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SUPER-REGULAR SEQUENCES

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Let $(R, \underline{m})$ be a local ring with associated graded ring $grR = R/\underline{m} \oplus \underline{m}/\underline{m}^2 \oplus \underline{m}^2/\underline{m}^3 \oplus \cdots$. This paper deals with the problem of finding properties of R which lead to good properties in grR. There are two main results in this paper which give techniques for recognizing when the maximal homogeneous ideal of grR contains regular elements. Applications of these results give examples of local Cohen-Macaulay rings which have Cohen-Macaulay associated graded rings.

Let $(R, \underline{m})$ be a local ring with associated graded ring $grR = R/\underline{m} \oplus \underline{m}/\underline{m}^2 \oplus \underline{m}^2/\underline{m}^3 \oplus \cdots$. Information about grR gives some measure of the singularity at R since properties of grR yield data about the Hilbert function of R and about monoidal transforms of R. However it is often difficult to compute grR and it is seldom true that properties of R are carried over to grR. Thus, it is important to recognize characteristics of R which lead to good properties in grR.

There are two main results in this paper which give techniques for recognizing good properties in grR. The first shows when the initial forms of a regular sequence in $\underline{m}/\underline{m}^2$ form a regular sequence in grR; the given regular sequence is then called super-regular. (All our applications use homogeneous regular sequences of degree one in grR, so we have avoided some complications by primarily considering this case. If $R/\underline{m}$ is infinite and if $gr\underline{m}$ contains a regular sequence of length t, it contains a homogeneous regular sequence of degree one of length t.) The second result shows that for certain local Cohen-Macaulay rings the question of whether grR is Cohen-Macaulay can be reduced to the same question for such Cohen-Macaulay rings of dimension one. The paper concludes with several applications of these results. The applications give examples of local Cohen-Macaulay rings which have Cohen-Macaulay associated graded rings.

We begin with some definitions. For any nonzero element x in the local ring $(R, \underline{m})$, *let* $\bar{x}$ denote the initial form of x in grR, i.e., if $x \in \underline{m}^s \backslash \underline{m}^{s+1}$, $\bar{x} = x + \underline{m}^{s+1} \in \underline{m}^s/\underline{m}^{s+1}$. We will say that x has order s and that $\bar{x}$ has degree s. Let $\bar{0}$ be the zero element in grR. 0 has infinite order and $\bar{0}$ has infinite degree. A system of elements of R (of grR) has order s (degree s) if each element has order s (degree s).

Recall that if S is any commutative ring, a sequence of elements $x_1, \cdots, x_t$ of S is a regular sequence if $(x_1, \cdots, x_t)S \neq S$ and if, for $i = 1, \cdots, t$, x_i is not a zero divisor on $S/(x_1, \cdots, x_{i-1})S$.

For the case of a local ring $(R, \underline{m})$ it is important to know when $gr\underline{m}$, the maximal homogeneous ideal of grR, contains a regular element. For the existence of a regular element in $gr\underline{m}$ permits a change of rings which reduces dimension. This is described in the following lemma.

LEMMA 0.1. *Let $(R, \underline{m})$ be a local ring with associated graded ring grR. Suppose that $gr\underline{m}$, the maximal homogeneous ideal of grR, contains a regular element. Then $gr\underline{m}$ contains a homogeneous regular element $\bar{x}$ and, if x is any element of R with initial form $\bar{x}$, there is an isomorphism of graded rings $grR/\bar{x}grR \cong gr(R/xR)$ induced by the natural map of local rings $R \rightarrow R/xR$.*

Proof. By [8, Ch. VII, §2], the prime ideals $P_1, \cdots, P_h$ in Ass grR are homogeneous. By [8, pg. 286, footnote], $gr\underline{m} \nsubseteq \cup P_i$ implies that there is a homogeneous element $\bar{x}$ of positive degree s, say, such that $\bar{x} \notin P_i$ for $i = 1, \cdots, h$. Let x be any element of R with initial form $\bar{x}$. Consider the natural homomorphism $\nu: R \rightarrow R/xR$. Since $\nu(m^n) \subseteq (m/xR)^n$, the induced homomorphism $gr\nu: grR \rightarrow gr(R/xR)$ given by $(gr\nu)_n: \underline{m}^n/\underline{m}^{n+1} \rightarrow (\underline{m}/xR)^n/(\underline{m}/xR)^{n+1}$ is a homomorphism of graded rings, cf. [1, Ch. 3, §2, no. 4]. To show that $gr\nu$ is surjective with kernel $\bar{x}grR$, it is enough to show that $(gr\nu)_n$ is surjective with kernel $(\bar{x}grR)_n$. It is clear that $(gr\nu)_n$ is surjective. If $\bar{w} \in \underline{m}^n/\underline{m}^{n+1}$ and $(gr\nu)_n\bar{w} = 0$, then $w \in (\underline{m}^{n+1} + xR) \cap \underline{m}^n = \underline{m}^{n+1} + (xR \cap \underline{m}^n)$ for any $w \in \bar{w}$. Since $\bar{x}$ is regular in grR, $xR \cap \underline{m}^n = x\underline{m}^{n-s}$, where a nonpositive power of $\underline{m}$ is understood to be R. Thus $w = z + xy$ with $z \in \underline{m}^{n+1}$ and $y \in \underline{m}^{n-s}$. Again using that $\bar{x}$ is regular in grR, we have that $\bar{x}\bar{y} = \overline{xy} = \overline{z + xy} = \bar{w}$. This proves that $\bar{w} \in \bar{x}grR$ and completes the proof of the lemma.

We will say that an element x in $\underline{m}$, the maximal ideal of a local ring $(R, \underline{m})$, is *super-regular* if $\bar{x}$ is a regular element in grR. A sequence of elements $x_1, \cdots, x_t$ in R will be called a *super-regular sequence* if $\bar{x}_1, \cdots, \bar{x}_t$ is a regular sequence in grR.

Several properties of super-regular sequences follow easily. A super-regular sequence $x_1, \cdots, x_t$ is a regular sequence. For if $\bar{x}_1$ is a regular element in $gr\underline{m}$, x_1 is a regular element in R and, by (0.1), $gr(R/x_1R) \cong grR/\bar{x}_1grR$. By induction, the images of $x_2, \cdots, x_t$ form a regular sequence in R/x_1R so $x_1, x_2, \cdots, x_t$ is a regular sequence in R. A similar isomorphism shows that a sequence $x_1, \cdots, x_t$ of elements of R is super-regular if and only if there is some j, $1 \leqq j \leqq t$, such that $x_1, \cdots, x_j$ is super-regular in R and the image of $x_{j+1}, \cdots, x_t$ is super-regular in $R/(x_1, \cdots, x_j)R$. It is also true that any permutation of a super-regular sequence is super-regular.

In addition to the notation and definitions introduced above, the following notation will be used. $\lambda(A) = \lambda_R(A)$ denotes the length of an R-module A. If I is an ideal in a local ring $(R, \underline{m})$, $v(I)$ denotes the number of generators in a minimal basis of I. $v(\underline{m})$ is the embedding dimension of R. $e = e(R)$ is the multiplicity of R. We will often use $\underline{x}$ to denote a system of elements $x_1, \cdots, x_t$ in R.

Let $\underline{x} = x_1, \cdots, x_t$ be a regular sequence in $(R, \underline{m})$. If we say that $f(x_1, \cdots, x_t)$ is a form of degree s in $x_1, \cdots, x_t$, we mean that $f(X_1, \cdots, X_t)$ is a degree s homogeneous polynomial in the polynomial ring $R[X_1, \cdots, X_t]$ and $f(x_1, \cdots, x_d)$ is the image of $f(X_1, \cdots, X_t)$ in R under the homomorphism sending X_i to x_i.

We will use the notion of minimal reduction of the maximal ideal of a local ring $(R, \underline{m})$, cf. [3], but we wish to take a more restrictive definition than in [3]. If $(R, \underline{m})$ is a d-dimensional local ring, a system $\underline{x} = x_1, \cdots, x_d$ of d elements of R is a *minimal reduction* of $\underline{m}$ if there is a positive integer r such that $\underline{m}^{r+1} = \underline{x}\underline{m}^r$. If $R/\underline{m}$ is infinite, minimal reductions exist, as a minimal reduction of $\underline{m}$ is the preimage in R of a degree one homogeneous system of parameters in grR. The existence of a minimal reduction is hardly ever a troublesome hypothesis because the change of rings $R \to R(U) = R[U]_{\underline{m}R[U]}$, U an indeterminate, is faithfully flat and $\underline{m}R(U)$ has minimal reductions.

