]$. For $1 \leqq i \leqq n$ let $\alpha_i = |\inf X_i|$, $\beta_i = |\sup X_i|$ and

$$J(X_i) = \{u_{i,1}, \cdots, u_{i,r_i}\} .$$

(As we have mentioned before, the sets $J(X_i)$ are finite since

$d(X_i) < 1$ by hypothesis.) Let

$$\gamma = \max\{1, \alpha_1, \beta_1, \cdots, \alpha_n, \beta_n\}$$

and let ε be any positive number. Since f is approximable there exists a polynomial q_ε, with integral coefficients, in the functions $\pi_1, \cdots, \pi_n$ such that

$$\|f - q_\varepsilon\| < \varepsilon .$$

Thus for every point $(u_{1,j_1}, \cdots, u_{n,j_n})$ in $\prod_{i=1}^n J(X_i)$ we have

$$|f(u_{1,j_1}, \cdots, u_{n,j_n}) - q_\varepsilon(u_{1,j_1}, \cdots, u_{n,j_n})| < \varepsilon . \tag{2}$$

Let $\lambda_1, \cdots, \lambda_n$ be any n nonnegative integers. If in (2) we multiply through by $u_{1,j_1}^{\lambda_1}, \cdots, u_{n,j_n}^{\lambda_n}$ and sum over all possible value of j_i we have

$$\begin{aligned}\Bigg|\sum_{j_1=1}^{r_1} \cdots \sum_{j_n=1}^{r_n} u_{1,j_1}^{\lambda_1} \cdots u_{n,j_n}^{\lambda_n} f(u_{1,j_1}, \cdots, u_{n,j_n})\\ -\sum_{j_1=1}^{r_1} \cdots \sum_{j_n=1}^{r_n} u_{1,j_1}^{\lambda_1} \cdots u_{n,j_n}^{\lambda_n} q_\varepsilon(u_{1,j_1}, \cdots, u_{n,j_n})\Bigg| < \varepsilon\gamma^{\lambda_1+\cdots+\lambda_n} r_1 \cdots r_n .\end{aligned} \tag{3}$$

We claim that the second sum in (3) is an integer. Indeed, for each i $(1 \leqq i \leqq n)$, the set $\{u_{i,1}, \cdots, u_{i,r_i}\} = J(X_i) = J_0(X_i)$ where $J_0(X_i)$ is the union of all complete sets of conjugate algebraic integers which lie completely in X_i. See Ferguson [1] for the last equality. The minimal polynomial of each conjugate set has integral coefficients; hence the entire set $J_0(X_i)$ is the set of roots of a monic polynomial with integral coefficients. Thus the elementary symmetric polynomials in $\{u_{i,1}, \cdots, u_{i,r_i}\}$ are integers. Notice that each sum

$$\sum_{j_n=1}^{r_n} u_{n,j_n}^{\lambda_n} q_\varepsilon(u_{1,j_1}, \cdots, u_{n,j_n}) \tag{4}$$

can be viewed as a symmetric polynomial in the variables $\{u_{n,1}, \cdots, u_{n,r_n}\}$ with coefficients in $Z[u_{1,j_n}, \cdots, u_{n-1,j_{n-1}}]$. By the standard theorem on symmetric polynomials (cf, Jacobson [4, Thm. 9, p. 109]), it can be written as a polynomial in the elementary symmetric polynomials of $\{u_{n,1}, \cdots, u_{n,r_n}\}$ with coefficients in $Z[u_{1,j_1}, \cdots, u_{n-1,j_{n-1}}]$. Since, as we have seen, these elementary symmetric polynomials are integers, it is clear that the sum in (4) is of the form $q'(u_{1,j_1}, \cdots, u_{n-1,j_{n-1}})$ where q' is a polynomial with integral coefficients. Proceeding by induction we see that the entire second sum in (3) is an integer, as claimed. Next notice that the first sum in (3) is independent of ε hence is a limit of a sequence of integers, hence an integer itself. If we take ε to be small enough to make the right hand side of (3) less than unity, the two sums in (3) will be equal. Thus setting

$$\varepsilon = \frac{1}{2\gamma^{r_1+\cdots+r_n-n}r_1\cdots r_n}$$

forces

$$\sum_{j_1=1}^{r_1} u_{1 j_1}^{\lambda_1}\{\sum_{j_2=1}^{r_2}\cdots\sum_{j_n=1}^{r_n} u_{2 j_2}^{\lambda_2}\cdots u_{n j_n}^{\lambda_n}(f(u_{1,j_1},\cdots,u_{n,j_n}) -q_\varepsilon(u_{1,j_1},\cdots,u_{n,j_n}))\} = 0 \tag{5}$$

