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FINITE HEREDITARY NEAR-RING-SEMIGROUPS

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We generalize the concept of a ring-semigroup to that of a near-ring-semigroup, thus obtaining a much larger class of semigroups. Our main result will be the classification of finite hereditary near-ring-semigroups.

A multiplicative semigroup $(S, \cdot)$ is called a ring-semigroup if addition, $+$, can be defined on S so that $(S, +, \cdot)$ is a ring. It is clear that not every semigroup is a ring-semigroup, thus, one seeks to study ring-semigroups with additional restrictions. Some of the recent activities along this direction are as follows: Ligh classified in [8] all the ring-semigroups in which each subsemigroup containing 0 is also a ring-semigroup. On the opposite end of Ligh's work, H. J. Shyr [12] showed that every subsemigroup of a free semigroup with zero is not a ring-semigroup. Using Ligh's result in [8], the present authors [5] determined all the semigroups that are not ring-semigroups but each proper subsemigroup containing zero is a ring-semigroup.

For a survey of ring-semigroups, see [9].

2. Preliminaries.

DEFINITION 1. A (left) near-ring R is a system with two binary operations, $+$ and $\cdot$, such that (i) $(R, +)$ is a group, (ii) $(R, \cdot)$ is a semigroup, (iii) $x(y + z) = xy + xz$ for all $x, y, z \in R$, and (iv) $0x = 0$ for all $x \in R$.

For basic facts about near-rings, see [10]. Note that (right) near-rings are considered in [10].

DEFINITION 2. Let $(S, \cdot)$ be a multiplicative semigroup. Then S is called a near-ring-semigroup (NR-semigroup) if addition, $+$, can be defined on S so that $(S, +, \cdot)$ is a near-ring. An NR-semigroup is said to be hereditary if every subsemigroup containing 0 is an NR-semigroup.

REMARK 1. Suppose S is a hereditary NR-semigroup and T is a subsemigroup of S. The near-ring T need not be a sub-near-ring of S. The problem of characterizing all the rings $(R, +, \cdot)$ in which each subsemigroup of $(R, \cdot)$ is a subring was begun in [3] and a complete solution is given in [6] and [8]. Motivated by the above problem, Ligh [8] obtained a complete classification of hereditary ring-semigroups.

PROPOSITION 1. [8] *Let S be a ring-semigroup. Then S is hereditary if and only if S is either a zero semigroup or $S\backslash 0$ is a cyclic group of order, n, where* $n = 1, 2, 4, 6, 8, 12, 16, 24, 48$ *and a Mersenne prime.*

The above result was instrumental in obtaining a classification of those semigroups which are not ring-semigroups but each proper subsemigroup is a ring-semigroup. For details, see [5].

REMARK 2. Since a zero, 0, can be adjoined to any semigroup S, we shall adopt the convention that all semigroups will contain the zero element.

EXAMPLE 1. Let S be a semigroup with the property that $ab = b$ for each $a \neq 0, b \in S$. Then S is an NR-semigroup. But S is a ring-semigroup if and only if S has one element or two elements.

EXAMPLE 2. Let G be any group and $T(G)$ be the set of functions from G into G which leave the identity of G fixed. Under the operation, $*$, of composition, $(T(G), *)$ is an NR-semigroup where addition is pointwise. Clearly $T(G)$ is a ring-semigroup if and only if the order of G is one or two.

EXAMPLE 3. Let S be the semigroup consisting of four elements $0, a, b, c$ with the defining relation: $0x = ax = 0$, $bx = cx = x$ for all x in S. Then S is a ring-semigroup with $(S, +) \cong K$, where K is the Klein group. On the other hand, S can be considered as a near-ring semigroup, where $(S, +) \cong Z_4$. But the near-ring $(S, +, \cdot)$ is not a ring.

EXAMPLE 4. Let S be a commutative semigroup with no nilpotent elements. If S is an NR-semigroup, then S is necessarily a ring-semigroup. This follows from the fact that a commutative near-ring with no nilpotent elements is a ring.

Since an infinite cyclic semigroup (or group) is not a ring-semigroup, the following result is a consequence of Example 4.

PROPOSITION 2. *An infinite cyclic semigroup (or group) is not an* NR-*semigroup.*

3. Finite near-rings with no zero divisors. All near-rings considered in this section are finite. We wish to discuss briefly the three different types of near-rings with no zero divisors.

