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APPLICATIONS OF TOPOLOGICAL TRANSVERSALITY TO DIFFERENTIAL EQUATIONS I. (SOME NONLINEAR DIFFUSION PROBLEMS.)

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In this paper, topological techniques are used to establish existence results for some boundary value problems arising in diffusion theory. Questions of uniqueness are also treated. Our topological arguments are based on the topological transversality theorem rather than the Leray-Schauder theory. An important feature of our approach is that some of the results obtained cannot be deduced by a direct application of the latter theory. Further applications of topological transversality to differential equations will be given in forthcoming parts of the paper.

0. **Introduction.** In this paper, we treat questions of existence and uniqueness for the solutions to certain systems of differential equations each of which models a steady state, one dimensional diffusion process. Conservation of mass considerations lead to the following system of differential equations for the unknow concentration $C = C(x)$ of the diffusing substance and the velocity $v=v(x)$ of the diffusing medium (see [1]):

$$(\mathscr{P})\qquad \begin{cases} -(D(x, C(x))C'(x))' + (v(x)C(x))' = f(x, C(x), C'(x)),\ 0 \leqq x \leqq 1\ , \\ -D(0, C(0))C'(0) + v_0 C(0) = L,\ C(1) = c_1\ , \\ \qquad v'(x) = -J(x, C(x))\ , \\ \qquad v(0) = v_0\ . \end{cases}$$

Here $D(x, C)$ is the diffusion coefficient which we suppose to be continuously differentiable and to satisfy,

$$(1)\qquad 0 < \varepsilon \leqq D(x, C) \text{ on } [0, 1] \times [0, \infty)\ .$$

Also, c_1, v_0, L are given constants with $c_1, L \geqq 0$, and the source term $f(x, C, C')$ is continuous and satisfies,

$$(2)\qquad \begin{cases} 0 \leqq f(x, C, C') \leqq A + B(|C|^{\alpha} + |C'|^{\alpha})\ , \\ \text{on } [0, 1] \times [0, \infty) \times (-\infty, \infty)\ , \end{cases}$$

where $A, B \geqq 0$, $0 \leqq \alpha < 1$. Also,

$$\|h\| = \max_{0 \leqq x \leqq 1} |h(x)|$$

for any h in $C[0, 1]$. Finally, we assume that $J(x, C)$ is continuous and satisfies:

$$(3)\quad \begin{cases} \text{for each } x \text{ in } [0, 1],\ J(x, C) \text{ is nonincreasing for } C \text{ in } [0, \infty) \\ \qquad\qquad \text{and, } |J(x, C)| \leqq A_1 + B_1|C|^\beta \\ \text{on } [0, 1] \times [0, \infty) \text{ for some } \beta \geqq 0 \text{ and } \alpha + \beta < 1\,. \end{cases}$$

It is clear on physical grounds that $(\mathscr{P})$ should have a positive solution. We shall prove: If the diffusion process satisfies (1), (2) and (3), then $(\mathscr{P})$ has at least one positive solution. Furthermore, if f is independent of C' and $f_C \leqq 0$, the system $(\mathscr{P})$ has a unique solution.

A specialization of problem $(\mathscr{P})$ was treated in [6]; namely, the case D constant, $v_0 = L = 0$, and $f = f(x)$ independent of C and C'. Part of our existence argument follows the approach in [6]. Uniqueness for the special case of $(\mathscr{P})$ noted above is also established in [6]; the analysis there uses an abstract uniqueness theorem based on the Schauder fixed point theorem. Our uniqueness analysis is different.

Our existence argument uses topological transversality (Pertinent results are summarized in § 1.) For this purpose, we establish a priori bounds on all possible solutions (C, v) to $(\mathscr{P})$ and to a related one parameter family of problems $(\mathscr{P}_\lambda)$ introduced in § 2. The sublinear growth restrictions on f and J are crucial for this—see the example in § 5. On the other hand, the remaining assumptions on $D(x, C)$, $f(x, C, C')$, and $J(x, C)$ are dictated by the physical situation.