1. When are regular sequences super-regular? If x is a regular element of order one in the local ring $(R, \underline{m})$, it is clear that x is super-regular if and only if $(x) \cap \underline{m}^{i+1} = x\underline{m}^i$ for all $i \geqq 0$. The assumption that x is regular cannot be dropped as the example $k[[x, y]]/(y^2, xy)$, k a field, shows. An analogous characterization holds for a regular sequence $\underline{x} = x_1, \cdots, x_t$ of order one. It will be useful to prove a little more. We will see that l intersection equalities $(\underline{x}) \cap \underline{m}^{i+1} = \underline{x}\underline{m}^i$ for $i \leqq l$ means that $\underline{x}$ is super-regular "up to $\underline{m}^{l+1}$."

THEOREM 1.1. *Let $(R, \underline{m})$ be a local ring and let $\underline{x} = x_1, \cdots, x_t$ be a regular sequence of order one. Then,*

$$(\underline{x}) \cap \underline{m}^{i+1} = \underline{x}\underline{m}^i ,$$

for all positive integers $i \leqq$ some positive integer l if and only if

$$((\bar{x}_1, \cdots, \bar{x}_{j-1}) \colon \bar{x}_j) \subseteqq (\bar{x}_1, \cdots, \bar{x}_{j-1}) + (gr\underline{m})^l$$

for $1 \leqq j \leqq t$.

We first note the following.

LEMMA 1.2. *Let $(R, \underline{m})$ be a local ring and let $\underline{x} = x_1, \cdots, x_t$ be a regular sequence of order one. If for all positive integers $i \leqq$ some positive integer l, $(\underline{x}) \cap \underline{m}^{i+1} = \underline{x}\underline{m}^i$, then*

$$(\underline{x})^j \cap \underline{m}^{i+1} = (\underline{x})^j \underline{m}^{i+1-j} ,$$

for $1 \leqq j \leqq i + 1$ and $i \leqq l$.

Proof. We prove the lemma by induction on j. Let $j>1$. Let $f_j(x_1, \cdots x_t)$, a form of degree j in $x_1, \cdots, x_t$, be in $(\underline{x})^j \cap \underline{m}^{i+1} \subset (\underline{x})^{j-1} \cap \underline{m}^{i+1} = \underline{x}^{j-1}\underline{m}^{i+1-j+1}$. $f_j(x_1, \cdots, x_t) = g_{j-1}(x_1, \cdots, x_t)$, where g_{j-1} is a form of degree $j - 1$ in $x_1, \cdots, x_t$ with coefficients in $m^{i+1-j+1}$. Since $(x_1, \cdots, x_t)^{j-1}/(x_1, \cdots, x_t)^j$ is free over $R/(x_1, \cdots, x_t)R$, the coefficients of g_{j-1} are in $(\underline{x}) \cap \underline{m}^{i+1-j+1} = \underline{x}\underline{m}^{i+1-j}$, so $f_j(x_1, \cdots, x_t) \in (\underline{x})\underline{m}^{ji+1-j}$.

Proof of Theorem 1.1. Assume that $(\underline{x}) \cap \underline{m}^{i+1} = \underline{x}\underline{m}^i$ for $i \leqq l$. We will prove that $((\bar{x}_1, \cdots, \bar{x}_{j-1}): \bar{x}_i) \subseteqq (\bar{x}_1, \cdots, \bar{x}_{j-1}) + (gr\underline{m})^l$ for $1 \leqq j \leqq t$, by induction on t. The proof for $t = 1$ is immediate so we will assume that $t > 1$. First, we show that $(0: \bar{x}_1) \subseteqq (gr\underline{m})^l$, i.e., that $x_1 z \in \underline{m}^{i+1}$ implies that $z \in \underline{m}^i$ for $i \leqq l$. If $x_1 z \in \underline{m}^2$, then $x_1 z = x_1 a_1 + \cdots + x_t a_t$ with $a_1, \cdots, a_t \in \underline{m}$. Thus $z - a_1 \in (x_2, \cdots, x_t)$ and $z \in \underline{m}$. We claim that the following relation holds:

$$(*) \qquad (m^{i+1}: x_1) \cap (\underline{x})^j \underline{m} \subseteqq (\underline{x})^j \underline{m}^{i+1-j-1} + (\underline{x})^{j+1}\underline{m} ,$$

for $2 \leqq i \leqq l$ and $0 \leqq j \leqq i - 2$. Suppose (*) has been proved. Let $x_1 z_0 \in \underline{m}^{i+1}$ for $i \geqq 2$. Then, with $j = 0$ in (*), we get $z_0 \in \underline{m}^i + \underline{x}\underline{m}$. $z_0 = \mu_i + z_1$ with $\mu_i \in \underline{m}^i$ and $z_1 \in (\underline{m}^{i+1}: x_1) \cap \underline{x}\underline{m}$. Now we apply (*) with $j = 1$ and continue in this way until we have $z_0 = \tilde{\mu}_i + z_{i-2}$ with $\tilde{\mu}_i \in \underline{m}^i$ and $z_{i-2} \in (\underline{m}^{i+1}: x_1) \cap (\underline{x})^{i-2}\underline{m}$. Finally, we apply (*) with $j = i - 2$, to get $z_{i-2} \in (\underline{x})^{i-2}\underline{m}^2 + (\underline{x})^{i-1}\underline{m} \subseteqq \underline{m}^i$.

Now we prove (*). Let $j = 0$. Let $x_1 z \in \underline{m}^{i+1}$ with $i \geqq 2$. $x_1 z \in \underline{m}^{i+1} \cap (\underline{x}) = \underline{x}\underline{m}^i$, so $x_1 z = a_1 x_1 + \cdots + a_t x_t$ with $a_1, \cdots, a_t \in \underline{m}^i$. $z - a_1 \in (x_2, \cdots, x_t)$ since $x_1, \cdots, x_t$ is a regular sequence. In fact, $z - a_1 \in (x_2, \cdots, x_t)\underline{m}$. For $(z - a_1)x_1 \in (x_2, \cdots, x_t)x_1 \cap \underline{m}^{i+1} \subseteqq (\underline{x})^2 \cap \underline{m}^{i+1} = (\underline{x})^2\underline{m}^{i-1}$ by (1.2).

Assume that $j > 0$. Let $z \in (\underline{m}^{i+1}: x_1) \cap (\underline{x})^j\underline{m}$. $z = g_j(x_1, \cdots, x_t)$ is a homogeneous polynomial in $x_1, \cdots, x_t$ of degree j with coefficients in $\underline{m}$. $zx_1 \in \underline{m}^{i+1} \cap (\underline{x})^{j+1} = (\underline{x})^{j+1}\underline{m}^{i+1-j-1}$ by (1.2), so

$$(**) \qquad zx_1 = g_j(x_1, \cdots, x_t)x_1 = h_{j+1}(x_1, \cdots, x_t)$$

with h_{j+1} a homogeneous polynomial in $x_1, \cdots, x_t$ of degree $j + 1$

and coefficients in $\underline{m}^{i+1-j-1}$. Equating coefficients of like monomials of degree $j+1$ in (**), we see that the coefficients of $g_j(x_1, \cdots, x_t)$ are in $\underline{m}^{i+1-j-1} + (\underline{x})$. Thus $g_j(x_1, \cdots, x_t) = \xi + f_{j+1}(x_1, \cdots, x_t)$ with $\xi \in (\underline{x})^j \underline{m}^{i+1-j-1}$ and $f_{j+1}(x_1, \cdots, x_t) \in (\underline{x})^{j+1}$. But $x_1(\underline{x})^{j+1} \cap \underline{m}^{i+1} \subseteqq (\underline{x})^{j+2} \cap \underline{m}^{i+1} = (\underline{x})^{j+2}\underline{m}^{i+1-j-2}$ by (1.2). Consequently, $(g_j(x_1, \cdots, x_t) - \xi)x_1 = f_{j+1}(x_1, \cdots, x_t)x_1 \in (\underline{x})^{j+2}\underline{m}^{i+1-j-2}$ and $f_{j+1}(x_1, \cdots, x_t) \in (\underline{x})^{j+1}\underline{m}$. Thus $g_j(x_1, \cdots, x_t) \in (\underline{x})^j\underline{m}^{i+1-j-1} + (\underline{x})^{j+1}\underline{m}$. This completes the proof that $(0\colon \bar{x}_1) \subseteqq (gr\underline{m})^l$.

Pass to $(\tilde{R}, \tilde{\underline{m}}) = (R/x_1R, \underline{m}/x_1R)$. The images $\tilde{x}_2, \cdots, \tilde{x}_t$ form a regular sequence in $\tilde{R}$ and $\tilde{\underline{m}}^{i+1} \cap (\tilde{x}_2, \cdots, \tilde{x}_t)\tilde{R} = (\tilde{x}_2, \cdots, \tilde{x}_t)\tilde{\underline{m}}^i$ for $i \leqq l$. By induction on t, we have that $((\bar{\tilde{x}}_2, \cdots, \bar{\tilde{x}}_{j-1})\colon \bar{\tilde{x}}_j) \subseteqq (\bar{\tilde{x}}_2, \cdots, \bar{\tilde{x}}_{j-1}) + (gr\tilde{\underline{m}})^l \cong (\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_{j-1})grR/\bar{x}_1grR + ((gr\underline{m})^l, \bar{x}_1)grR/\bar{x}_1grR$. Thus, $((\bar{x}_1, \cdots, \bar{x}_{j-1})\colon \bar{x}_j) \subseteqq (\bar{x}_1, \cdots, \bar{x}_{j-1}) + (gr\underline{m})^l$.

