PACIFIC JOURNAL OF MATHEMATICS
Vol. 88, No. 2, 1980

THE SYMMETRY OF SESSILE AND PENDENT DROPS

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Let X denote a bounded, open, and connected subset of R^{n+1} ($n \geq 1$) which we consider to represent the interior of a liquid drop (when $n=2$). The principal result of this paper will be to show that under suitable conditions X is an axially symmetric drop in the sense that there is a vertical line (axis) such that any nonempty intersection of X with a horizontal hyperplane is an open disk whose center lies on the axis. *Condition* **1: X adheres to a horizontal hyperplane, Σ (i.e., $X \cap \Sigma = \Phi$ but $\bar{X} \cap \Sigma \neq \Phi$), with the mean curvature, H, of the liquid-air interface, Ω, a differentiable function of the vertical coordinate and the angle of contact, α, of Ω with Σ a constant along $\partial\Omega$, $0 \leq \alpha \leq \pi$, (Theorem 1. 1).** *Condition* **2: X adheres to Σ with the mean curvature a smooth function of height and the contact region of $\bar{X}$ with Σ a disk (special case of Theorem 1. 2).**

1. **Introduction.** Let $(x_1, \cdots, x_n, u)$ be a Euclidean coordinate system for R^{n+1}. Theorem 1.1, which we now state, corresponds to the equilibrium state of a homogeneous pendent (or sessile) drop adhering to a horizontal hyperplane, Σ.

THEOREM 1.1. *Let X be a bounded, open, and connected subset of R^{n+1} which is adhering to the hyperplane Σ: $\{u = 0\}$. Suppose that the boundary of X, $\partial X = \Sigma_X \cup \Omega$ where $\Sigma_X = \Sigma \cap \bar{X}$ and Ω, the liquid-air interface, is a hypersurface with boundary of class C^2 embedded in R^{n+1} such that $\Gamma \equiv \partial\Omega = \Omega \cap \Sigma$. Suppose that the mean curvature, H, of Ω measured relative to the exterior normal is the restriction to Ω of a C^1-function on R^{n+1} depending on the u-coordinate alone. Finally, suppose that the angle of contact, α, of Ω with Σ measured interior to X is a constant along $\partial\Omega$ where $0 \leqq \alpha \leqq \pi$. Then there is a vertical line about which X is axially symmetric such that any nonempty intersection of X with a horizontal hyperplane is an open disk with center on the axis.*

The physical case of $X \subset R^3$ and mean curvature $H = -ku + c$ ($k > 0$) corresponds to a sessile drop when X is above Σ, a pendent drop when X is below Σ. The case $k = 0$ of constant mean curvature is the situation of no gravity.
J. Serrin [8] treated the case where the liquid-air interface, Ω, may be expressed nonparametrically, $u = u(x_1, \cdots, x_n)$ with H a linear function of height. It turns out that the method of proof used there may be adapted to the present situation. The key tools are

the E. Hopf maximum principle [5], the Hopf boundary point lemma [6], and Serrin's boundary point lemma at a corner [8]. We shall state these "lemmas" in § II. The method of proof is to take a vertical hyperplane, T_0, which initially lies outside of $\bar{X}$ and move it towards and into X through the one parameter family of hyperplanes, T, parallel to T_0. As T moves into X one takes that part of Ω through which T has passed and reflects it about T forming the reflected surface, $\Omega'(T)$. Initially $\Omega'(T)$ lies inside X and we look for a first time when this will fail. At this point one applies one of the touching lemmas to conclude that $\Omega'(T)$ is identical to the unreflected portion of Ω.

This device was first introduced by Alexandrov [1] who was able to show that the only embedded compact hypersurface of constant mean curvature is a sphere. The procedure was then refined by Serrin in [7] and [8].

I became interested in this problem through the work of P. Concus and R. Finn [2] who made a study of axially symmetric pendent drops. Their work induced the author to investigate the stability properties of such drops [9]. A detailed study of the axially symmetric sessile drop has recently been done by Finn [3]. I should also like to mention the paper of E. Gonzalez [4], in which it is proven that for any prescribed volume and any angle of contact α, $0 < \alpha \leqq \pi$, there exists a sessile drop of minimum energy. By a symmetrization argument, such a drop must be axially symmetric. I am indebted to S. T. Yau who brought up the problem considered in this paper and suggested that the method of Alexandrov might work.