for $0 \leqq \lambda_i \leqq r_i - 1$ $(1 \leqq i \leqq n)$. Now we view (5) as a system of equations where the "unknowns" are the quantities within the braces and the coefficients are $u_{1 j_1}^{\lambda_1}$. The u_{1,j_1} $(1 \leqq j_1 \leqq r_1)$ are distinct by the way they were defined and $\det(u_{1 j_1}^{\lambda_1})$ $(1 \leqq j_1 \leqq r_1,\ 0 \leqq \lambda_1 \leqq r_1 - 1)$ is a Vandermonde determinant, hence nonzero. It follows that the quantities within the braces are all zero. Repeating this argument $n-1$ times, we obtain

$$f(u_{1,j_1},\cdots,u_{n,j_n}) = q_\varepsilon(u_{1,j_1},\cdots,u_{n,j_n})$$

for all $(u_{1,j_1},\cdots,u_{n,j_n})$ in $\prod_{i=1}^n J(X_i)$.

This theorem can be strengthened in two ways. We will first relax the condition that $d(X_i) < 1$, all i. Then we will also allow the factor spaces to be infinite in number. We need the following generalization of Håstad [2, Thm. 7] to infinite dimensions. It may be of interest in its own right.

THEOREM 4. *Let* $\{X_\lambda\}_{\lambda\in\Lambda}$ *be a family of compact subsets of* $\boldsymbol{R}$ *with* $d(X_\lambda) \geqq 1$ *for all* $\lambda \in \Lambda$ *and set* $X = \prod_{\lambda\in\Lambda} X_\lambda$. *We define the projections* $\{\pi_\lambda\}_{\lambda\in\Lambda}$ *and the family* $\mathscr{F}$ *as above. Then* $J(X) = X$, *hence an element of* $C(X, \boldsymbol{R})$ *is approximable if and only if it already lies in* $\boldsymbol{Z}[\mathscr{F}]$.

Proof. Suppose that $q \in B(X)$. Then q is a polynomial in only finitely many π_λ, say $\pi_{\lambda_1},\cdots,\pi_{\lambda_n}$. We must prove that $q \equiv 0$. We proceed by induction on n. If $n = 1$ then q is an element of $\boldsymbol{Z}[x_{\lambda_1}]$ with $\|q\|_{X_{\lambda_1}} < 1$. As we can see in Ferguson [1], since $d(X_{\lambda_1}) \geqq 1$ we have $q \equiv 0$ on X_{λ_1}. For $n > 1$ we assume that $q \not\equiv 0$ and derive a contradiction. Write

$$q(x) = q_m(\tilde{x})x_{\lambda_n}^m + q_{m-1}(\tilde{x})x_{\lambda_n}^{m-1} + \cdots + q_0(\tilde{x})$$

where $\tilde{x}$ is a generic element of $\prod_{i=1}^{n-1} X_{\lambda_i}$, $q_i \in \boldsymbol{Z}[x_{\lambda_1},\cdots,x_{\lambda_{n-1}}]\backslash\{0\}$ $(0 \leqq i \leqq m)$ and x_{λ_n} is a generic element of X_{λ_n}. For a fixed $\tilde{x}$, q is an element of $\boldsymbol{Z}[x_{\lambda_n}]$ with uniform norm <1; hence its leading coefficient $q_m(\tilde{x})$ satisfies $|q_m(\tilde{x})| < 1$. Indeed, if not then we could divide q by its leading coefficient and thereby obtain a monic polynomial on X_{λ_n} with uniform norm <1, but this contradicts $d(X_{\lambda_n}) \geqq 1$. Thus $\|q_m\|_{\prod_{i=1}^{n-1} X_{\lambda_i}} < 1$ and by the induction hypothesis $q_m \equiv 0$ which is a contradiction.

After the following, we have established the conclusion of Theorem 3 for all possible values of the $d(X_i)$. Here we use that $J(X_\lambda) = X_\lambda$ if $d(X_\lambda) \geqq 1$. This result generalizes Håstad [2, Thm. 6]. The present method of proof is different from his.