For the converse we assume that $x_1, \cdots, x_t$ is a regular sequence in R with $((\bar{x}_1, \cdots, \bar{x}_{j-1})\colon \bar{x}_j) \subseteqq (\bar{x}_1, \cdots, \bar{x}_{j-1}) + (gr\underline{m})^l$ for $1 \leqq j \leqq t$ and again use induction on t. We have $(0\colon \bar{x}_1) \subseteqq (gr\underline{m})^l$ so $(x_1) \cap \underline{m}^{i+1} = x_1\underline{m}^i$ for $i \leqq l$. The proof is finished if $t = 1$ so we take $t > 1$. We have $(grR/\bar{x}_1grR)_j \cong (gr(R/x_1R))_j$ for $j = 0, \cdots, l+1$ since

$$(grR/\bar{x}_1grR)_j = \underline{m}^j/x_1\underline{m}^{j-1} + \underline{m}^{j+1}$$

and $(gr(R/x_1R))_j = (\underline{m}^j, x_1)/(\underline{m}^{j+1}, x_1) \cong \underline{m}^j/\underline{m}^{j+1} + (x_1) \cap \underline{m}^j$. This means that the required hypotheses are satisfied by $\tilde{x}_2, \cdots, \tilde{x}_t$, the images of $x_2, \cdots, x_t$ in $(\tilde{R}, \tilde{m}) = (R/x_1R, \underline{m}/x_1R)$. By induction on t we have $(\tilde{x}_2, \cdots, \tilde{x}_t) \cap \tilde{\underline{m}}^{i+1} = (\tilde{x}_2, \cdots, \tilde{x}_t)\tilde{\underline{m}}^j$ for $i \leqq l$, so that $(x_1, \cdots, x_t) \cap (\underline{m}^{i+1}, x_1) = (x_1, \cdots, x_t)\underline{m}^i + x_1R$. Let $w \in (x_1, \cdots, x_t) \cap \underline{m}^{i+1}$. $w = ax_1 + \mu_2x_2 + \cdots + \mu_tx_t$ with $\mu_2, \cdots, \mu_t \in \underline{m}^i$. Then $ax_1 \in \underline{m}^{i+1}$ so $a \in \underline{m}^i$ and $w \in (x_1, \cdots, x_t)\underline{m}^i$.

COROLLARY 1.3. *Let $\underline{x} = x_1, \cdots, x_t$ be a regular sequence in a local ring $(R, \underline{m})$ such that $(x_1, \cdots, x_t) \cap \underline{m}^{i+1} = (x_1, \cdots, x_t)\underline{m}^i$ for $i \leqq l$. Then $(x_1, \cdots, x_s) \cap \underline{m}^{i+1} = (x_1, \cdots, x_s)\underline{m}^i$ for $1 \leqq s \leqq t$ and $0 \leqq i \leqq l$.*

COROLLARY 1.4.[1] *Let $(R, \underline{m})$ be a local ring and let $\underline{x} = x_1, \cdots, x_t$ be a regular sequence. $x_1, \cdots, x_t$ is super-regular if and only if $(\underline{x}) \cap \underline{m}^{i+1} = \underline{x}\underline{m}^i$ for all $i \geqq 0$.*

Proof. If $(\underline{x}) \cap \underline{m}^{i+1} = \underline{x}\underline{m}^i$ for all $i \geqq 0$, then, by (1.1),

$$((\bar{x}_1, \cdots, \bar{x}_{j-1})\colon \bar{x}_j) \subseteqq \bigcap_{l=0}^{\infty} (\bar{x}_1, \cdots, \bar{x}_{j-1}) + (gr\underline{m})^l = (\bar{x}_1, \cdots, \bar{x}_{j-1}) .$$

The converse also follows immediately from (1.1).

[1] *Added in proof.* Corollary 1.4 is a special case of Corollary 2.7 of the paper "Form rings and regular sequences" by P. Valabrega and G. Valla which recently appeared in Nagaya Math. J., **72** (1978), 93-101.

COROLLARY 1.5. *Let $(R, \underline{m})$ be a d-dimensional local Cohen-Macaulay ring. grR is Cohen-Macaulay if there is a minimal reduction $\underline{x} = x_1, \cdots, x_d$ of $\underline{m}$ such that $(\underline{x}) \cap \underline{m}^{i+1} = \underline{x}\underline{m}^i$ for all $i \geqq 0$.*

Proof. By (1.4), $\bar{x}_1, \cdots, \bar{x}_d$ is a homogeneous system of parameters which is a regular sequence. By [2], grR is Cohen-Macaulay.

REMARK. (1.5) is similar to a special case of Theorem 4.4 in [2].

We take a moment to note the connection between super-regularity and the notion of analytic independence in $\underline{m}$. Recall, [3], that a system of elements $\underline{x} = x_1, \cdots, x_t$ in a local ring $(R, \underline{m})$ is analytically independent in $\underline{m}$ if given $f_s(X_1, \cdots, X_t)$, a homogeneous polynomial in $R[X_1, \cdots, X_t]$ of (arbitrary) degree s, such that $f_s(x_1, \cdots, x_t) \in \underline{m}^{s+1}$ then all the coefficients of f_s are in $\underline{m}$. Stated differently, $\underline{x} = x_1, \cdots, x_t$ is analytically independent in $\underline{m}$ if $(x_1, \cdots, x_t)^s \cap \underline{m}^{s+1} = (x_1, \cdots, x_t)^s m$ for all positive integers s. Note also that if a regular sequence $\underline{x} = x_1, \cdots, x_t$ in $(R, \underline{m})$ satisfies $(\underline{x}) \cap \underline{m}^{i+1} = \underline{x}\underline{m}^i$ for $i \leqq l$ and if $f_s(X_1, \cdots, X_t)$ is a form of degree s in $R[X_1, \cdots, X_t]$ with $f_s(x_1, \cdots, x_t) \in \underline{m}^{i+1}$ for $i \leqq l$, then all one can say about the coefficients of f_s is that they are in $(\underline{m}^{i+1-s}, x_1, \cdots, x_t)$ as the example $f(X_1, X_2) = x_2X_1 - x_1X_2$ shows.

Next we give a direct proof that intersection equalities as in (1.1) give information about minimal bases for powers of $\underline{m}$. This can also be done using the Hibert sum transform as in [7]. The idea is that $(\underline{x}) \cap \underline{m}^{i+1} = \underline{x}\underline{m}^i$ for $i \leqq l$ implies that multiplication of $\underline{m}^i/\underline{m}^{i+1}$ by each $\bar{x}_j$, for x_j in the regular sequence $\underline{x}$, is a monomorphism for $i \leqq l$.

We introduce some notation. Let $\underline{x} = x_1, \cdots, x_t$ be a regular sequence in the local ring $(R, \underline{m})$. Let $\mathscr{H}_i(\underline{x})$ denote the set of monomials of degree i in $x_1, \cdots, x_t$. If J is a set of elements of R, $\mathscr{H}_i(\underline{x})J$ denotes the set of products $\{\xi j \mid \xi \in \mathscr{H}_i(\underline{x}), j \in J\}$.