The question of symmetry also arises naturally in the "medicine dropper" problem. Again let X denote the interior of the drop in contact with the horizontal hyperplane, Σ. Suppose that $\Sigma_X = \bar{X} \cap \Sigma$ is a disk and the mean curvature of Ω is linear in height. The conjecture is that X is contractable and axially symmetric. The following theorem covers this case.

THEOREM 1.2. *Suppose Σ is the hyperplane $\{u = 0\}$ and let X be a bounded, open, and connected subset of R^{n+1} adhering to Σ. Suppose that $\Sigma_X = \Sigma \cap \bar{X}$ has nonempty interior in Σ which is symmetic about an $(n-1)$-plane, Π, in Σ with the property that the boundary, Γ, of Σ_X can be decomposed into two parts $\Gamma = \Gamma_+ \cup \Gamma_-$ where Γ_+ is the graph of a nonnegative C^2-function, g, from $\Pi_X = \Sigma_X \cap \Pi$ such that g is positive on the interior of its domain and vanishes on $\partial \Pi_X$. Γ_- is the reflection of Γ_+ about Π.*

Suppose that the boundary of X, $\partial X = \Omega \cup \Sigma_X$ where Ω is a hy-

persurface with boundary, $\Gamma = \partial\Omega = \Omega \cap \Sigma$, *which is of class* C^2 *on its interior and on that part of* $\partial\Omega$ *not touching* Π. *Suppose also that the mean curvature,* H, *of* Ω *is the restriction to* Ω *of a* C^1*-function of* u *alone (except perhaps on* $\Omega \cap \Pi$*). Let* T_Π *be the vertical hyperplane generated by* Π. *Under these conditions* X *is symmetric about* T_Π *and the nonempty intersection of* X *with any normal line to* T_Π *is a line segment with center on* T_Π.

Remark 1. Clearly, if $\partial\Omega \subset \Sigma$ is a circle we may apply Theorem 1.2 to conclude the axial symmetry of X as asserted earlier.

Remark 2. If $\partial\Omega \subset \Sigma \equiv R^2$ is a square region with rounded corners we may conclude that X is symmetric about the vertical hyperplanes generated by the two diagonals. However, if $\partial\Omega$ is a rectangle with smoothed corners, unequal sides, and ℓ_1, ℓ_2 are the lines of symmetry for $\partial\Omega$, then our theorem does not allow us conclude corresponding symmetry for X about the generated vertical hyperplanes. (We do not consider the case when $\partial\Omega$ may have corners off of Π although the arguments should work at least in certain cases.)

Remark 3. Consider the dumbell-shaped region $A_\varepsilon \subset \Sigma \equiv R^2$ consisting of the union of two disks $D_i = \{(x_1, x_2) | (x_1^2 \pm 2)^2 + x_2^2 \leqq 1\}$ connected by a narrow neck $R_\varepsilon = \{(x_1, x_2) \mid -2 \leqq x_1 \leqq 2,\ -\varepsilon \leqq x_2 \leqq \varepsilon\}$, and suppose X adheres to Σ with $\Sigma_X = A_\varepsilon$ (again with rounded corners). The results of Theorem 1.2 assert that X must be symmetric about the plane $x_2 = 0$. A_ε is also symmetric about $x_1 = 0$. However, in this case ∂A_ε cannot be represented by a graph plus its reflection and so we cannot conclude that X is symmetric about $x_1 = 0$. In fact, for the case of no gravity, with $H =$ constant, and large volume for X, one would not expect the equilibrium configuration for X of least area for Ω to possess such symmetry.

II. The touching principle. The theorems stated in this section are well-known results from the literature. We state them here for the sake of completeness and reference.

Let $w(x) = w(x_1, \cdots, x_n)$ be a differentiable function in some region of R^n. We shall write $w_i \equiv w_i(x_1, \cdots, x_n)$ to represent the partial derivative of w with respect to x_i. Higher order derivatives are represented similarly.