THEOREM 5. *If in Theorem 3 we relax the hypothesis $d(X_i)<1$, all i, to $d(X_i) < 1$ for at least one i, then the conclusion still holds.*

Proof. Let $X = \prod_{i=1}^n X_i$ and $\tilde{x} \in X \backslash \prod_{i=1}^n J(X_i)$. Then there exists i_0 such that $\tilde{x}_{i_0} \notin J(X_{i_0})$. Also $d(X_{i_0}) < 1$ since $d(X_i) \geqq 1$ implies $J(X_i) = X_i$. Without loss of generality we assume that $i_0 = 1$. Since $d(X_1) < 1$ there exists q in $\boldsymbol{Z}[t]$ with $\|q\|_{X_1} < 1$ and $J(X_1) = Z_q \cap X_1$ which can be established as in the beginning of the proof of Theorem 3. Then $\|q \cdot \pi_1\|_X < 1$ and $(q \cdot \pi_1)(\tilde{x}) \neq 0$ which shows that $\tilde{x} \in J(X)$. Thus we have established that

$$\prod_{i=1}^n J(X_i) \supset J(X) .$$

For the reverse inclusion let $q \in \boldsymbol{Z}[\mathscr{F}]$ with $\|q\|_X < 1$. Without loss of generality we assume that

$$d(X_i) < 1 \qquad 1 \leqq i \leqq k$$

and

$$d(X_i) \geqq 1 \qquad (k+1) \leqq i \leqq n .$$

Then as we have seen before, $J(X_i)$ is finite for $1 \leqq i \leqq k$ and we can write

$$J(X_i) = \{u_{i,1}, \cdots, u_{i,r_i}\} .$$

The elementary symmetric polynomials for the elements of $J(X_i)$ $(1 \leqq i \leqq k)$ are integers, as we saw in the proof of Theorem 3. Form the polynomial

$$(*) \qquad \sum_{j_1=1}^{r_1} \cdots \sum_{j_k=1}^{r_k} q^{2l}(u_{1,j_1}, \cdots, u_{k,j_k}, x_{k+1}, \cdots, x_n) .$$

For a sufficiently large positive integer l, this polynomial will have uniform norm <1 on X since $\|q\|_X < 1$. For each $i(1 \leqq i \leqq k)$ it is clearly a symmetric polynomial in $\{u_{i,1}, \cdots, u_{i,r_i}\}$ hence by the same induction argument as in the proof of Theorem 3 it is actually a polynomial $\tilde{q}$ in $\boldsymbol{Z}[x_{k+1}, \cdots, x_n]$. But

$$\|\tilde{q}\|_{\prod_{i=k+1}^{n} X_i} < 1$$

hence by Theorem 4 we have $\tilde{q} \equiv 0$. Since every summand in (*) is nonnegative, this implies that

$$q(u_{1,j_1}, \cdots, u_{k,j_k}, x_{k+1}, \cdots, x_n) = 0$$

for every possible choice of $j_1, \cdots, j_k$ and $x_{k+1}, \cdots, x_n$. Thus

$$J(X) \supset \prod_{i=1}^{k} J(X_i) \times \prod_{i=k+1}^{n} X_i = \prod_{i=1}^{n} J(X_i)$$

where the last equality follows from the fact that $d(X_i) \geqq 1$ implies $J(X_i) = X_i$.

The following generalizes Hewitt and Zuckerman [3, Thm. 6.8] who proved it when the factor spaces are intervals of length <4. Our proof is a modification of theirs.

THEOREM 6. *Let* $\{X_\lambda\}_{\lambda \in \Lambda}$ *be a family of compact subsets of* $\boldsymbol{R}$ *and define* $\mathscr{F}$ *as above. Then*

$$J(\prod_{\lambda \in \Lambda} X_\lambda) = \prod_{\lambda \in \Lambda} J(X_\lambda)\ . \tag{1}$$

Proof. Let $X = \prod_{\lambda \in \Lambda} X_\lambda$ and $J_\lambda = J(X_\lambda)$, $\lambda \in \Lambda$. Suppose first that some J_λ is empty, say for $\lambda = \tilde{\lambda}$. Then as at the beginning of the proof of Theorem 3, it is easy to see that there is a q in $\boldsymbol{Z}[t]$ such that $\|q\|_{X_{\tilde{\lambda}}} < 1$ and $q > 0$ on $X_{\tilde{\lambda}}$. Then $(q \cdot \pi_{\tilde{\lambda}}) \in \boldsymbol{Z}[\mathscr{F}]$, $\|q \cdot \pi_{\tilde{\lambda}}\|_X < 1$ and $(q \cdot \pi_{\tilde{\lambda}}) > 0$ on X which shows that both sides of (1) are empty hence equal. Thus we can assume that every J_λ is nonvoid. Select an element u_λ from each J_λ.

Given a finite subset Φ of Λ, let L_Φ be the set of all x in X such that $x_\lambda = \mu_\lambda$ for $\lambda \notin \Phi$. Let $K_\Phi = L_\Phi \cap (\prod_{\lambda \in \Lambda} J_\lambda)$.