THEOREM 1.6. *Let $(R, \underline{m})$ be a local ring and let $\underline{x} = x_1, \cdots, x_t$ be a regular sequence such that $x_1, \cdots, x_t, z_{t+1}, \cdots, z_v$ is a minimal basis for $\underline{m}$. Suppose that $(\underline{x}) \cap \underline{m}^{i+1} = \underline{x}\underline{m}^i$ for $i \leqq l$. Then, for each $i \leqq l$, the set $\mathscr{H}_i(\underline{x}) \cup \mathscr{H}_{i-1}(\underline{x})\mathscr{Z}_1 \cup \cdots \cup \mathscr{H}_1(\underline{x})\mathscr{Z}_{i-1}$ is a subset of a minimal basis for $\underline{m}^i$, where $\mathscr{Z}_j$ is the set of monomials in $z_{t+1}, \cdots, z_v$ needed to complete the set $\mathscr{H}_j(\underline{x}) \cup \cdots \mathscr{H}_1(\underline{x})\mathscr{Z}_{j-1}$ to a minimal basis of $\underline{m}^j$.*

Proof. We prove the theorem by induction on i and t. We may assume $i > 1$. Let $t = 1$. If

$$ax_1^i + bx_1^{i-1}\xi_{11} + \cdots + cx_1\xi_{i-1n_{i-1}} \in \underline{m}^{i+1}$$

where ξ_{pq}, $1 \leqq p \leqq i-1$, $1 \leqq q \leqq n_p$, are the monomials $z_{t+1}, \cdots, z_v$ completing the basis for $\underline{m}^p$, then $x_1(ax_1^{i-1} + bx_1^{i-2}\xi_{11} + \cdots + c\xi_{i-1n_{i-1}}) \in \underline{m}^{i+1}$. Thus, $ax_1^{i-1} + bx_1^{i-2}\xi_{11} + \cdots + c\xi_{i-1n_{i-1}} \in \underline{m}^i$ so, by induction on i, the coefficients $a, b, \cdots, c$ are in $\underline{m}$. Assume that $t > 1$. Let $\tilde{x}_2, \cdots, \tilde{x}_t$, $\tilde{z}_{t+1}, \cdots, \tilde{z}_v$ be the images of $x_2, \cdots, x_t$, $z_{t+1}, \cdots, z_v$ in $(\tilde{R}, \tilde{\underline{m}}) = (R/x_1R, \underline{m}/x_1R)$. Note that for $j < i$ we have that

$$\mathscr{H}_j(\tilde{x}_2, \cdots, \tilde{x}_t) \cup \mathscr{H}_{j-1}(\tilde{x}_2, \cdots, \tilde{x}_t)\tilde{\mathscr{Z}}_1 \cup \cdots \cup \mathscr{H}_1(\tilde{x}_2, \cdots, \tilde{x}_t)\tilde{\mathscr{Z}}_{j-1} \cup \tilde{\mathscr{Z}}_j$$

is a minimal basis for $\tilde{\underline{m}}^j$. This set is the image in $\tilde{R}$ of all the monomials in the chosen minimal basis for $\underline{m}^j$ which do not contain x_1. It is clear that this set is a generating set for $\tilde{\underline{m}}^j$. It is minimal because

$$\begin{aligned}\dim_{R/m} \tilde{\underline{m}}^j/\tilde{\underline{m}}^{j+1} &= \dim_{R/m} \underline{m}^j/\underline{m}^{j+1} + x_1\underline{m}^{j-1} \\ &= \dim_{R/m} \underline{m}^j/\underline{m}^{j+1} - \dim_{R/m} x_1\underline{m}^{j-1} + \underline{m}^{j+1}/\underline{m}^{j+1}\end{aligned}$$

and $\dim_{R/m} x_1\underline{m}^{j-1} + \underline{m}^{j+1}/\underline{m}^{j+1}$ is the number of monomials in the chosen basis for $\underline{m}^j$ which contain x_1.

Suppose that

$$(*) \qquad ax_1^{\alpha_1} \cdots x_t^{\alpha_t} + \cdots + bx_1^{\beta_1} \cdots x_t^{\beta_t}\xi_{11} + \cdots + cx_t\xi_{i-1n_{i-1}} \in \underline{m}^{i+1}$$

is a relation mod $\underline{m}^{i+1}$ among the elements of the set $\mathscr{H}_i(\underline{x}) \cup \mathscr{H}_{i-1}(\underline{x})\mathscr{Z}_1 \cup \cdots \cup \mathscr{H}_1(\underline{x})\mathscr{Z}_{i-1}$. By passing to $\tilde{R}$ and using induction we get that all the coefficients of monomials in (*) which do not contain x_1 are in $\underline{m}$. So we have a relation

$$(**) \qquad ax_1^{\alpha_1} \cdots x_t^{\alpha_t} + \cdots + bx_1^{\beta_1} \cdots x_t^{\beta_t}\xi_{11} + \cdots + cx_1\xi_{i-1n_{i-1}} \in \underline{m}^{i+1}$$

with x_1 appearing in each monomial. Thus,

$$x_1(ax_1^{\alpha_1-1} \cdots x_t^{\alpha_t} + \cdots + bx_1^{\beta_1-1} \cdots x_t^{\beta_t}\xi_{11} + \cdots + c\xi_{i-1n_{i-1}}) \in \underline{m}^{i+1} .$$

By (1.3), $(x_1) \cap \underline{m}^{i+1} = x_1\underline{m}^i$ for $i \leqq l$, so

$$ax_1^{\alpha_1-1} \cdots x_t^{\alpha_t} + \cdots + bx_1^{\beta_1-1} \cdots x_t^{\beta_t}\xi_{11} + \cdots + c\xi_{i-1n_{i-1}} \in \underline{m}^i .$$

By induction, all the coefficients $a, b, \cdots, c$ are in $\underline{m}$.

EXAMPLE. We illustrate (1.6) with an example. Let R be the ring $k[[t^7, t^8, t^{13}, t^{19}]]$ with k a field. $\underline{m} = (t^7, t^8, t^{13}, t^{19})$. $(t^7) \cap \underline{m}^{i+1} = t^7\underline{m}^i$ for $i = 1, 2, 3$ but $(t^7) \cap \underline{m}^5 \supsetneqq t^7\underline{m}^4$ as $t^{40} = t^7 \cdot t^7(t^{13})^2$ is in $\underline{m}^5$ but $t^7(t^{13})^2$ is not in $\underline{m}^4$. We can apply (1.6) to get that $t^7\{t^7, t^8, t^{13}, t^{19}\}$ is part of a minimal basis for $\underline{m}^2 = (t^7\underline{m}, t^{16})$ and $t^7\{t^{14}, t^{15}, t^{20}, t^{26}, t^{16}\}$ is part of a minimal basis for $\underline{m}^3 = (t^7\underline{m}^2, t^{24})$.

2. Reduction to lower dimension. Let $(R, \underline{m})$ be a d-dimensional local Cohen-Macaulay ring of multiplicity e and embedding dimension v. If $v = d$, $d+1$ or $e+d-1$ then grR is Cohen-Macaulay, cf. [4]. The property $\mathscr{P}_k$ for a local Cohen-Macaulay ring to have embedding dimension $e+d-k$ is preserved under passage to $R(U) = R[U]_{\underline{m}R[U]}$, U an indeterminate, and is preserved under reduction modulo the ideal generated by any element of minimal reduction of $\underline{m}$. Given a local Cohen-Macaulay ring with a property $\mathscr{P}$ which is preserved under both types of change of rings mentioned above, (2.4) below shows that if every 1-dimensional local Cohen-Macaulay ring with $\mathscr{P}$ has Cohen-Macaulay associated graded ring then so does every d-dimensional local Cohen-Macaulay ring with $\mathscr{P}$.

Thus to show that Cohen-Macaulay local rings of embedding dimension d, $d+1$ or $e+d-1$ have Cohen-Macaulay associated graded rings, it sufficient to show that 1-dimensional local Cohen-Macaulay rings of embedding dimension 1, 2 or e have Cohen-Macaulay associated graded rings. (However, the usual proof for embedding dimension d or $d+1$ is more direct.) Another application of (2.4) follows in §3.

As the techniques in the proof of (2.4) have other uses, we put this result in a more general setting. In §1 we saw that "super-regularity" can be expressed in terms of certain intersection equalities. What we need are conditions which allow intersection equalities in dimension $d-1$ to be lifted to dimension d.

PROPOSITION 2.1. *Let $(R, \underline{m})$ be a local ring and $\underline{x} = x_1, \cdots, x_t$ be a regular sequence. The following statements are equivalent.*

1. $(\underline{x}) \cap \underline{m}^{i+1} = \underline{x}\underline{m}^i$ *for $i \leqq$ some positive integer l.*

2. (a) $(\underline{m}/x_1R)^{i+1} \cap (\tilde{x}_2, \cdots, \tilde{x}_t)R/x_1R = (\tilde{x}_2, \cdots, \tilde{x}_t)(\underline{m}/x_1R)^i$ *for $i \leqq l$, where $\sim$ denotes image in R/x_1R;*

(b) *If f_s is a homogeneous polynomial in the polynomial ring $R[X_1, \cdots, X_t]$ of (arbitrary) degree s with some coefficient a unit such that $rf_s(x_1, \cdots, x_t) \in \underline{m}^{i+1}$ for some $r \in R$ and any $i \leqq l$, then $r \in \underline{m}^{i+1-s}$.*

3. (a) $(\underline{m}/x_1R)^{i+1} \cap (\tilde{x}_2, \cdots, \tilde{x}_t)R/x_1R = (\tilde{x}_2, \cdots, \tilde{x}_t)(\underline{m}/x_1R)^i$ *for $i \leqq l$, where $\sim$ denotes image in R/x_1R;*

(b) $(x_1) \cap \underline{m}^{i+1} = x_1\underline{m}^i$ *for $i \leqq l$.*

EXAMPLE. Let k be a field and let $R = k[[X, Y, Z]]/(Y^3 - XZ) = k[[x, y, z]]$ with $\underline{m} = (x, y, z)$. R is Cohen-Macaulay and x, z is a regular sequence. In $R/xR \cong k[[Y, Z]]/(Y^3)$, we have $(\underline{m}/xR)^{i+1} \cap zR/xR = z(m/xR)^i$ for $i \geqq 0$. Similarly, in R/zR we have $(\underline{m}/zR)^{i+1} \cap xR/zR = x(\underline{m}/zR)^i$ for $i \geqq 0$. But $\underline{m}^3 \cap (x, z) \neq (x, z)\underline{m}^2$ as $xz \in \underline{m}^3$. Thus the image of z is super-regular in R/xR, the image of x is super-regular in R/zR but neither x nor z is super-regular in R.