Let $(R, +, \cdot)$ be a near-ring with no zero divisors. If $ab = b$ for each $a \neq 0$ in R, then R is called trivial. If R has at least one nonzero element that is not a left identity, then R is called a near integral domain. If R has a unique left identity, then R is a near field.

We now summarize some of the results concerning near integral domains which will be needed later.

Let R be a near integral domain and x be an element of R. Since R has no zero divisors, there is a positive integer n such that $x^n = x$ and $(x^{n-1}) = e = e^2$. For each idempotent e in $R^* = R\backslash 0$, let $G_e = \{r \in R^*\colon re = r\}$.

PROPOSITION 3. [1] *Let $(R, +, \cdot)$ be a near integral domain. Then*

(1) *each $(G_e, \cdot)$ is a group with identity e;*

(2) *the family $\{G_e\}$, $e \in R^*$, is pairwise disjoint;*

(3) $R^* = \bigcup G_e$, $e \in R^*$;

(4) $G_e \cong G_{e'}$, $e, e' \in R^*$;

(5) *each e is a left identity for $(R, +, \cdot)$.*

4. Finite hereditary NR-semigroups (trivial and near-field cases). Let S be a finite hereditary NR-semigroup. There are four cases to consider: (i) $(S, +, \cdot)$ is trivial, (ii) $(S, +, \cdot)$ is a near-field, (iii) $(S, +, \cdot)$ is a near integral domain and (iv) $(S, +, \cdot)$ has proper zero divisors.

If the near-ring $(S, + \cdot)$ is trivial, then any subset of S is a subsemigroup. Since this multiplication, i.e., $ab = b$ for $a \neq 0, b$ in S, can always be defined on any group to get a near-ring, it follows that there is no other restriction on S.

Suppose the near-ring $(S, +, \cdot)$ is a near-field. Recall that $(S\backslash 0, \cdot)$ is a group and any subsemigroup of S is indeed a subgroup. By a previous paper [7], S is called a hereditary near-field group and a complete classification was obtained in [7].

PROPOSITION 4 [7]. *A finite group G is a hereditary* NR-*semigroup if and only if G is one of the following:*

(i) *G is cyclic of orders* 1, 2, 4, 6, 8, 12, 16, 24, 48 *and a Mersenne prime,*

(ii) $G = Q_8$,

(iii) $G = M$,

(iv) $G = N$.

NOTE. Q_8 is the quaternion group of order 8. The non-commutative group M is metacyclic of order 24 and all the subgroups of M are cyclic of orders 2, 3, 4, 6, 8 and 12. The noncommutative group N is of order 24 and all the subgroups are either cyclic of orders 2, 3, 4 and 6, or Q_8.

Henceforth Z_n will denote the cyclic group of order n.

5. Finite hereditary NR-semigroups (near integral domain case). Let S be a finite hereditary NR-semigroup and $(S, +, \cdot)$ be a near integral domoin that is not a near-field. Recall from § 3 that for each idempotent e in $S^* = S\backslash 0$, the set $G_e = \{s \in S^*: se = s\}$ is a group. Furthermore $S^* = \bigcup G_e$ and there are at least two such G_e's, say G_e and $G_{e'}$. The idempotents e and e' are left identities of S and $G_e \cong G_{e'}$. Since S is hereditary, each G_e is the multiplicative semigroup of a near-ring F_e. Thus F_e is a near-field and by Proposition 4, we know precisely what each G_e can possibly be. Since there are at least two such G_e's, our task now is to determine exactly the structure of G_e and how many pieces.

REMARK 3. Let $(N, +, \cdot)$ be a finite near-ring and $x \in N$ such that x is not a zero divisor. The map $x^*: (N, +) \rightarrow (N, +)$ defined by $x^*(y) = xy$ for each y in N is an automorphism of $(N, +)$.

LEMMA 0. *Let $(G, +)$ be a group. If the order of G is 10, then G does not have an automorphism of order 3. If the order of G is 33, then G does not have an automorphism of order either 8 or 16.*

Proof. The two groups of order 10 are Z_{10} and D_{10} and their automorphism groups are of order 4 and 20 respectively. There is only one group of order 33, namely, Z_{33}, and its automorphism group is of order 20. Hence the result follows.