Suppose that $f \in C(X, \boldsymbol{R})$ and f is approximable on X. Then for every $\varepsilon > 0$ there is a q_ε in $\boldsymbol{Z}[\mathscr{F}]$ such that $\|q_\varepsilon - f\|_X < \varepsilon$. Let Λ_0 be the set of indices of the variables occurring in q_ε. Let Φ be any finite subset of Λ and $S' = \Phi \cup \Lambda_0$. We define a function f' in the variables $\{x_\lambda\}_{\lambda \in S'}$ as follows. For x' in $X_0 = \pi_{\lambda \in S} X_\lambda$ let $f'(x') = f(x)$ where $x_\lambda = x'_\lambda$ if $\lambda \in \Phi$ and $x_\lambda = u_\lambda$ if $\lambda \notin \Phi$. Write $S = \{\lambda_1, \cdots, \lambda_n, \cdots, \lambda_k\}$ where $d(X_{\lambda_i}) < 1$ for $1 \leqq i \leqq n$ and $d(X_{\lambda_i}) \geqq 1$ for

$(n+1) \leqq i \leqq k$. As in the proof of Theorem 3, we set

$$J(X_{\lambda_i}) = \{u_{i,1}, \cdots, u_{i,r_i}\} \qquad 1 \leqq i \leqq n .$$

Continuing as in that proof we have

$$(*) \qquad \begin{aligned} &|\sum u_{1}^{\lambda_1}{}_{j_1} \cdots u_{n}^{\lambda_n}{}_{j_n} f'(u_{1,j_1}, \cdots, u_{n,j_n}, x_{n+1}, \cdots, x_k) \\ &\quad - \sum u_{1}^{\lambda_1}{}_{j_1} \cdots u_{n}^{\lambda_n}{}_{j_n} q'_\varepsilon(u_{1,j_1}, \cdots, u_{n,j_n}, x_{n+1}, \cdots, x_k)| \\ &\quad < \varepsilon \gamma^{\lambda_1 + \cdots + \lambda_n} r_1 \cdots r_n \end{aligned}$$

where $x_{n+1}, \cdots, x_k$ are generic elements of $X_{\lambda_{n+1}}, \cdots, X_{\lambda_k}$ (respectively), the λ_i's are arbitrary nonnegative integers and the summations are over all possible j_i's as before. By essentially the same induction argument about symmetric polynomials as before, we see that the second sum in (*) is an element of $\boldsymbol{Z}[x_{n+1}, \cdots, x_k]$, say $\hat{q}_\varepsilon$. For all sufficiently small ε, if q'_ε and $\tilde{q}'_\varepsilon$ satisfy (*), then the corresponding elements $\tilde{q}'_\varepsilon$ and $\hat{\tilde{q}}'_\varepsilon$ of $\boldsymbol{Z}[x_{n+1}, \cdots, x_k]$ satisfy

$$\| \tilde{q}'_\varepsilon - \hat{\tilde{q}}'_\varepsilon \| < 1$$

where the norm is taken over $\tilde{X} = \prod_{i=n+1}^{k} X_{\lambda_i}$. By Theorem 4, $J(\tilde{X}) = \tilde{X}$ hence $q'_\varepsilon - \hat{\tilde{q}}'_\varepsilon \equiv 0$ on $\tilde{X}$. Thus the first sum in (*) equals the second sum for any choice of $x_{n+1}, \cdots, x_k$ and sufficiently small ε. Continuing the argument as in the proof of Theorem 3, we finally conclude that

$$f' \equiv q'_\varepsilon \quad \text{on} \quad \prod_{i=1}^{k} J(X_{\lambda_i}) \times \prod_{i=n+1}^{k} X_{\lambda_i} ,$$

i.e.,

$$f' \equiv q'_\varepsilon \quad \text{on} \quad \prod_{\lambda \in S} J_\lambda$$

since $X_{\lambda_i} = J_{\lambda_i}$, $(n+1) \leqq i \leqq k$. Thus

$$f(x) = q_\varepsilon(x) \qquad x \in K_S$$

and, in particular, f is interpolated on K_Φ.

We claim that $K_\Phi \subset J(X)$. Indeed, if not, then there is a point y in $K_\Phi \backslash J(X)$. Let γ be any transcendental number. There is an element φ of $C(X, \boldsymbol{R})$with $\varphi(y) = \gamma$ and $\varphi(J(X)) = \{0\}$, by Tietze's extension theorem. Such a φ is approximable by Theorem 1. By the above paragraph, φ is interpolated by an element of $\boldsymbol{Z}[\mathscr{F}]$ on K_Φ but this is a contradiction since $q(y)$ is an algebraic number for every q in $\boldsymbol{Z}[\mathscr{F}]$. This contradiction establishes the claim.

Thus we have

$$\prod_{\lambda \in \Lambda} J_\lambda = (\cup K_\Phi)^{-} \subset J(X)$$

where the union is taken over all finite subsets Φ of Λ. The reverse inclusion is easily established as in the proof of the previous theorem.

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