Proof of 2.1. Suppose (1) holds. (2) (a) follows immediately. We prove (2) (b) by induction on t and on $i+1-s$. There is no difficulty if $t=1$ so we assume $t>1$ and we may assume also that $i+1-s>0$. Let f_s and r be as in (2) (b). Let

$$(*) \qquad f_s(x_1, \cdots, x_t) = a x_1^{\alpha_1} \cdots x_t^{\alpha_t} + \cdots + c x_1^{\gamma_1} \cdots x_t^{\gamma_t} ,$$

with a a unit. We may assume that r is not a multiple of any x_i. If $\alpha_1 = 0$ in (*), by passing to R/x_1R where all the hypotheses of (1) hold for the images of $x_2, \cdots, x_t$, we get $r \in (\underline{m}^{i+1-s}, x_1)$. $r = \mu + r'x_1$ with $\mu \in \underline{m}^{i+1-s}$ and $r'x_1 f_s(x_1, \cdots, x_t) \in \underline{m}^{i+1}$. Since $x_1 f_s$ has degree $s+1$ and $i+1-(s+1) < i+1-s$, $r' \in m^{i+1-(s+1)}$ and $r \in \underline{m}^{i+1-s}$. Thus we may assume that x_1 appears in each monomial with unit coefficient in (*). Pass to the ring $R(U) = R[U]_{\underline{m}R[U]}$, U an indeterminate and let $x_1' = x_1 + Ux_2$. Since $R \to R(U)$ is faithfully flat, $x_1', x_2, \cdots, x_t$ is a regular sequence in $R(U)$ and $(x_1', x_2, \cdots, x_t) \cap (\underline{m}R(U))^{i+1} = (x_1', x_2, \cdots, x_t)(\underline{m}R(U))^i$ for $i \leqq l$. We have

$$f_s(x_1, \cdots, x_t) = a(x_1' - Ux_2)^{\alpha_1} \cdots x_t^{\alpha_t} + \cdots + c(x_1' - Ux_2)^{\gamma_1} \cdots x_t^{\gamma_t} ,$$

with $rf_s(x_1, \cdots, x_t) \in (\underline{m}R(U))^{i+1}$. We have

$$(**) \qquad \begin{aligned} f_s(x_1, \cdots, x_t) &= a x_1'^{\alpha_1} \cdots x_t^{\alpha_t} + \cdots \\ &- aU^{\alpha_1} x_2^{\alpha_1+\alpha_2} \cdots x_t^{\alpha_t} + \cdots - cU^{\gamma_1} x_2^{\gamma_1+\gamma_2} \cdots x_t^{\gamma_t} . \end{aligned}$$

Since f_s has terms with x_1' missing, we want to consider f_s as a form in $x_1', \cdots, x_t$ and apply the same argument as above. However, there may be some collapsing in (**) so we have to collect terms and write f_s as a sum of distinct monominals of degree s in $x_1', \cdots, x_t$. If all the other monomials in (**) are distinct from $x_2^{\alpha_1+\alpha_2} \cdots x_t^{\alpha_t}$ then, by passing to $R(U)/x_1'R(U)$ and using the same reasoning as above, we see that $rf_s \in (\underline{m}R(U))^{i+1}$ implies that $r \in (\underline{m}R(U))^{i+1-s} \cap R = \underline{m}^{i+1-s}$. So we assume there is some collapsing and let $a, a_{i_1}, \cdots, a_{i_g}$ be unit coefficients in (*) such that the monomials $x_2^{\alpha_{i_j1}+\alpha_{i_j2}} \cdots x_t^{\alpha_{i_jt}}$ in (**) having coefficients $a_{i_j}U^{\alpha_{i_j}}$ are equal to $\mathscr{H} = x_2^{\alpha_1+\alpha_2} \cdots x_t^{\alpha_t}$. Note that $x_2^{\alpha_{i_j1}+\alpha_{i_j2}} \cdots x_t^{\alpha_{i_jt}} = \mathscr{H}$ implies that $\alpha_{i_j3} = \alpha_3, \cdots, \alpha_{i_jt} = \alpha_t$ and $\alpha_{i_j1} + \alpha_{i_j2} = \alpha_1 + \alpha_2$. Thus $\alpha_{i_j1} \neq \alpha_1$ since the monomials in (*) are distinct. Let b be the coefficient (in $\underline{m}$ and possibly zero) of $\mathscr{H}$ in (*). The new (collected) coefficient of $\mathscr{H}$ is

$$A_{\mathscr{H}} = -aU^{\alpha_1} - a_{i_1}U^{\alpha_{i_1}1} - \cdots - a_{i_g}U^{\alpha_{i_g}1} + b .$$

Since the powers of U are distinct, $A_{\mathscr{H}}$ is a unit. Thus f_s may be written as a form of degree s in $x_1', x_2, \cdots, x_t$ with coefficient of the monomial $\mathscr{H}$ a unit. Since x_1' is missing from $\mathscr{H}$, we apply the same argument as above to show that $r \in \underline{m}^{i+1-s}$. This concludes the proof that $(1) \Rightarrow (2)$.

Clearly (2) $\Rightarrow$ (3). Suppose (3) holds. If $t = 1$, then $j = 1$ and (1) holds. Assume $t > 1$. Let $w = ax_1 + g(x_2, \cdots, x_t) \in (\underline{x}) \cap \underline{m}^{i+1}$. With $\sim$ denoting images in R/x_1R, we have $\tilde{g}(\tilde{x}_2, \cdots, \tilde{x}_t) \in (\tilde{x}_2, \cdots, \tilde{x}_t) \cap (\underline{m}/x_1R)^{i+1} = (\tilde{x}_2, \cdots, \tilde{x}_t)(\underline{m}/x_1R)^i$. Thus $g(x_2, \cdots, x_t) = h(x_2, \cdots, x_t) + x_1r$ where $h(x_2, \cdots, x_t) \in (x_2, \cdots, x_t)\underline{m}^i$. We have $w = ax_1 + rx_1 + h(x_2, \cdots, x_t) \in \underline{m}^{i+1}$ so $(a + r)x_1 \in \underline{m}^{i+1} \cap (x_1) = x_1\underline{m}^i$. Since x_1 is a non-zero divisor; $a + r \in \underline{m}^i$ and $w \in \underline{x}\underline{m}^i$.

(2.2) below is the technical result needed to reduce dimension. To eliminate excess notation, homomorphic images of the regular sequence $\underline{x} = x_1, \cdots, x_t$ will be denoted by the same letters, and if $U_1, \cdots, U_p$ are indeterminates, we denote

$$R(U_1, \cdots, U_p) = R[U_1, \cdots, U_p]_{\underline{m}R[U_1,\cdots,U_p]}$$

by $R(U)$.

THEOREM 2.2. *Let $(R, \underline{m})$ be a local ring and let $\underline{x} = x_1, \cdots, x_t$ be a regular sequence with $t > 1$. Suppose that*

$$(\underline{m}/x_1R)^{i+1} \cap (x_2, \cdots, x_t)R/x_1R = (x_2, \cdots, x_t)(\underline{m}/x_1R)^i$$

for $i \leqq$ some positive integer l. Suppose that for any finite set of indeterminates $U_1, \cdots, U_p$, there are elements $x_{i_1}, \cdots, x_{i_p} \in \{x_2, \cdots, x_t\}$ and a linear polynomial $g(U_1, \cdots, U_p) = x_1 + x_{i_1}U_1 + \cdots + x_{i_p}U_p$ such that

$$\begin{aligned}(x_2, \cdots, x_t)(R(U)/gR(U)) &\cap (\underline{m}R(U)/gR(U))^{i+1} \\ &= (x_2, \cdots, x_t)(\underline{m}R(U)/gR(U))^i, \quad \text{for } i \leqq l .\end{aligned}$$

Then,

$$(x_1, \cdots, x_t) \cap \underline{m}^{i+1} = (x_1, \cdots, x_t)\underline{m}^i, \quad \text{for } i \leqq l .$$

Proof. We prove that (2)(b) of (2.1) holds for any local ring with a regular sequence satisfying the hypotheses of the theorem by induction on $i + 1 - s$, where s, f_s and r are as in the statement of (2)(b). We may assume that $i + 1 - s > 0$. Let

$$(*) \qquad f_s = ax_1^{\alpha_1} \cdots x_t^{\alpha_t} + \cdots + cx_1^{\gamma_1} \cdots x_t^{\gamma_t}, \quad \text{with } a \text{ a unit} .$$

If $\alpha_1 = 0$, pass to R/x_1R. By (2.1) (1) $\Rightarrow$ (2)(b), we can conclude that $r \in (\underline{m}^{i+1-s}, x_1)$. $r = \mu + r'x_1$ with $\mu \in \underline{m}^{i+1-s}$. But then $r'x_1f_s \in \underline{m}^{i+1}$ and x_1f_s has degree $s + 1$. By induction, we have $r' \in \underline{m}^{i+1-(s+1)}$, and $r \in \underline{m}^{i+1-s}$. Thus we may assume that x_1 appears in each monomial in (*) with unit coefficient. By hypothesis, for the indeterminate U, there is an element, say x_2 in $\{x_2, \cdots, x_t\}$ such that

$$(x_2, \cdots, x_t)(R(U)/x_1'R(U)) \cap (\underline{m}R(U)/x_1'R(U))^{i+1} = (x_2, \cdots, x_t)(\underline{m}R(U)/x_1'R(U))^i ,$$

for $i \leqq l$, where $x_1' = x_1 + Ux_2$.