In § 6, we consider a variant of problem $(\mathscr{P})$ and establish existence results under much weaker growth rate restrictions on f and J. On the other hand, f and J are assumed to satisfy certain inequalities whose physical significance is only partially clear. However, there are reasonable choices for f and J which do satisfy these restrictions; for instance, see [7].

We will use the following notation. For each $u \in C^1[0, 1]$ define, $||u||_1 = \max(||u||, ||u'||)$, where $||u|| = \max\{|u(x)|: 0 \leqq x \leqq 1\}$. Also, K^1 will denote the cone of nonnegative functions in $C^1[0, 1]$ and K^1_R the functions in K^1 with $||u||_1 \leqq R$.

1. Topological preliminaries. We begin with a brief review of the topological results to be used throughout this paper. For further details see [2] and [3]. Let C be a convex subset of a Banach space E, X a metric space, and $F: X \to C$ a continuous map. We say that F is *compact* if $F(X)$ is contained in a compact subset of C. F is *completely continuous* if it maps bounded subsets in X

into compact subsets of C. A homotopy $\{H_t: X \to C\}_{0 \leq t \leq 1}$ is said to be compact provided the map $H: X \times [0, 1] \to C$ given by $H(x, t) = H_t(x)$ for (x, t) in $X \times [0, 1]$ is compact.

Let $U \subset C$ be open in C. A compact map $F: \bar{U} \to C$ is called *admissible* if it is fixed point free on the boundary, ∂U, of U. The set of all such maps will be denoted by $\mathscr{K}_{\partial U}(\bar{U}, C)$.

DEFINITION 1.1. A map F in $\mathscr{K}_{\partial U}(\bar{U}, C)$ is *inessential* if there is a fixed point free compact map $G: \bar{U} \to C$ such that $G|\partial U = F|\partial U$. A map F is $\mathscr{K}_{\partial U}(\bar{U}, C)$ which is not inessential is called *essential*.

THEOREM 1.2. *Let p be an arbitrary point in U and F be in $\mathscr{K}_{\partial U}(\bar{U}, C)$ be the constant map $F(x) = p$ for x in $\bar{U}$. Then F is essential.*

Theorem 1.2 is an elementary consequence of the Schauder fixed point theorem.

DEFINITION 1.3. Two maps F and G in $\mathscr{K}_{\partial U}(\bar{U}, C)$ are called *homotopic* ($F \sim G$) if there is a compact homotopy $H_t: \bar{U} \to C$ for which $F = H_0$, $G = H_1$, and H_t is admissible for each t in $[0, 1]$.

The following simple characterization of inessential maps is important.

LEMMA 1.4. *A map F in $\mathscr{K}_{\partial U}(\bar{U}, C)$ is inessential if and only if it is homotopic to a fixed point free map.*

Proof. If F is inessential and G is a fixed point free map such that $F|\partial U = G|\partial U$, then a compact homotopy joining F and G is given by $H_t(x) = tF(x) + (1 - t)G(x)$.

Suppose $H_0: \bar{U} \to C$ is a fixed point free map and that $H_t: \bar{U} \to C$ is an admissible homotopy joining H_0 to F. We will show that each H_t (and in particular $H_1 = F$) is an inessential map. To this end, consider the map $H: \bar{U} \times [0, 1] \to C$ and define a set $B \subset \bar{U}$ by the condition $B = \{x \in \bar{U}: H(x, t) = x$ for some t in $[0, 1]\}$. Assuming without loss of generality that B is nonempty, note that B is closed and disjoint from ∂U. Take an Urysohn function $\lambda: \bar{U} \to [0, 1]$ with $\lambda(a) = 1$ for $a \in \partial U$ and $\lambda(b) = 0$ for $b \in B$ and put $H_t^*(x) = H(x, \lambda(x)t)$ for $(x, t) \in \bar{U} \times [0, 1]$. It is now easily seen that $\{H_t^*: \bar{U} \to C\}_{0 \leq t \leq 1}$ is a fixed point free compact homotopy such that $H_t^* | \partial U = H_t | \partial U$ for each $t \in [0, 1]$. Consequently, each H_t is inessential and the proof is complete.