Let $M(w)$ be a linear differential operator in some open set $G \subset R^n$.

$$M(w) = \sum_{i,j} a_{ij}(x) w_{ij} + \sum_i b_i(x) w_i \,. \tag{2.1}$$

We assume that $a_{ij}(x)$ and $b_i(x)$ are continuous in G, that $a_{ij}(x) = a_{ji}(x)$, with the summations in (2.1) for $1 \leqq i, j \leqq n$. $M(w)$ is elliptic on G if

$$\sum_{i,j} a_{ij}(x)\xi_i\xi_j > 0 \quad \text{for all} \quad x \in G \tag{2.2}$$

and $\xi = (\xi_1, \cdots, \xi_n) \neq (0, \cdots, 0)$. It is uniformly elliptic on G with ellipticity constant $\kappa > 0$ if

$$\sum_{i,j} a_{ij}(x)\xi_i\xi_j \geqq \kappa |\xi|^2 \quad \text{for all} \quad x \in G\,. \tag{2.3}$$

LEMMA 2.1. (E. Hopf Maximum Principle [5]). *Let $w \in C^2(G)$ satisfy $M(w) \geqq 0$ where M is an elliptic operator on G. If there is a point $x_0 \in G$ with $w(x_0) \geqq w(x)$ for all $x \in G$, then $w(x)$ is constant on G.*

LEMMA 2.2. (Hopf's Boundary Point Lemma [6]). *Let G be a region in R^n and suppose that in a neighborhood of $x_0 \in \partial G$, the boundary of G is of class C^1. Let $M(w)$ be a uniformly elliptic operator on $\bar{G}$ and suppose that $w(x) \in C^2(G) \cap C^1(\bar{G})$ satisfies $M(w) \geqq 0$ on G. If $w(x_0) \geqq w(x)$ for all $x \in G$ then either $w(x)$ is a constant on $\bar{G}$ or the inward normal derivative $\partial w/\partial \nu < 0$ at x_0.*

LEMMA 2.3. (Serrin's Boundary Point Lemma at a Corner [8]). *Let $G \subset R^n$ be a bounded region which has a C^2 boundary in a neighborhood of $x_0 \in \partial G$. Let T be a normal plane to ∂G at x_0 and let G^+ be that component of G lying on one side of T which contains x_0 in its closure. Let $M(w)$ be a uniformly elliptic differential operator on $\bar{G}^+$ which satisfies (2.3) for some $\kappa > 0$ on $\bar{G}^+$. Suppose also that*

$$\Big|\sum_{i,j} a_{ij}(x)\xi_i\eta_j\Big| \leqq K[|(\xi\cdot\eta)| + |\xi| d]$$

for some constant $K > 0$, all $x \in \bar{G}^+$, any $\xi = (\xi_1, \cdots, \xi_n)$, where $\eta = (\eta_1, \cdots, \eta_n)$ is a unit normal to T, and where d is the distance from x to T.

Let $w \in C^2(\bar{G}^+)$ satisfy $M(w) \geqq 0$ on $\bar{G}^+$ and suppose that $w(x_0) \geqq w(x)$ for all $x \in \bar{G}^+$. If $w(x)$ is not constant on $\bar{G}^+$, then either $\partial w/\partial s < 0$ or $\partial^2 w/\partial s^2 < 0$ in any direction which enters G^+ non-tangentially at x_0.

There is a touching principle corresponding to each of these maximum principles. The proofs are well known and similar. We sketch the proof in the first instance.

LEMMA 2.4. (Interior Touching Principle). *Let $M(w)$ be an elliptic operator on G as described in Lemma 2.1. Suppose there is a function $w(x) \in C^2(G)$ which satisfies*

$$L(w) \equiv M(w) + c(x)w \geqq 0 \tag{2.4}$$

on G where $c(x)$ is continuous. If $w(x) \leqq 0$ on G and $w(x_0) = 0$, then $w(x) \equiv 0$ on G.