We have $f_s = a(x_1' - Ux_2)^{\alpha_1}x_2^{\alpha_2} \cdots x_t^{\alpha_t} + \cdots + c(x_1' - Ux_2)^{\gamma_1}x_2^{\gamma_2} \cdots x_t^{\gamma_t}$. The same argument as in the proof of (2.1) (1) $\Rightarrow$ (2)(b) shows that f_s can be written as a homogeneous polynomial, in the regular sequence $x_1', x_2, \cdots, x_t$, of degree s, having a monomial with unit coefficient in which x_1' does not appear: $f_s = A(U)x_2^{\eta_2} \cdots x_t^{\eta_t} + \cdots + B(U)x_1'^{\tau_1}x_2^{\tau_2} \cdots x_t^{\tau_t}$, with $A(U)$ a unit in $R(U)$. $rf_s \in (\underline{m}R(U))^{i+1}$. By passing to $R(U)/x_1'R(U)$ and applying (2.1) (1) $\Rightarrow$ (2)(b), we get $r \in ((\underline{m}R(U))^{i+1-s}, x_1')$ so $r = \mu + r'x_1'$ with $\mu \in (\underline{m}R(U))^{i+1-s}$ and $r' \in R(U)$. Now $R(U)$ and the regular sequence $x_1', x_2, \cdots, x_t$ satisfy the hypotheses of the theorem. Since $r'x_1'f_s \in (\underline{m}R(U))^{i+1}$ and $i + 1 - (s + 1) < i + 1 - s$, we have by induction that $r' \in (\underline{m}R(U))^{i+1-(s+1)}$. Thus $r \in (\underline{m}R(U))^{i+1-s} \cap R = \underline{m}^{i+1-s}$.

We will say that a property $\mathscr{P}$ of a local ring $(R, \underline{m})$ is *firm* if $\mathscr{P}$ is preserved under the change of rings $R \to R(U)$, U an indeterminate. We will say that a firm property $\mathscr{P}$ of a d-dimensional local Cohen-Macaulay ring $(R, \underline{m})$ is *stable for the minimal reduction* $\underline{x} = x_1, \cdots, x_d$ *of* $\underline{m}$ if for any finite set of indeterminates $U_1, \cdots, U_p$, $\mathscr{P}$ is preserved under reduction of $R(U_1, \cdots, U_p)$ modulo the ideal generated by any subset of generators of $(x_1, \cdots, x_d)R(U_1, \cdots, U_p)$. We will say that a firm property $\mathscr{P}$ of a d-dimensional local Cohen-Macaulay ring $(R, \underline{m})$ is *stable* if $\mathscr{P}$ is preserved under reduction modulo the ideal generated by any element of a minimal reduction of $\underline{m}$.

EXAMPLES. If $(R, \underline{m})$ is a local Cohen-Macaulay ring, the property of being Gorenstein or regular is clearly stable. Let e be a fixed positive integer. The property $\mathscr{P}$ for a local Cohen-Macaulay ring to have multiplicity e is a stable property. As mentioned in the first paragraph of §2, the property $\mathscr{P}$ for a local Cohen-Macaulay ring of dimension d and multiplicity e to have embedding dimension $e + d - k$ is stable. As example of a property stable for a particular minimal reduction $\underline{x}$ is given in §3.

THEOREM 2.3. *Let $(R, \underline{m})$ be a d-dimensional local Cohen-Macaulay ring with a property $\mathscr{P}$ stable for the minimal reduction $\underline{x} = x_1, \cdots, x_d$ of $\underline{m}$. Suppose that 1-dimensional local Cohen-Macaulay rings (S, n) with $\mathscr{P}$ satisfy $(x) \cap \underline{n}^{i+1} = x\underline{n}^i$ for $i \leqq$ some positive integer l and any minimal reduction x of $\underline{n}$. Then $(\underline{x}) \cap \underline{m}^{i+1} = \underline{x}\underline{m}^i$ for $i \leqq l$.*

Proof. The proof is by induction on d. We take $d > 1$. Let $U = U_1, \cdots, U_p$ be a finite set of indeterminates. Let $g(U) = x_1 + U_1x_2 + \cdots + U_px_2$. By hypothesis, $R(U)/g(U)R(U)$ has $\mathscr{P}$ and is stable for the minimal reduction $\tilde{x}_2, \cdots, \tilde{x}_d$ which is the image of $x_2, \cdots, x_d$ in $R(U)/g(U)R(U)$. By induction, $(\underline{m}R(U)/g(U)R(U))^{i+1} \cap (\tilde{x}_2, \cdots, \tilde{x}_d) = (\tilde{x}_2, \cdots, \tilde{x}_d)(\underline{m}R(U)/g(U)R(U))^i$ for $i \leqq l$. By (2.2), $m^{i+1} \cap (x_1, \cdots, x_d) = (x_1, \cdots, x_d)\underline{m}^i$ for $i \leqq l$.

COROLLARY 2.4. *Let $(R, \underline{m})$ be a d-dimensional local Cohen-Macaulay ring with a property $\mathscr{P}$ stable for the minimal reduction $\underline{x} = x_1, \cdots, x_d$ of $\underline{m}$. If grS is Cohen-Macaulay for every 1-dimensional local Cohen-Macaulay ring $(S, \underline{n})$ with $\mathscr{P}$, then grR is Cohen-Macaulay.*

For further use it will be convenient to have a variant of (2.3) which requires us to test only some of the minimal reductions in dimension 1. We need the same types of stability for properties of a minimal reduction. Let $(R, \underline{m})$ be a d-dimensional local Cohen-Macaulay ring. Let U be an indeterminate. Let $\underline{x} = x_1, \cdots, x_d$ be a minimal reduction of $\underline{m}$. Let $\mathscr{Q}$ be a property of $\underline{x}R$. We will say that $\mathscr{Q}$ is *firm* if $\underline{x}R(U)$ has $\mathscr{Q}$. We will say that a firm property $\mathscr{Q}$ if *stable* if, given any element g of a minimal generating set for $\underline{x}R(U)$, say $\underline{x}R(U) = (g, y_2, \cdots, y_d)R(U)$, the image $(y_2, \cdots, y_d)(R(U)/gR(U))$ has $\mathscr{Q}$.

For example, let $(R, \underline{m})$ be a d-dimensional local Cohen-Macaulay ring and, for positive integers j, let $\mathscr{Q}_j$ be the property for minimal reductions $\underline{x}$ of $\underline{m}$ that $\underline{m}^j \subseteqq \underline{x}R$. Then $\mathscr{Q}_j$ is stable. Let $\mathscr{Q}'_j$ be the property that $\underline{m}^j \nsubseteqq \underline{x}R$. Then $\mathscr{Q}'_j$ is also stable.

THEOREM 2.5. *Let $(R, \underline{m})$ be a d-dimensional local Cohen-Macaulay ring with a property $\mathscr{P}$ stable for the minimal reduction $\underline{x} = x_1, \cdots, x_d$ of $\underline{m}$. Assume in addition that $\underline{x}$ has a stable property $\mathscr{Q}$. Suppose that 1-dimensional local Cohen-Macaulay rings $(S, \underline{n})$ with $\mathscr{P}$ satisfy $(x) \cap \underline{n}^{i+1} = x\underline{n}^i$ for $i \leqq$ some positive integer l and all minimal reductions x of $\underline{n}$ which have the stable property $\mathscr{Q}$. Then $(\underline{x}) \cap \underline{m}^{i+1} = \underline{x}\underline{m}^i$ for $i \leqq l$.*

Proof. The proof is the same as the proof of (2.3).