As a consequence of Lemma 1.4 we have,

THEOREM 1.5. *Let F and G be in $\mathscr{K}_{\partial U}(\bar{U}, C)$ be homotopic maps, $F \sim G$. Then one of these maps is essential iff the other is.*

2. **Existence.** Integration of the first equation in ($\mathscr{P}$) yields,

$$-D(x, C)C' + v(x)C - L = \int_0^x f(\tau, C(\tau), C'(\tau))d\tau .$$

If

$$m(x) = \int_x^1 \frac{v(z)dz}{D(z, C)}$$

we obtain,

$$(e^{m(x)}C)' = \frac{-Le^{m(x)}}{D(x, C)} - \frac{e^{m(x)}}{D(x, C)}\int_0^x f(\tau, C(\tau), C'(\tau))d\tau .$$

A further integration from x to 1 yields,

$$\begin{aligned} C(x) &= c_1 e^{-m(x)} \\ &\quad + \int_x^1 \frac{e^{m(t)-m(x)}}{D(t, C(t))}\left\{L + \int_0^t f(\tau, C(\tau), C'(\tau))d\tau\right\}dt . \end{aligned} \tag{4}$$

We define an operator $T: K^1 \to K^1$ by

$$\begin{aligned} (TC)(x) &= c_1 e^{-m(x)} \\ &\quad + \int_x^1 \frac{e^{m(t)-m(x)}}{D(t, C(t))}\left\{L + \int_0^t f(\tau, C(\tau), C'(\tau))d\tau\right\}dt . \end{aligned}$$

Evidently, T maps K^1 into its interior if $c_1 + L > 0$ so any fixed points are strictly positive. Assume momentarily that $c_1 + L > 0$.

A priori bounds in K^1 on positive fixed points C of T are obtained: First, for $t \geqq x$,

$$\begin{aligned} m(t) - m(x) &= \int_x^t \frac{-v(z)}{D(z, C(z))}dz \\ &= \int_x^t \frac{1}{D(z, C(z))}\left\{-v_0 + \int_0^z J(\tau, C(\tau))d\tau\right\}dz \\ &\leqq \max_{z\in[0,1]}\left\{-v_0 + \int_0^z J(\tau, 0)d\tau\right\}\int_x^t \frac{dz}{D(z, C(z))} . \end{aligned}$$

So, by (1),

$$m(t) - m(x) \leqq G(t - x)$$

for a fixed constant G. Since $m(1) = 0$, use of this result in (4) together with (1) and (2) yields,

$$\|C\| \leqq G_1 + G_2(\|C\|^\alpha + \|C'\|^\alpha) \tag{5}$$

for constants G_1 and G_2 independent of C in K^1. Below $G_3, G_4, \cdots$ also denote constants independent of C in K^1. Since,

$$C'(x) = \frac{v(x)C(x)}{D(x, C(x))} - \frac{1}{D(x, C(x))}\left\{L + \int_0^x f(\tau, C(\tau), C'(\tau))d\tau\right\}$$

we find,

$$\|C'\| \leqq \frac{\|v\|\ \|C\|}{\varepsilon} + \frac{1}{\varepsilon}\{G_3 + G_4(\|C\|^\alpha + \|C'\|^\alpha)\}\ , \tag{6}$$

for positive constants G_3, G_4. Now,

$$|v(x)| \leqq |v_0| + \int_0^x |J(\tau, C(\tau))|d\tau \leqq G_5 + G_6\|C\|^\beta$$

for positive constants G_5, G_6 by (3). So,

$$\|v\| \leqq G_5 + G_6\|C\|^\beta\ . \tag{7}$$

Use of (5) and (7) in (6) yields,

$$\begin{aligned}\|C'\| \leqq G_7 + G_8(\|C\|^\alpha + \|C'\|^\alpha) + G_9\|C\|^\beta \\ + G_{10}\|C\|^{\alpha+\beta} + G_{11}\|C'\|^\alpha\|C\|^\beta\end{aligned} \tag{8}$$

for positive constants $G_7, G_8, \cdots, G_{11}$. If $p, q > 1$ and $1/p + 1/q = 1$, we have,

$$\|C'\|^\alpha\|C\|^\beta \leqq \frac{1}{p}\|C'\|^{\alpha p} + \frac{1}{q}\|C\|^{\beta q}\ . \tag{9}$$