Proof. It is sufficient to prove the lemma on any relatively compact neighborhood, U, of x_0 where $\bar{U} \subset G$. We set $w(x) \equiv e^{\beta x_1} u(x)$ where $\beta > 0$ and x_1 is the first coordinate. A direct calculation yields $L(w) = e^{\beta x_1} \hat{L}(u)$ where $\hat{L}(u) = \hat{M}(u) + \hat{c}(x)u$. Here $\hat{M}(u)$ has the form (2.1) with $\hat{a}_{ij}(x) = a_{ij}(x)$ and $\hat{c}(x) = \beta^2 a_{11}(x) + \beta a_1(x) + c(x)$. Thus $\hat{M}(u)$ is uniformly elliptic on U and for large enough β, $\hat{c}(x) > 0$ on U. Therefore $\hat{M}(u) \geqq -\hat{c}(x)u \geqq 0$ on U, and $u(x)$ has a maximum at x_0 showing that $u(x)$ [and thus $w(x)$] is identically 0 on U.

LEMMA 2.5. (Boundary Point Touching Principle). *Let G, x_0 be as in Lemma 2.2. Suppose that $w(x) \in C^2(G) \cap C^1(\bar{G})$ satisfies (2.4) where $M(w)$ is a uniformly elliptic operator on $\bar{G}$ and $c(x)$ is continuous on $\bar{G}$. If $w(x_0) = 0$, $w(x) \leqq 0$ for $x \in \bar{G}$, and the inward normal derivative $\partial w/\partial \nu = 0$ at x_0, then w is identically 0.*

LEMMA 2.6. (Boundary Point Touching Principle at a Corner [8]). *Let G, G^+, T, and x_0 be as in Lemma 2.3. Suppose that $w(x) \in C^2(\bar{G}^+)$ satisfies the differential inequality (2.4) on $\bar{G}^+$ where the uniformly elliptic operator, $M(w)$, on $\bar{G}^+$ satisfies the conditions of Lemma 2.3 and $c(x)$ is continuous on $\bar{G}^+$. Let $w(x_0) = 0$, $w(x) \leqq 0$ for $x \in G^+$, and suppose that for any nontangential direction entering G^+ at x_0 we have $\partial w/\partial s = \partial^2 w/\partial s^2 = 0$. Then $w(x)$ vanishes on $\bar{G}^+$.*

Now let $u(x)$ and $v(x)$ be two solutions to the same prescribed mean curvature equation

$$\operatorname{div}(Tu) = nH(x, u) \,, \quad Tu = \nabla u/(1 + |\nabla u|^2) \tag{2.5}$$

in a region, G. The operator $\operatorname{div}(Tu)$ is quasi-linear and may be written in the form

$$\begin{aligned} \operatorname{div}(Tu) &= \sum_{i,j} a_{ij}(x, u, \nabla u) u_{ij} \\ &= (1/W) \sum_i u_{ii} - (1/W^3) \sum_{i,j} u_i u_j u_{ij} \end{aligned} \tag{2.6}$$

where $W^2 = 1 + |\nabla u|^2 \equiv 1 + |p|^2$. It follows that

(2.7) $\sum_{i,j} a_{ij}(x, u, \nabla u)\xi_i\xi_j = (1/W^3)[(1 + p^2)|\xi|^2 - \sum_{i,j} p_i p_j \xi_i \xi_j] \geqq (1/W^3)|\xi|^2 .$

Now let $w(x) = u(x) - v(x)$. Then, as is well known, $w(x)$ is a solution to a homogeneous linear elliptic P.D.E. of the form $M(w) + c(x)w \equiv 0$. Here $M(w)$ is in the form (2.1) and the principle part of $M(w)$ is

$$a_{ij}(x) = \int_0^1 a_{ij}(x, u + t(v - u), \nabla u + t(\nabla v - \nabla u))dt \tag{2.8}$$

where $a_{ij}(x, u, p)$ is given by (2.6) and (2.7). In particular, $M(w)$ is elliptic and on any bounded domain is uniformly elliptic with ellipticity constant

$$\kappa = 1/\max(W_0^3, W_1^3)$$

where $W_0^2(x) = 1 + |\nabla u|^2$ and $W_1^2(x) = 1 + |\nabla v|^2$.

These remarks lead to the following conclusions.

Application 1. Let $u(x)$ and $v(x)$ be two C^2 solutions to the same differential equation of prescribed mean-curvature, (2.5), on a region $G \subset R^n$ where $H(x, u)$ is continuously differentiable on $G \times R$. Suppose that $u(x) \leqq v(x)$ on G and $u(x_0) = v(x_0)$ for some $x_0 \in G$. Then $u(x) \equiv v(x)$ on G.