LEMMA 1. *The group G_e cannot be Q_8, Z_{16} or Z_{48}.*

Proof. Suppose $G_e \cong Q_8$. Since there are at least two such G_e's, let $N^* = G_e \cup G_{e'}$ where $G_e \cong G_{e'}$ by Proposition 3. It can be checked that N^* is a subsemigroup of S^*, thus, $N = N^* \cup 0$ is a near-ring. Hence $(N, +) \cong Z_{17}$ and the automorphisms of $(N, +)$ commute with each other. Suppose a, b are in G_e. Define the maps a^* and b^* as in Remark 3. Hence $(a^* \circ b^*)(e) = (b^* \circ a^*)(e)$ and it follows that $abe = bae$ and $ab = ba$, contradicting Q_8 being a noncommutative group.

Suppose $G_e \cong Z_{16}$. As in above, let $N^* = G_e \cup G_{e'}$. Thus $N = N^* \cup 0$ is a near-ring of order 33. If x is a generator of Z_{16}, then

x^* is an automorphism of $(N, +)$ of order 16, contradicting Lemma 0.

Suppose $G_e \cong Z_{48}$. Let A be the subgroup of order 16 of G_e. Then Ae' is the subgroup of order 16 in $G_{e'}$. Thus $A \cup Ae'$ is a subsemigroup of order 32. From above this is impossible.

LEMMA 2. *The group G_e cannot be the group N described in Proposition* 4.

Proof. Recall that the group N has Q_8 as one of its subgroups. Since there are at least two G_e's, say G_e and $G_{e'}$, then $Q_8 \cup Q_8e'$ is a subsemigroup of order 16 that supports a near-ring. Follow the first part of the proof of Lemma 1; this is impossible.

LEMMA 3. *If the order of G_e is a Mersenne prime $p = 2^q - 1$, then $p = 3$.*

Proof. Let $R^* = G_e \cup G_{e'}$. Thus R^* is a subsemigroup of S^* and hence $R = R^* \cup 0$ is a near-ring of order $2p + 1 = 2(2^q - 1) + 1$. Suppose $2^{q+1} - 1 = p_1^{n_1} \cdots p_j^{n_j}$ where each p_i is an odd prime. There is an element w of order p_i in $(R, +)$. Without loss of generality, suppose w is in G_e. Since G_e is cyclic, w is a generator, hence each element in G_e has additive order p_i. Also $(-w)$ has order p_i and since $-w \neq w$, there are an even number of elements of order p_i. Since G_e has an odd number of elements, there is an element z of order p_i in $G_{e'}$. By the above argument, each element in $G_{e'}$ has order p_i. Thus $(R, +)$ is a p_i-group and the order of $(R, +)$ is $p_i^{n_i}$. Hence $2^{q+1} - 1 = p_i^{n_i}$. By [7, Lemma 1], $n_i = 1$ and $q + 1$ is a prime. Consequently, $q = 2$ and $p = 3$.

Now we are ready to determine the number of $G_{e'}$'s in each S^*. From the above lemmas and Proposition 4, we see that the only possible choices for the G_e's are: Z_n, $n = 2, 3, 4, 6, 8, 12, 24$ and the metacyclic group M of order 24.

LEMMA 4. *Suppose $G_e \cong Z_n$, $n = 3, 6, 12, 24$, or $G_e \cong M$. Then $S^* = G_e \cup G_{e'}$ where $G_e \cong G_{e'}$.*

Proof. Suppose $G_e \cong Z_3$. Recall that $\cup G_e$ is a subsemigroup of S^*, and if there were at least three idempotents, then $N = G_e \cup G_{e'} \cup G_{e''}$ supports a near integral domain N that has 10 elements. Let x be a generator of G_e. Then the map x^* is of order 3 and this is impossible by Lemma 0.

Next suppose $G_e \cong Z_n$, $n = 6, 12, 24$ or $G_e \cong M$. Observe that in each case, G_e has a cyclic subgroup of order 3. The above argument shows that S^* can have only two pieces.

LEMMA 5. *Suppose* $G_e \cong Z_8$. *Then* S^* *can have at most three* G_e*'s.*

Proof. If there were four G_e's, then $B = \cup G_e$ has 32 elements and the near-ring $(B, +, \cdot)$ has 33 elements. If x is a generator of G_e, then the automorphism x^* is of order 8. This is not possible by Lemma 0.