3. Applications.

THEOREM 3.1. *Let $(R, \underline{m})$ be a d-dimensional local Gorenstein ring of multiplicity e and embedding dimension $e + d - 3$. If $\underline{x} = x_1, \cdots, x_d$ is any minimal reduction of $\underline{m}$, $\underline{m}^3 \nsubseteqq \underline{x}R$ and $\underline{m}^4 \subset \underline{x}\underline{m}$.*

grR is Cohen-Macaulay if and only if $\underline{m}^4 = \underline{x}\underline{m}^3$ for some minimal reduction x of m.

Proof. Let $\underline{x} = x_1, \cdots, x_d$ be a minimal reduction of $\underline{m}$. Pass to $(\tilde{R}, \tilde{\underline{m}}) = (R/\underline{x}R, \underline{m}/\underline{x}R)$. Since $v(\tilde{\underline{m}}) = e - 3$ and $\tilde{\lambda}(\tilde{R}) = e$, we must have that $\tilde{\underline{m}}^4 = 0$. Since $\lambda(\text{socle } \tilde{R}) = 1$, $\tilde{\underline{m}}^3 = \text{socle } \tilde{R}$. Thus $\underline{m}^3 \nsubseteq \underline{x}R$ and $\underline{m}^4 \subset \underline{x}R$. By the analytic independence of $\underline{x}$, $\underline{m}^4 \subset \underline{x}\underline{m}$.

If, for some minimal reduction $\underline{x}$ of $\underline{m}$, $\underline{m}^4 \nsubseteq \underline{x}\underline{m}^3$, then $\bar{x}_1, \cdots, \bar{x}_d$ is a homogeneous system of parameters in grR that is not a regular sequence by (1.1) so that grR is not Cohen-Macaulay by [2].

To complete the proof we will show that if $(R, \underline{m})$ is a d-dimensional local Cohen-Macaulay ring of embedding dimension $e + d - 3$ and if $\underline{x}$ is a minimal reduction of $\underline{m}$ such that $\underline{m}^3 \nsubseteq \underline{x}R$ and $\underline{m}^4 = \underline{x}\underline{m}^3$ then grR is Cohen-Macaulay. Let $\mathscr{P}$ be the property for d-dimensional local Cohen-Macaulay rings $(S, \underline{n})$ that S has embedding dimension $e + d - 3$ and that there exists a minimal reduction $\underline{y}$ of $\underline{n}$ such that $\underline{n}^3 \nsubseteq \underline{y}R$ and $\underline{n}^4 = \underline{y}\underline{n}^3$. R has $\mathscr{P}$ and $\mathscr{P}$ is stable for the minimal reduction $\underline{x}$ of $\underline{m}$. Thus by (2.4), we may assume $d = 1$. Let $\underline{x} = x$ and let $\underline{m} = (x, w_1, \cdots, w_{e-3})$. There exist $p, q \in \{1, \cdots, e-3\}$ such that $\underline{m}^2 = (x\underline{m}, w_p w_q)$, $\underline{m}^3 = (x\underline{m}^2, w_p^2 w_q)$ and $w_p^2 w_q \notin x\underline{m}$. This follows from the fact that $(\underline{m}/xR)^2$ and $(\underline{m}/xR)^3$ are non-zero principal ideals. It is clear that $\underline{m}^{i+1} \cap (x) = x\underline{m}^i$ for $i \geqq 3$. We must show that $\underline{m}^3 \cap (x) = x\underline{m}^2$. Let $z \in \underline{m}^3 \cap (x)$. $z = x\mu + rw_p^2 w_q$ with $\mu \in \underline{m}^2$ and $r \in R$. Since $rw_p^2 w_q \in (x)$, $r \in \underline{m}$ and $rw_p^2 w_q \in \underline{m}^4 = x\underline{m}^3$. Thus $z \in x\underline{m}^2$ and $\underline{m}^3 \cap (x) = x\underline{m}^2$.

REMARKS. Note that, with the hypotheses of (3.1), if grR is Cohen-Macaulay it is not Gorenstein. (3.1) may also be proved directly, using just (1.5) instead of (2.4) and (1.5).

EXAMPLES. 1. Let k be a field. The rings $k[[t^5, t^6, t^9]]$, $k[[t^6, t^7, t^{10}, t^{11}]]$, $k[[t^6, t^{11}, t^{13}, t^{20}]]$ are Gorenstein and satisfy $v = e+d-3$ and $\underline{m}^4 = x\underline{m}^3$. Recall that the multiplicity e of a numerical semi-group ring $R = k[[t^{\alpha_1}, t^{\alpha_2}, \cdots, t^{\alpha_n}]]$ with $\alpha_1 < \alpha_2 < \cdots < \alpha_n$ and $gcd(\alpha_1, \alpha_2, \cdots, \alpha_n) = 1$ is just α_1. This follows for example, from [8, VIII, § 10, Thm. 24].

2. $k[[t^5, t^6, t^9]] = k[[X, Y, Z]]/(X^3 - YZ, Y^3 - Z^2)$ is a complete intersection with associated graded ring $k[X, Y, Z]/(YZ, Z^2, Y^4 - ZX^3)$ which is Cohen-Macaulay but not a complete intersection.

3. The hypothesis that R is Gorenstein cannot be omitted from (3.1). Let $R = k[[t^7, t^8, t^{11}, t^{13}, t^{17}]]$, with k a field. R satisfies $v(\underline{m}) = e + d - 3 = 7 + 1 - 3$ and $\underline{m}^4 = t^7\underline{m}^3$ but grR is not Cohen-Macaulay at $t^{17}\underline{m} \subset \underline{m}^3$.

It is possible that the hypothesis $\underline{m}^4 = \underline{x}\underline{m}^3$ is redundant in (3.1), i.e., it is possible that all d-dimensional local Gorenstein rings of embedding dimension $e+d-3$ have Cohen-Macaulay associated graded rings.[1] By (2.4) it is sufficient to prove this for $d=1$. It is true that if $(R, \underline{m})$ is a 1-dimensional analytically irreducible local Gorenstein domain with algebraically closed residue field and with embedding dimension $e+d-3$, then grR is Cohen-Macaulay.

Next, we want to show that Cohen-Macaulay local rings with certain Hilbert polynomials have Cohen-Macaulay associated graded rings. First, we recall definitions of the concepts involved. For any d-dimensional local ring $(R, \underline{m})$, the Hilbert function is defined for nonnegative integers n by $H_R(n) = \dim_{R/\underline{m}}(\underline{m}^n/\underline{m}^{n+1})$. The Hilbert sum transforms are defined inductively for nonnegative integers n by

$$H_R^0(n) = H_R(n) \quad \text{and} \quad H_R^i(n) = \sum_{j=0}^{n} H_R^{i-1}(j) .$$

For large n, $H_R^i(n)$ is a polynomial $P_R^i(n)$ of degree $d-1+i$. If $x \in \underline{m}$, then $H^1_{R/xR}(n) - H^0_R(n) = \lambda((\underline{m}^{n+1}\colon x)/\underline{m}^n)$, cf. [8], Lemma 3, VIII, §8. If x is super-regular, $H^1_{R/xR}(n) = H^0_R(n)$. If x just has the property that $(\underline{m}^{n+1}\colon x) = \underline{m}^n$ for all large n, we still get that $P^1_{R/xR}(n) = P^{\prime}_R(n)$. This prompts the definition of superficial element, cf. [8], VIII, §8. An element x in a local ring $(R, \underline{m})$ is superficial if there is a positive integer c such that $(\underline{m}^{n+1}\colon x) \cap \underline{m}^c = \underline{m}^n$, for all $n \geqq c$. If x is a superficial element and a nonzero divisor then $(\underline{m}^{n+1}\colon x) = \underline{m}^n$ for all large n. A superficial element is the preimage in R of an element of degree 1 in grR which does not lie in any prime belonging to 0 in grR except possibly grm. Superficial elements exist if $R/\underline{m}$ is infinite. If, in addition, $\underline{m}$ does not belong to 0 in R, there exists a superficial element which is also a nonzero divisor.

THEOREM 3.2. *Let $(R, \underline{m})$ be a d-dimensional local Cohen-Macaulay ring with $d \geqq 2$ and multiplicity e. The Hilbert polynomial $P_R^0(n)$ for R is*

$$P_R^0(n) = \binom{n+d-2}{n-1} e + \binom{n+d-2}{n}$$

if and only if R has embedding dimension $e+d-1$ in which case grR is Cohen-Macaulay.

Proof. It was proved in [4] that $v(\underline{m}) = e+d-1$ implies that grR is Cohen-Macaulay and in [5] that $v(\underline{m}) = e+d-1$ implies that $H_R^0(n) = \binom{n+d-2}{n-1} e + \binom{n+d-2}{n}$, for all $n \geqq 0$.