Fix p such that

$$\frac{1}{1-\beta} < p < 1/\alpha$$

which is possible because $\alpha + \beta < 1$. Let $1/q = 1 - 1/p$. Then one easily checks that $\beta q < 1$ and consequently in view of (8) and (9) there is a constant γ, $0 < \gamma < 1$, such that

$$\|C'\| \leqq G_{12} + G_{13}(\|C\|^\gamma + \|C'\|^\gamma)\ . \tag{10}$$

It follows easily from (5) and (10) that there is an $R > 0$ such that

$$T(C) = C \text{ implies } \|C\|_1 < R\ . \tag{11}$$

Now consider the family of problems

$$(\mathscr{P}_\lambda)\qquad \begin{cases} -(D(x, C)C')' + (\lambda v(x)C)' = \lambda f(x, C(x), C'(x)) , \\ D(0, C(0))C'(0) + \lambda v_0 C(0) = \lambda L,\ C(1) = \lambda c_1 , \\ \qquad v'(x) = -J(x, C(x)) , \\ \qquad v(0) = \lambda v_0 , \end{cases}$$

for $0 \leqq \lambda \leqq 1$. Since this family of problems arises from $(\mathscr{P})$ by replacing v by λv, f by λf, v_0 by λv_0, L by λL and c_1 by λc_1, one sees at once that solutions to $(\mathscr{P}_\lambda)$ satisfy (4) with m replaced by λm, L by λL, c_1 by λc_1 and f by λf. After these replacements are made, the right side of (4) defines an operator,

$$T_\lambda\colon K^1 \longrightarrow K^1 ,$$

for which,

$$T_\lambda(C) = C \text{ implies } \|C\|_1 < R , \tag{12}$$

with the same constant R which appears in (11).

We now regard $T_\lambda\colon K_R^1 \to K^1$. This map is fixed point free on $\|x\|_1 = R$ and $T(x, \lambda) = T_\lambda x\colon K_R^1 \times [0, 1] \to K^1$ is a compact homotopy. (It is easy to check that $\|T_\lambda(C)''\|$ is bounded independent of λ for C in a bounded set in K^1.) Since T_0 is the zero map, which is essential, the Topological Transversality Theorem 1.5 implies that $T_1 = T$ is essential. Consequently, T has a fixed point, i.e., $(\mathscr{P})$ has a solution.

This proves the following theorem when $c_1 + L > 0$.

THEOREM 2.1. *Assume* (1), (2), *and* (3) *hold. Then the system* $(\mathscr{P})$ *has at least one nonnegative solution, which is strictly positive if* $c_1 + L > 0$.

As a corollary of the proof, we note the following result essentially obtained in [6] for a specialization of $(\mathscr{P})$.

THEOREM 2.2. *Assume that* (1) *holds, that* f *is independent of* C *and* C' (i.e., (2) *holds with* $\alpha = 0$), *and that* $J(x, C)$ *is nonincreasing in* C *for each* x. *Then* $(\mathscr{P})$ *has at least one solution.*

Proof. The bound (5) above reduces to $\|C\| \leqq G_1$. For such C, we easily find that

$$|v(x)| \leqq |v_0| + \max_{[0\ 1]\times[0\ G_1]} |J(x, C)|$$

and hence that (7) holds with $\beta = 0$. Then $\|C'\| \leqq G'$ follows for some $G' < \infty$. The proof is concluded as above.

To remove the additional restriction on c_1, $L \geqq 0$ that $c_1 + L > 0$ made above, we reason as follows. If C_n is a solution to $(\mathscr{P})$ with $c_1 = 1/n$ and $L = 0$, say, then using the estimates above it follows easily that $\| C_n \|_1 \leqq M < \infty$ for a fixed constant M. A compactness argument now yields a solution to $(\mathscr{P})$ when $c_1 = L = 0$. Thus, Theorem 2.1 is established.

3. Uniqueness. The following uniqueness result holds for $(\mathscr{P})$. The notation used in § 2 is maintained. Recall that any solutions to $(\mathscr{P})$ must be nonnegative.