LEMMA 6. *Suppose* $G_e \cong Z_4$. *Then* S^* *can have at most four* G_e*'s.*

Proof. If there were five G_e's, then $B = \cup G_e$ has 20 elements and the near-ring $(B, +, \cdot)$ has 21 elements. Thus $(B, +)$ has a normal Sylow subgroup W of order 7, so W is a characteristic subgroup. Let x be the generator of G_e. The automorphism x^* of $(B, +)$ has to be an automorphism of $W \cong Z_7$. But this is not possible since Z_7 does not have an automorphism of order 4.

LEMMA 7. *Suppose* $G_e \cong Z_2$. *Then* S^* *can have any number of pieces.*

Now summarizing the above lemmas we have the following necessary conditions for a finite semigroup to be a hereditary NR-semigroup.

THEOREM 1. *Let* S *be a finite hereditary* NR*-semigroup. If* $(S, +, \cdot)$ *is a near integral domain, then* $S^* = S\backslash 0$ *is a union of isomorphic groups* G_e *with the number* n *of possible pieces given as follows:*

(i) $G_e \cong Z_2$, $n = 2, 3, \cdots$,
(ii) $G_e \cong Z_3, Z_6, Z_{12}, Z_{24}$ *and* M, $n = 2$,
(iii) $G_e \cong Z_4$, $n = 2, 3, 4$,
(iv) $G_e \cong Z_8$, $n = 2, 3$.

6. Finite hereditary NR-semigroups (with zero divisors). Let S be a finite hereditary NR-semigroup and $(S, +, \cdot)$ be the near-ring. We have already considered the cases where $(S, +, \cdot)$ has no proper zero divisors. Thus the only case left is the existence of proper zero divisors in $(S, +, \cdot)$. Our first goal is to show that $xS = S$ or $xS = 0$ for each x in S. This will follow from a series of technical lemmas.

LEMMA 8. *If* x *is a nilpotent element of* S, *then* $x^2 = 0$.

Proof. Suppose $n > 2$ is the smallest positive integer such that $x^n = 0$. Consider the cyclic semigroup $C(x)$ generated by x. Then $A = C(x) \cup 0$ is a NR-semigroup. Since A is commutative, every element in A is right distributive and hence $(-x)^2 = x^2$ for each x in A. If $x + x^2 = 0$, then $x^2 = -x$ and $x^2 \cdot x^2 = (-x)^2 = x^2$. Let $t = n - 2$. Then $0 = x^n = x^{t+2} = x^t$. This contradicts the minimality of n. If $x + x^2 \neq 0$, then $x + x^2 = x^j$, $j > 2$. Let $t + 2 = n$. Then $x^t(x + x^2) = x^t \cdot x^j = 0$ and $x^{t+1} + x^{t+2} = 0$. Thus $x^{t+1} = 0$, a contradiction. Hence $x^2 = 0$.

LEMMA 9. *If $e^2 = e \neq 0$, then $ex = x$ for each x in S.*

Proof. First suppose $ey = 0$ for some $y \neq 0$. Consider the subsemigroup $A(e, y)$ generated by e and y. Thus $A(e, y) = \{0, e\} \cup \{y^k : k \in N\} \cup \{y^j e : j \in N\}$. Since $A(e, y)$ is an NR-semigroup, $e - y = y^k$ or $e - y = y^j e$ for the appropriate k or j. But $e - y = y^k$ implies that $e(e - y) = ey^k = 0$. That is, $e^2 = 0$. Similarly, $e - y = y^j e$ implies $e = 0$. This contradiction establishes that $ey \neq 0$ for any $y \neq 0$ in S.

If x is any element in S, then $e(x - ex) = 0$. From above $x - ex = 0$ and $x = ex$.

LEMMA 10. *If $xz = 0$ for some $z \neq 0$, then $x^2 = 0$.*

Proof. If $x^2 \neq 0$, then $x^n \neq 0$ for all n by Lemma 8. Thus there is an integer k such that $x^k = x^{2k}$; let $e = x^k$. Then $e \neq 0$ and by Lemma 9, $ez = z$. But $ez = x^k z = x^{k-1}xz = 0$, a contradiction. Hence $x^2 = 0$.

LEMMA 11. *If $x^2 = 0$ for all x in S, then $xy = 0$ for each x, y in S.*

Proof. Let x, y be in S. Then $(x + y)^2 = 0$ implies that $(x+y)x + (x + y)y = 0$ and $x(x + y)x + x(x + y)y = x \cdot 0 = 0$. Thus $xyx + xyy = 0$ and $xyx = 0$. By symmetry, $yxy = 0$.