Application 2. Let G, x_0 be as in Lemma 2.2. Suppose that $u(x)$ and $v(x) \in C^1(\bar{G}) \cap C^2(G)$ are both solutions to the same prescribed mean-curvature differential equation, (2.5), where $H(x, u)$ is continuously differentiable on $\bar{G} \times R$. If $u(x_0) = v(x_0)$, $u(x) \leqq v(x)$ for $x \in G$, and the inward normal derivatives $\partial u/\partial \nu = \partial v/\partial \nu$ at x_0, then $u(x) \equiv v(x)$ on $\bar{G}$.

Application 3. Let G, G^+, T and x_0 be as in Lemma 2.3. Suppose that $u(x)$ and $v(x) \in C^2(\bar{G}^+)$ are both solutions to the same prescribed mean-curvature differential equation, (2.5), on $\bar{G}^+$ where $H(x, u)$ is continuously differentiable on $\bar{G}^+ \times R$. If $u(x_0) = v(x_0)$, $u(x) \leqq v(x)$ for $x \in \bar{G}^+$, and if for any nontangential direction pointing into G^+ at the corner x_0 we have $\partial u/\partial s = \partial v/\partial s$ and $\partial^2 u/\partial s^2 = \partial^2 v/\partial s^2$, then $u(x) \equiv v(x)$ on $\bar{G}^+$.

III. Proofs of the main theorems.

Proof of Theorem 1.1. Following the procedure of Alexandrov and Serrin we let T_0 be a vertical hyperplane in R^{n+1} which lies outside of $\bar{X}$. We move T_0 through a one-parameter family of parallel hyperplanes, T, towards and into X. Once T has cut into

X, let $\Omega'(T)$ be the reflection about T of that part of Ω through which T has passed. We adopt the convention that $\Omega'(T)$ is closed so that $\Omega \cap T \subset \Omega'(T)$. Similarly, we let $\Omega(T)$ be that part of Ω through which T has not passed. Again $\Omega(T)$ is assumed closed so that $\Omega \cap T \subset \Omega(T)$.

When T first cuts into X, the interior of $\Omega'(T)$ will be contained in X. For $0 \leqq \alpha \leqq \pi$, this will continue to be true until at least one of the following possibilities occur for some $T = T_1$.

1. $\Omega'(T_1)$ will be internally tangent to $\Omega(T_1)$ at a point, P, off of Σ and away from T_1.

2. At some point, P, on $\Omega \cap T_1$ but off of Σ the normal, $n(P)$, to Ω at P will be parallel to T_1.

3. $\Omega'(T_1)$ will touch $\Omega(T_1)$ internally at a point, P, on Σ but away from T_1.

4. At some point, P, on $\Omega \cap T_1$ lying on Σ the exterior normal, $m(P)$, to $\partial\Omega \equiv \Gamma$ in Σ will be parallel to T_1.

We first show that there is a first time, $T = T_1$, where at least one of these possibilities occur.

For each $Q \in T_0$ let $\angle(Q)$ be the normal half line to T_0 from Q directed towards X. Let P_1 be the initial contact point of $\angle(Q)$ with Ω if such exists. Now set $a(Q)$ to be the distance from Q to P_1 if P_1 exists, otherwise set $a(Q) = +\infty$. $a(Q)$ is a lower semi-continuous function on T_0.

Next let $Q \in T_0$ be a point off of Σ such that $\angle(Q)$ meets Ω. If $\angle(Q)$ cuts through Ω transversally at P_1, let P_2 be the second time that $\angle(Q)$ meets Ω and set $b(Q)$ to be the distance from Q to P_2. If the normal, $n(P_1)$, to Ω at P_1 is parallel to T_0 then set $P_2 = P_1$ and $b(Q) = a(Q)$. Again, if $\angle(Q)$ fails to meet Ω, set $b(Q) = +\infty$.