Suppose that $(R, \underline{m})$ is a local Cohen-Macaulay ring with Hilbert polynomial $P_R^0(n) = \binom{n+d-2}{n-1} e + \binom{n+d-2}{n}$. We will show

[1] *Added in proof.* The author has recently verified that this is the case.

that $v(\underline{m}) = e + d - 1$ by induction on d. Let $d = 2$. We may assume that $R/\underline{m}$ is infinite and take a superficial element x with x a nonzero divisor. Then, for $n \geqq n_0$, some integer n_0, $H^1_{R/xR}(n) = H^0_R(n) = ne + 1$. R/xR is a 1-dimensional local Cohen-Macaulay ring so, for all $n \geqq 1$, $H^1_{R/xR}(n) = 1 + e - j_1 + e - j_2 + \cdots + e - j_n$ with nonnegative integers j_i having the property that if $j_i = 0$, then $j_k = 0$ for $k \geqq i$. Since for $n \geqq n_0$, $H^1_{R/xR}(n) = 1 + ne - \sum_{k=1}^{n} j_k = 1 + ne$, it must be true that $\sum_{k=1}^{n} j_k = 0$. Thus $v(\underline{m}/xR) = e$ and $v(\underline{m}) = e + 1$.

Suppose $d > 2$. Again, assuming $R/\underline{m}$ infinite, if necessary, we take a superficial element x which is also a nonzero divisor. For large n, $H^1_{R/xR}(n) = H^0_R(n) = \binom{n+d-2}{n-1}e + \binom{n+d-2}{n}$. Since $H^0_{R/xR}(n) = H^1_{R/xR}(n) - H^1_{R/xR}(n-1)$, we have for large n,

$$\begin{aligned} H^0_{R/xR}(n) &= \binom{n+d-2}{n-1}e + \binom{n+d-2}{n} \\ &\quad - \binom{n-1+d-2}{n-2}e - \binom{n-1+d-2}{n-1} \\ &= \binom{n+(d-1)-2}{n-1}e + \binom{n+(d-1)-2}{n}. \end{aligned}$$

Thus, by induction, $v(\underline{m}/xR) = e + (d-1) - 1$. Therefore, $v(\underline{m}) = e + d - 1$.

REMARK. Clearly, the hypothesis $d \geqq 2$ is necessary in (3.2) as every 1-dimensional local Cohen-Macaulay ring $(R, \underline{m})$ has $P^0_R(n) = e$.

THEOREM 3.3. *Let (R, m) be a d-dimensional local Cohen-Macaulay ring with $d \geqq 2$, and multiplicity e. The Hilbert polynomial $P^0_R(n)$ for R is*

$$P^0_R(n) = \binom{n+d-2}{n-1}e + \binom{n+d-3}{n}$$

if and only if R has embedding dimension $e + d - 2$ and grR is Cohen-Macaulay.

We need two lemmas. Lemma 3.4 was given in [6]. We give a proof below using the results of §1.

LEMMA 3.4. *Let $(R, \underline{m})$ be a d-dimensional local Cohen-Macaulay ring of multiplicity e and embedding dimension $e + d - 2$. Let $\underline{x} = x_1, \cdots, x_d$ be a minimal reduction for $\underline{m}$. Then $\underline{m}^3 \subset (\underline{x})$ and grR is Cohen-Macaulay if and only if $\underline{m}^3 = \underline{x}\underline{m}^2$, for some minimal* *reduction* $\underline{x}$ *of* $\underline{m}$.

Proof. Pass to $(\tilde{R}, \tilde{\underline{m}}) = (R/\underline{x}R, \underline{m}/\underline{x}R)$. $\lambda(\tilde{R}) = e$ and $v(\tilde{\underline{m}}) = e - 2$. Counting lengths, we see that $\tilde{\underline{m}}^2 \neq 0$ and $\tilde{\underline{m}}^3 = 0$. Thus $\tilde{\underline{m}}^3 \subset (\underline{x})$. Suppose that $\underline{m}^3 = \underline{x}\underline{m}^2$. Then $\underline{m}^{i+1} \cap (\underline{x}) = \underline{x}\underline{m}^i$ for $i \geqq 2$. But, by the analytic independence of $\underline{x}$, cf. [3], we also have $\underline{m}^2 \cap (\underline{x}) = \underline{x}\underline{m}$. Thus grR is Cohen-Macaulay by (1.5). If, on the other hand, $\underline{m}^3 \not\subseteq \underline{x}\underline{m}^2$, then by (1.1) there is some $j, 1 \leqq j \leqq d$, such that$(\bar{x}_1, \cdots, \bar{x}_{j-1}: \bar{x}_j) \not\subseteq (\bar{x}_1, \cdots, \bar{x}_{j-1}) + (gr\underline{m})^2$ so that grR is not Cohen-Macaulay.

LEMMA 3.5. *Let* $(R, \underline{m})$ *be a d-dimensional local Cohen-Macaulay ring with* $d \geqq 2$. *Let* $\underline{x} = x_1, \cdots, x_d$ *be a minimal reduction for* $\underline{m}$. *Suppose that* $(\underline{m}/x_1R)^3 \cap (x_2, \cdots, x_d)R/x_1R = (x_2, \cdots, x_d)(\underline{m}/x_1R)^2$ *and that* $(\underline{m}/x_2R)^3 \cap (x_1, x_3, \cdots, x_d)R/x_2R = (x_1, x_3, \cdots, x_d)(\underline{m}/x_2R)^2$. *Then*

$$\underline{m}^3 \cap (x_1, \cdots, x_d)R = (x_1, \cdots, x_d)\underline{m}^2 .$$

Proof. Let $w = a_1x_1 + \cdots + a_dx_d \in \underline{m}^3 \cap (x_1, \cdots, x_d)$. Passing to R/x_1R and R/x_2R in turn, we get that $a_1, a_2, \cdots, a_d \in (\underline{m}^2, \underline{x})$. Thus $w = f_2(x_1, \cdots, x_d) + w'$, with $w' \in (x_1, \cdots, x_d)\underline{m}^2$ and $f_2(x_1, \cdots, x_d)$ a form of degree 2 in $x_1, \cdots, x_d$. $f_2(x_1, \cdots, x_d) \in \underline{m}^3$, by the analytic independence of $\underline{x}$, $f_2(x_1, \cdots, x_d) \in (\underline{x})^2\underline{m}$ and $w \in \underline{x}\underline{m}^2$, as desired.

Proof of 3.3. The proof is by induction on d. We may assume that $R/\underline{m}$ is infinite and take a minimal reduction $\underline{x} = x_1, \cdots, x_d$ of $\underline{m}$ having the property that each x_i is a superficial element. Let $d = 2$. For large n and for $i = 1, 2$, $H^1_{R/x_iR}(n) = H^0_R(n) = ne$. Since R/x_iR is a 1-dimensional local Cohen-Macaulay ring, for every $n \geqq 1$, $H^1_{R/x_iR}(n) = 1 + e - j_1 + \cdots + e - j_n$, with nonnegative integers j_k having the property that if $j_l = 0$, then $j_k = 0$ for $k \geqq l$. For large n, $H^1_{R/x_iR}(n) = 1 + ne - \sum_{k=1}^{n} j_k = ne$. Thus, $\sum_{k=1}^{n} j_k = 1$ so $j_1 = 1$ and $j_k = 0$ for $k > 1$. It follows that $v(\underline{m}/x_iR) = e - 1$ and $v((\underline{m}/x_iR)^l) = e$ for $l > 1$. Therefore, $v(\underline{m}) = e$ and $(\underline{m}/x_1R)^3 = x_2(\underline{m}/x_1R)^2$ and $(\underline{m}/x_2R)^3 = x_1(\underline{m}/x_2R)^2$. By (3.5), we have $\underline{m}^3 \cap (x_1, x_2) = (x_1, x_2)\underline{m}^2$ and by (3.4), grR is Cohen-Macaulay.

Assume that $d > 2$. For each x_i, $i = 1, \cdots, d$, and for large n, we have $H^1_{R/x_iR}(n) = H^0_R(n) = \binom{n+d-2}{n-1}e + \binom{n+d-2}{n}$. Since, $H^0_{R/x_iR}(n) = H^1_{R/x_iR}(n) - H^1_{R/x_1R}(n-1)$, it follows that

$$H^0_{R/x_iR}(n) = \binom{n+(d-1)-2}{n-1}e + \binom{n+(d-1)-2}{n},$$

for large n. We may apply induction to R/x_iR to get $v(\underline{m}/x_iR) =$

$e+(d-1)-2$ and $v(\underline{m}) = e+d-2$. Also, by induction, $(\underline{m}/x_iR)^3 = (x_1, \cdots, x_i, \cdots, x_d)(\underline{m}/x_iR)^2$, so, by (3.5) and (3.4), $\underline{m}^3 = (x_1, \cdots, x_d)\underline{m}^2$ and grR is Cohen-Macaulay.

REMARK. (3.3) answers a question D. Mumford asked the author.

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