Now assume that $x \cdot y \neq 0$. Then $N = \{0, x, y, xy, yx\}$ is an NR-semigroup with five elements. But according to the tables given in [2], each near-ring on Z_5 has a left identity. A quick check shows that none of the elements in N can be a left identity. Thus $x \cdot y = 0$.

LEMMA 12. *If $xz = 0$ for some $z \neq 0$, then $xy = 0$ for each y in S.*

Proof. By Lemma 10, $x^2 = 0$. If there is an element r in S such that $r^n \neq 0$ for all n, then $r^k = r^{2k}$ for some k. Let $e = r^k$. Then $xe = 0$. For if not, the set $\{0, x, e, xe\}$ is an NR-semigroup. Thus $e + x = 0$ or xe. In either case, $x(e + x) = 0$ and it follows that $xe = 0$, a contradiction. New let y be in S. By Lemma 9, $ey = y$ and it follows that $xy = x(ey) = (xe)y = 0y = 0$.

On the other hand, if every element in S is nilpotent, then $xy = 0$ by Lemma 11.

Summarizing the above, we have the following:

THEOREM 2. *For each x in S, either $xS = S$ or $xS = 0$.*

Let $e^2 = e \neq 0$ be in S. As before, define the set $G_e = \{s \in S^*: se = s\}$. Each G_e is a group and $G_e \cong G_{e'}$ for nonzero idempotents e and e'. Before we can describe S, we need to show that the order of G_e except $G_e \cong Z_2$, is the same as the order of $G_e a = \{ga: g \in G_e\}$, where $a^2 = 0$.

LEMMA 13. *Suppose $a^2 = 0, a \neq 0$ and $x^n \neq 0$ for each n. If $xa = a$, then either x is an idempotent, say $x = e$, or $x^2 = e$.*

Proof. Suppose $x \neq e$. Let k be the minimal integer such that $x^k = e$ and $es = s$ for all s in S by Lemma 9.

Case 1. k is odd and $k \geqq 3$. Using Lemma 12, the set $B = \{0, a, x, x^2, \cdots, x^k\}$ is a subsemigroup with $k + 2$ elements. Since $k + 2$ is odd, $a + a \neq 0$ in the near-ring $(B, +, \cdot)$. Hence $a + a = x^j$, $1 \leqq j \leqq k$. But $x(a + a) = x \cdot x^j = x^{j+1}$. Since $xa = a$, it follows that $a + a = x^{j+1}$. Thus $x^j = x^{j+1}$, a contradiction.

Case 2. k is even and $k \geqq 4$.

(i) $k = 2t$. Suppose there is an odd prime p such that $p \mid k$. Then there is an integer i such that x^i is of order p. Let $x^i = y$. Following the argument in Case 1, considering the subsemigroup $\{0, a, y, y^2, \cdots, y^p\}$, we reach a contradiction.

(ii) $k = 2^m$. Since $k \geqq 4$, there is an element, say x^i, of order 4. Let $y = x^i$. Then the subsemigroup $B = \{0, a, y, y^2, y^3, y^4\}$ supports a near-ring. The map $f: B \to B$ defined by $f(w) = yw$ is an automorphism of $(B, +)$ of order 4. But $(B, +)$ is either Z_6 or S_3 and neither one has an automorphism of order 4.

Combining Cases 1 and 2, we see that $x^2 = e$.

LEMMA 14. *Suppose $e^2 = e$ and $a^2 = 0$. Let the group $G_e = \{s \in S: se = s\}$ be any one of the groups in Proposition 4 except Z_2. If*

$wa = ya$ for $w, y \in G_e$, then $w = y$. Hence the orders of G_e and $G_e a$ are the same.

Proof. $wa = ya$ implies that $y^{-1}wa = a$. If $y^{-1}w = e$, then $y = w$. If $y^{-1}w \neq e$, by Lemma 13, $(y^{-1}w)^2 = e$.

Since the group G_e supports a near-ring, it follows that $(G_e, +, \cdot)$ is a near-field. By [10, p. 239], $x^2 = e$ in a near-field implies that $x = e$ or $x = -e$. Thus $y^{-1}w = e$ or $-e$.

Suppose $y^{-1}w = -e$. Then $w = y(-e)$ and $y(-e)a = wa = ya$. Thus $(-e)a = a$.