Now suppose $Q \in T_0 \cap \Sigma$ with $\angle(Q)$ meeting Ω for the first time at P_1. Suppose that the normal, $m(P_1)$, to $\partial\Omega$ in Σ is not parallel to T_0. If $0 < \alpha < \pi$ then the normal vector, $n(P_1)$, to Ω at P_1 also is not parallel to T_0. It follows that $\angle(Q)$ will cut through Ω and there will be a second point, P_2, where $\angle(Q)$ meets Ω. Observe that this will remain true for points $Q' \in T_0$ near Q for which $\angle(Q)$ meets Ω. As above we set $b(Q)$ to be the distance from Q to P_2.

Now suppose that $\alpha = 0$ or $\alpha = \pi$. In this case we observe that the prescribed mean-curvature function, $H(u)$, for Ω must satisfy $H(0) \neq 0$. If $H(0) = 0$, it would follow that in a neighbornood of P_1 both Ω and Σ could be expressed nonparametrically in the form $u = u(x)$ as solutions to the same prescribed mean-curvature equation (2.5). It follows from Application 2 that $\Omega \equiv \Sigma$, a contradiction. Since $\alpha = 0$ or π and the mean curvature of Ω at P_1 is not zero it follows that the normal curvature, $k(P_1)$ of $\Omega \cap N$ where N is the normal 2-plane to $\partial\Omega$ at P_1, is different from zero.

This implies that for points P' near P_1 on Ω but off of Σ, the normal vector, $n(P')$, to Ω at P' is not parallel to T_0. Once again this means that for all points Q' near Q on T_0 any directed normal line, $\ell(Q')$, which meets Ω will cut through Ω and thus will meet a second time at a point, P_2. As above we let $b(Q)$ to be the distance from Q to P_2. Finally, if $m(P_1)$ is parallel to T_0 then set $P_2 = P_1$ and $a(Q) = b(Q)$.

Our discussion allows us to conclude that $b(Q)$ is a lower semi-continuous function on T_0. Now let $c(Q) = [a(Q) + b(Q)]/2$. $c(Q)$ is also lower semi-continuous and so there is a point $Q^* \in T_0$ where $c(Q)$ takes on a positive minimum. This minimum value is precisely the distance through which we must move T_0 to reach the hyperplane, T_1, where at least one of the conditions (1)-(4) first apply.

We now consider each of the four possibilities.

Possibility 1. Choose a Euclidean coordinate system $(x_1, \cdots, x_n, u)$ with the origin at P such that the tangent space to Ω at P is $u = 0$ and so that the u-axis is directed into X. In a neighborhood of $x = \bar{O}$ both $\Omega(T_1)$ and $\Omega'(T_1)$ may be represented non-parametrically in the form $u(x)$ and $v(x)$ respectively where both functions satisfy the same prescribed mean-curvature equation, (2.5), for some C'-function $H(x, u)$. We also have $u(\bar{O}) = v(\bar{O})$, $u(x) \leqq v(x)$ and so by Application 1 $u(x) \equiv v(x)$ and $\Omega(T_1) = \Omega'(T_1)$.

Possibility 2. Choose a Euclidean coordinate system to that P is the origin, $u = 0$ is the tangent space to Ω at P, the hyperplane, T_1, is given by $x_1 = 0$, with the positive u-axis pointing into X, and the positive x_1-axis pointing towards $\Omega'(T_1)$. There is a neighborhood, U, of the origin in $(x_1, \cdots, x_n)$ space such that on the domain $G = \bar{U} \cap \{x_1 \geqq 0\}$ both $\Omega(T_1)$ and $\Omega'(T_1)$ may be represented nonparametrically by C^2-functions $u(x)$ and $v(x)$ both satisfying the same prescribed mean-curvature equation, (2.5), on $\bar{G}$. By construction we have $u(\bar{O}) = v(\bar{O})$, $u(x) \leqq v(x)$ for $x \in \bar{G}$, and $\partial u/\partial x_1 = \partial v/\partial x_1 = 0$ at $x = \bar{O}$. By Application 2 it follows that $u(x) \equiv v(x)$ on $\bar{G}$ and so $\Omega(T_1) = \Omega'(T_1)$.

Possibility 3. The argument is similar to Possibility 2. Choose a Euclidean coordinate system $(x_1, \cdots, x_n, u)$ centered at P so that $u = 0$ is the tangent space to Ω at P, with $u = x_1 = 0$ the tangent space to $\partial\Omega$ at P lying in Σ, so that the positive x_1-axis is directed towards Ω, and the positive u-axis heads into X.