There is an element z in G_e such that $z^k = e$ with $k > 2$. Now if $za \neq a$, then $\{0, e, -e, a, za\}$ is a subsemigroup that supports a near-ring. Examining all the semigroups of the near-rings that are defined on $(Z_5, +)$ [2], the above semigroup is not one of them. Hence $za = a$. By Lemma 13, either $z = e$ or $z^2 = e$, a contradiction.

Thus $y^{-1}w = e$ and $y = w$.

REMARK 4. If the group $G_e \cong Z_2 = \{x, e\}$, then it is possible that $xa = ea = a$ for some nilpotent element a. One such example is [2, 2.1, #7].

Let us summarize and see what is happening at this point. Let S be a finite hereditary NR-semigroup. Then for each x in S, either $xS = S$ or $xS = 0$. Clearly if $xS = 0$ for each x in S, then S is a zero semigroup. If S is not a zero semigroup, for $e = e^2 \neq 0$ in S, the set $G_e = \{s \in S: se = s\}$ is a group and $G_e \cong G_{e'}$ for $e' = (e')^2 \neq 0$. If a is a zero divisor, then $a^2 = 0$ and the order of G_e is the same as the order of $G_e a = \{ga: g \in G_e\}$ with the only exception of G_e being isomorphic to Z_2. Clearly the intersection of G_e and $G_e a$ is empty. Since each G_e is a hereditary NR-semigroup, the structure of G_e is given by Proposition 4. In § 5 we have determined how many pieces of G_e's S can have in case that S is a near integral domain. Now applying the same techniques, we can determine the number of pieces for S in the case that S has proper zero divisors. The following lemma eliminates a few possibilities.

LEMMA 15. *Suppose S is a finite hereditary* NR*-semigroup and S is not a zero semigroup. Let $e = e^2$ be in S. Then the order of G_e cannot be* 6, 12, or 24.

Proof. Let $a \neq 0$ be a zero divisor of S. By Lemma 14, the two sets G_e and $G_e a$ have the same number of elements. If the order of G_e is either 6, 12, or 24, then G_e has a subgroup T of order 6. Thus T has a subgroup B of order 3, and $B \cup Ta$ is a subsemigroup with 9 elements. But this implies that the additive group $(B \cup Ta, +)$ has an automorphism of order 3, contradicting Lemma 0.

THEOREM 3. *Let S be a finite hereditary* NR*-semigroup such that the near ring $(S, +, \cdot)$ is not a near integral domain. Then either S is a zero semigroup or S is a union of groups and zero semigroups as follows*:

(i) *If $G_e \cong Z_2$, then S^* is a union of any number of G_e's and $G_e a$ where $a^2 = 0$.*

(ii) *If $G_e \cong Z_3$, then $S^* = G_e \cup G_e a$, $a^2 = 0$.*

(iii) *If $G_e \cong Z_4$, then $S^* = G_e \cup G_e a$ or $S^* = G_e \cup G_e a \cup G_e b$ or $S^* = G_e \cup G_{e'} \cup G_e a \cup G_{e'} b$ or $S^* = G_e \cup G_e a \cup G_e b \cup G_e c$ where $a^2 = b^2 = c^2 = 0$.*

(iv) *If $G_e \cong Z_8$, then $S^* = G_e \cup G_e a$ or $S^* = G_e \cup G_e a \cup G_e b$, $a^2 = b^2 = 0$.*

7. **Sufficient conditions.** In the previous sections we have determined the necessary conditions for a finite semigroup to be a hereditary NR-semigroup. In this section we shall provide the sufficient conditions. This will be accomplished, with a couple of exceptions, by applying the following result to our construction of NR-semigroups.

Let $(G, +)$ be a group of order n and f be a fixed-point-free automorphism of $(G, +)$ of order m. Suppose there is a positive integer k such that $n = km + 1$. Let $e_1 \in G$. Then the set $A_1 = \{f^j(e_1): j = 1, 2, \cdots, m\}$ has m distinct elements. There is $e_2 \in G$ such that $e_2 \notin A_1$. Define A_2 in a similar manner. Then it follows that the family $\{A_t: t = 1, 2, \cdots, k\}$ is pairwise disjoint and $(G \backslash 0) = \bigcup_{t=1}^{k} A_t = \bigcup_{t=1}^{k} f^j(e_t)$, $j = 1, 2, \cdots, m$. Define the operations, $*$ and $\sharp$, on G as follows:

$$f^i(e_s) * f^l(e_t) = f^{i+l}(e_t),\ 1 \leqq s, t \leqq k;\ 1 \leqq i,\ l \leqq m\ .$$

Let q be a fixed integer such that $1 \leqq q < k$.

$$f^i(e_s) \sharp f^l(e_t) = \begin{cases} f^{i+l}(e_t),\ 1 \leqq s \leqq q < k;\ 1 \leqq i,\ l \leqq m\ . \\ 0,\ q < s \leqq k\ . \end{cases}$$

THEOREM 4. *Let $(G, +)$ and the operations, $*$ and $\sharp$, be defined as above. Then $(G, +, *)$ is a near integral domain where each e_i is a left identity and $(G \backslash 0)$ is a union of cyclic groups of order m. Also, $(G, +, \sharp)$ is a near-ring and $(G \backslash 0)$ is a union of cyclic groups of order m and a zero semigroup of order $(k - q)m + 1$.*

Proof. The proof is straightforward so will be omitted.

Since Lemma 14 (see Remark 4 also) did not treat the case when $G_e \cong Z_2$, the following result fills the gap. That is, we shall show how to construct a near-ring from a semigroup S which is a union of Z_2's and zero semigroups. This is the sufficiency of (i) of Theorem 3.

LEMMA 16. *Suppose S is a union of Z_2's and zero semigroups of the form Z_2a with $a^2 = 0$. Then S is an* NR-*semigroup.*

Proof. Recall from Remark 4, it is possible that $xa = a$ for $x \neq e$ in Z_2. So we have two cases to consider:

Case 1. Suppose there is an element a in S such that $a^2 = 0$, and $xa = a, x \neq e$ is in Z_2. Then S cannot have another element b such that $b^2 = 0$ and $xb = b$. This follows from the fact that no near-ring can be defined on the semigroup $\{0, x, e, a, b\}$. Thus S is a union of $\{0, x, e, a\}$ and any number of Z_2's and Z_2b, $b^2 = 0$. This shows that S is of order $2k$. Let $(S, +) \cong (Z_{2k}, +)$. Then $(S, +)$ has a unique element of order two and using the map $f(x) = -x$, and the operation $\sharp$ as defined in Theorem 4, it is easy to check that $(S, +, \sharp)$ is a near-ring. Hence S is a NR-semigroup.

Case 2. Suppose $xa \neq a$ for each a in S satisfying $a^2=0$. Then S is a union of Z_2's and zero semigroups Z_2a's; thus the order of $(S, +)$ is $2k + 1$. Let $(S, +) \cong Z_{2k+1}$ and the map $f(x) = -x$ satisfies Theorem 4. This completes the proof.

Recall from Theorem 1 that if S is a finite hereditary NR-semigroup and if $(S, +, \cdot)$ is a near integral domain, then $S\backslash 0$ is a union of isomorphic groups G_e. If $G_e = M$, the metacyclic group of order 24, then $S\backslash 0 = M \cup M'$ with $M \cong M'$, $M \cap M' = \phi$, hence $(S, +)$ must be a group with 49 elements. Since the automorphism group of Z_{49} is commutative, it follows that $(S, +) = Z_7 \times Z_7$. If a near integral domain can be defined on $S = Z_7 \times Z_7$ so that $S\backslash 0 = M \cup M'$, it is necessary for the automorphism group of S to contain a subgroup isomorphic to M. The next result will show that such a near integral domain cannot exist.

First recall [4, Theorem 9.4.3], the metacyclic group M is generated by x and y of orders 3 and 8 respectively with the relation $xy = yx^2$. Let $S = Z_7 \times Z_7$. Then the automorphism group of S is isomorphic to GL (2, 7), the 2 by 2 nonsingular matrices over Z_7 [11, p. 130].

THEOREM 5. *If $S = Z_7 \times Z_7$, then the automorphism group, $A(S)$, of S does not contain a subgroup isomorphic to the metacyclic group M of order* 24.

Proof. Suppose x and y are in $A(S) = \mathrm{GL}\ (2, 7)$ such that $x^3 = y^8 = 1$ and $xy = yx^2$. Since the Jordan normal form of a matrix of order 3 in GL (2, 7) is diagonal, x could be taken in diagonal form. If the two diagonal entries of x were equal, x would be central in GL (2, 7), and it follows that $x^2 = 1$, a contradiction. If the two diagonal entries are different, then the centralizer of x in GL (2, 7) would consist of all the diagonal matrices and thus be isomorphic to $Z_6 \times Z_6$. From the relation $xy = yx^2$, it follows that $y^{-2}xy^2 = y^{-1}x^2y = y^{-1}xyy^{-1}xy = x^4 = x$. Hence $xy^2 = y^2x$ and y^2 is in the centralizer of x. Since y^2 is of order 4, we arrive at a contradiction.