Since the angle of contact, α, is constant along $\partial\Omega$ it follows that the hyperplane $u = 0$ is the common tangent space to $\Omega(T_1)$ and $\Omega'(T_1)$ at P. Since Ω is of class C^2 with boundary, it follows that in a neighborhood of P, $\Omega(T_1)$ and $\Omega'(T_1)$ are represented nonparametrically by C^2-functions $u(x)$ and $v(x)$ respectively where $u(x)$ and $v(x)$ are defined on domains $G_1, G_2 \subset R^n$ where ∂G_i is a C^2-surface in R^n containing the origin with the x_1-axis normal to ∂G_i at $x = \bar{O}$ and pointing into G_i. From our construction we have $G_1 \supset G_2$ if $O \leqq \alpha < \pi/2$, $G_1 = G_2 = \{x_1 > 0\}$ if $\alpha = \pi/2$, and $G_1 \subset G_2$ if $\alpha > \pi/2$. We let $G = G_1 \cap G_2$ and use Application 2 again. On $\bar{G}$, $u(x)$ and $v(x)$ are solutions to the same prescribed mean-curvature equation, (2.5), with $u(\bar{O}) = v(\bar{O})$, $u(x) \leqq v(x)$ for $x \in \bar{G}$, and $\partial u/\partial x_1 = \partial v/\partial x_1 = 0$ at $x = \bar{O}$. By Application 2 we conclude that $u(x) \equiv v(x)$ and hence $\Omega(T_1) = \Omega'(T_1)$.

Possibility 4. Choose a Euclidean coordinate system $(x_1, \cdots, x_n, x)$ with the origin at P, so that $u = 0$ is the tangent space to Ω at P with the positive u-axis directed into X, so that $x_1 = 0$ is the reflecting plane, T_1, with the positive x_1-axis pointing towards $\Omega(T_1)$, and so that $x_n = u = 0$ is the tangent space to $\partial\Omega$ at P in Σ with the positive x_n-axis directed towards Ω.

Relative to this coordinate system the surface, Ω, in a neighborhood of P, is represented nonparametrically by a function $u(x)$ of class C^2 on a domain $\bar{G} \subset R^n$ where $\bar{O} \in \partial G$ and ∂G is the graph of a C^2-function, $x_n = \phi(x_1, \cdots, x_{n-1})$ satisfying $\phi(\bar{O}) = 0$, $\phi_j(\bar{O}) = 0$ for $j = 1, \cdots, n-1$, and G lies above the graph of $\phi(x)$. $\Omega(T_1)$ is represented by this function, $u(x)$, on $\bar{G}_1^+ \equiv \bar{G} \cap \{x_1 > 0\}$ while the reflected surface, $\Omega'(T_1)$, is represented by the function $v(x)$ on $\bar{G}_2^+$ where $v(x_1, x_2 \cdots, x_n) = u(-x_1, x_2, \cdots, x_n)$ for $x_1 \geqq 0$ and $\bar{G}_2^+$ is the reflection of $\bar{G}^-$ about $x_1 = 0$. Observe that $\bar{G}_2^+ \subset \bar{G}_1^+$ if $0 \leqq \alpha < \pi/2$, $\bar{G}_1^+ = \bar{G}_2^+$ if $\alpha = \pi/2$, and $\bar{G}_1^+ \subset \bar{G}_2^+$ if $\pi/2 < \alpha \leqq \pi$.

If we let $\bar{G}^+ = \bar{G}_1^+ \cap \bar{G}_2^+$, then $u(x)$ and $v(x)$ are both $C^2(\bar{G}^+)$ solutions to the same precribed mean-curvature equation (2.5). Furthermore, $u(\bar{O}) = v(\bar{O})$, $u(x) \leqq v(x)$ for $x \in \bar{G}^+$, and $\partial u/\partial s = \partial v/\partial s = 0$ at $x_0 = \bar{O}$ in any nontangential direction entering G^+.