The above proof, due to the referee, is a considerable improvement over that of the authors.

Finally we have the classification of finite hereditary NR-semigroups.

THEOREM 6. *Let S be a finite semigroup. Then S is a hereditary* NR-*semigroup if and only if S is one of the following*:

(I) *S is a zero semigroup.*

(II) *S is trivial* (i.e., $ab = b$ *for* $a, b \in S$).

(III) (i) $S = Z_n$, $n = 2, 4, 6, 8, 12, 16, 24, 48$ *and a Mersenne prime*;

(ii) $S = Q_8$;

(iii) $S = M$, (*see Proposition* 4);

(iv) $S = N$.

(IV) *S is a union of groups G_e with the number of pieces n*:

(i) $G_e \cong Z_2$, $n = k \geqq 2$, *any positive integer*;

(ii) $G_e \cong Z_3, Z_6, Z_{12}$ *and* Z_{24}, $n = 2$;

(iii) $G_e \cong Z_4$, $n = 2, 3, 4$;

(iv) $G_e \cong Z_8$, $n = 2, 3$.

(V) *S is a union of groups G_e and zero semigroups G_ea, where $a^2 = 0$, and n is the total number of pieces of G_e and G_ea*:

(i) $G_e \cong Z_2$, $n = k \geqq 2$, *any positive integer*;

(ii) $G_e \cong Z_3$, $n = 2$;

(iii) $G_e \cong Z_4$, $n = 2, 3, 4$;

(iv) $G_e \cong Z_8$, $n = 2, 3$.

Proof. The proof of (I) and (II) are obvious. The proof of (III) is given by Proposition 4. The necessities for (IV) and (V) are given by Theorem 1 and Theorem 3 respectively. The sufficiency for (i) of Theorem 3 is given by Lemma 16; while Theorem 5 demonstrates the nonexistence of a near integral domain with 49 elements whose nonzero elements are the union of two isomorphic groups, the metacyclic cyclic group M of order 24. For the sufficiencies of the rest of (IV) and (V), we shall employ Theorem 4. For convenience, we present it in the table below. Column one denotes the structure of G_e, while columns two and three denote the number of pieces in (IV) and (V) respectively. Column four lists the additive group required by Theorem 4. Finally column five exhibits the required fixed-point-free automorphisms dictated by Theorem 4.

We shall use 1 as the generator of Z_n, and any matrix representation of the automorphisms of $Z_n \times Z_n$ is relative to the basis (1, 0) and (0, 1). The automorphism f in column five is of order equal to that of G_e, as required by Theorem 4. The number k denotes any positive integer.

G_e	(IV) n	(V) n	$(S, +)$	$f \in A(S)$
Z_2	k	k	Z_{2k}, Z_{2k+1}	$f(1) = -1$
Z_3	2	2	Z_7	$f(1) = 2$
Z_4	2	2	$Z_3 \times Z_3$	$f = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$
Z_4	3	3	Z_{13}	$f(1) = 5$
Z_4	4	4	Z_{17}	$f(1) = 4$
Z_6	2	none	Z_{13}	$f(1) = 4$
Z_8	2	2	Z_{17}	$f(1) = 2$
Z_8	3	3	$Z_5 \times Z_5$	$f = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$
Z_{12}	2	none	$Z_5 \times Z_5$	$f = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$
Z_{24}	2	none	$Z_7 \times Z_7$	$f = \begin{pmatrix} 2 & 3 \\ 6 & 4 \end{pmatrix}$

8. Concluding remarks. Hereditary ring-semigroups were classified completely in [8] and there were no infinite nontrivial ones. The situation in hereditary NR-semigroups is quite different: (1) it is not known whether there are infinite hereditary near-field groups [7]; (2) there exists infinite hereditary NR-semigroups S. This can be seen by taking $(S, +) = (Z, +)$ and defining $*$ on Z as in Theorem 4, using $f(x) = -x$. Hence we may conclude this paper with the following:

PROBLEM: Classify all the infinite hereditary NR-semigroups.

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