We now show that $\partial^2 u/\partial s^2 = \partial^2 v/\partial s^2$ at $x_0 = \bar{O}$ in any nontangential direction entering G^+. It suffices to show that $u_{ij}(\bar{O}) = v_{ij}(\bar{O})$ for $1 \leqq i, j \leqq n$. From the definition of $v(x)$ it follows at once that $u_{ij}(\bar{O}) = v_{ij}(\bar{O})$ for $2 \leqq i, j \leqq n$ or if $i = j = 1$. Since $u(x_1, x_2, \cdots, x_n) \leqq u(-x_1, x_2, \cdots, x_n)$ when $x_1 \geqq 0$, it follows that $u_1(0, x_2, \cdots, x_n) \leqq 0$ and so $u_{1j}(\bar{O}) = v_{1j}(\bar{O}) = 0$ for $j = 2, \cdots, n-1$. Since $x_n \geqq 0$ for $x \in G$ we must argue differently for $u_{1n}(\bar{O})$.

However, Ω intersects the hyperplane, Σ, at a constant angle, α. The unit normal to Ω is given by $N = (\nabla u, -1)/W$ and the unit normal to Σ may be written $\xi = (0, \cdots, 0, b, a)$ where $a^2 + b^2 = 1$. It follows that $\cos\alpha = (bu_n - a)/W$. Substitute $x_n = \phi(x_1, \cdots, x_{n-1})$

into this equation and differentiate with respect to x_1. We find $0 = b[u_{1n}(\bar{O}) + u_{nn}(\bar{O})\phi_1(\bar{O})]$. Since $\phi_1(\bar{O}) = 0$ we conclude that $u_{1n}(\bar{O}) = 0$ if $b \neq 0$. However, if $b = 0$ then $\alpha = 0$ or $\alpha = \pi$. In this case we have $u_n(x_1, \cdots, x_{n-1}, \phi(x_1, \cdots, x_{n-1})) \equiv 0$. Again differentiate this expression with respect to x_1, set $x = \bar{O}$ and we find that $u_{1n}(\bar{O}) = v_{1n}(\bar{O}) = 0$ in this case also.

We have verified all the conditions of Application 3. We conclude that $u(x) = v(x)$ for $x \in G$ and hence $\Omega(T_1) = \Omega'(T_1)$.

Proof of Theorem 1.2. Let $T_0 \subset R^{n+1}$ be a vertical hyperplane which is exterior $\bar{X}$ and parallel to T . As in the proof of Theorem 1.1 we consider the possibility of moving T_0 through the one-parameter of hyperplanes, T, parallel to T_0 into X. For $Q \in T_0$ we define the functions $a(Q)$ and $b(Q)$ as previously if Q is off of Σ. Let $Q \in T_0 \cap \Sigma$ and suppose the normal half line, $\ell(Q)$, intersects Ω. If $\ell(Q)$ first meets $\partial\Omega$ at a point, P_1, off of Π then, since $\partial\Omega$ is represented by a graph at P_1, the normal, $m(P_1)$, to $\partial\Omega$ in Σ is not parallel to T_0 and so $\ell(Q)$ will meet Ω a second time at a point, P_2. We set $a(Q) = d(Q, P_1)$ and $b(Q) = d(Q, P_2)$. If $\ell(Q)$ first meets Ω at a point P_1 on Π then we set $b(Q) = a(Q) = d(Q, P_1)$. As before, it follows that both $a(Q)$ and $b(Q)$ are lower semi-continuous functions which implies that $[a(Q) + b(Q)]/2$ takes on a positive minimum.

Let T_1 be the corresponding hyperplane. If T_1 is not T_Π then it follows that either Possibility 1 or 2 occurs at a point $P \in \Omega$ and $P \notin \Sigma$. By the appropriate touching principle, Application 1 or 2, we conclude that $\Omega(T_1) = \Omega'(T_1)$, an impossibility unless $T_1 = T_\Pi$. Therefore $T_1 = T_\Pi$.

The same conclusion must hold if we had initially chosen T_0 to lie on the other side of T_Π. The only way for this to be true is if X itself is symmetric about T_Π and such that any nonempty intersection of X with a normal line through T_Π is a segment whose center lies on T_Π.

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Received November 1, 1979. A substantial portion of this paper was completed while the author was a Visiting Member of SFB 40 at the University of Bonn, Germany.

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