THEOREM 3.1. *Assume that* (1) *holds, that* $f = f(x, C)$ *is independent of* C', *and that both* $f(x, C)$ *and* $J(x, C)$ *are continuously differentiable and nonincreasing with respect to* C. *Then* $(\mathscr{P})$ *has at most one solution.*

Proof. Suppose that (C_i, v_i) for $i = 1, 2$ are two solution pairs to $(\mathscr{P})$. Then,

$$-D(x, C_i)C_i' + v_i C_i = \int_0^x f(t, C_i(t))dt + L .$$

Thus,

$$-D(x, C_1)C_1' + D(x, C_2)C_2' + v_1 C_1 - v_2 C_2 = \int_0^x [f(t, C_1) - f(t, C_2)]\, \mathrm{dt}$$

or

$$\begin{aligned} -D(x, C_1)[C_1 - C_2]' - [D(x, C_1) - D(x, C_2)]C_2' + v_1(C_1 - C_2) \\ + (v_1 - v_2)C_2 = \int_0^x f_C(t, \rho(t))[C_1(t) - C_2(t)]\, \mathrm{dt} , \end{aligned} \tag{13}$$

where $\rho(t)$ is a bounded function determined by the mean-value theorem. Also,

$$D(x, C_1(x)) - D(x, C_2(x)) = D_C(x, \phi(x))[C_1(x) - C_2(x)] , \tag{14}$$

and,

$$\begin{aligned} v_1(x) - v_2(x) &= -\int_0^x [J(t, C_1(t)) - J(t, C_2(t))]\, \mathrm{dt} \\ &= -\int_0^x J_C(t, \psi(t))[C_1(t) - C_2(t)]\, \mathrm{dt}, \end{aligned} \tag{15}$$

where $\phi(x)$ and $\psi(t)$ are bounded functions determined by the mean-value theorem.

Define,

$$\Delta(x) = C_1(x) - C_2(x) .$$

Then (13), (14), and (15) give,

$$-D(x, C_1)\Delta' - D_C(x, \phi(x))C_2'\Delta + v_1\Delta = C_2\int_0^x J_C(t, \psi(t))\Delta(t)\,\mathrm{dt}$$
$$+ \int_0^x f_C(t, \rho(t))\Delta(t)\,\mathrm{dt} ,$$

or,

$$\Delta' + h(x)\Delta = k(x)\int_0^x p(t)\Delta(t)\,\mathrm{dt} + l(x)\int_0^x q(t)\Delta(t)\,\mathrm{dt} \tag{16}$$

where

$$h(x) = \frac{C_2'(x)D_C(x, \phi(x)) - v_1(x)}{D(x, C_1(x))} , \tag{17}$$

is bounded,

$$k(x) = \frac{C_2(x)}{D(x, C_1(x))} \geqq 0$$

because $C_2(x) \geqq 0$,

$$p(x) = -J_C(x, \psi(x)) \geqq 0 ,$$
$$l(x) = \frac{1}{D(x, C_1(x))} > 0 ,$$

and,

$$q(x) = -f_C(x, \rho(x)) \geqq 0 .$$

We will show that (16) together with $\Delta(1) = 0$ implies that $\Delta \equiv 0$ which proves uniqueness. Observe first that (14) and (17) show that $h(x)$ is continuous on any interval on which $\Delta \neq 0$. Suppose $\Delta(0) \neq 0$. Then we may assume $\Delta(0) > 0$ and since $\Delta(1) = 0$,

$$x_0 = \inf \{x \in [0, 1]: \Delta(x) \leqq 0\} ,$$

is defined. Clearly, $0 < x_0 \leqq 1$, $\Delta(x_0) = 0$, and $\Delta > 0$ on $[0, x_0)$. Let,

$$\mu(x) = \exp\left(\int_0^x h(t)\,\mathrm{dt}\right) > 0 .$$

Then,

$$(\mu(x)\Delta)' = \mu(x)\left[k(x)\int_0^x p(t)\Delta(t)\,\mathrm{dt} + l(x)\int_0^x q(t)\Delta(t)\,\mathrm{dt}\right] ,$$

and integration from x to x_0 gives,

$$-\mu(x)\Delta(x) = \int_x^{x_0}\mu(t)\left[\, k(t)\int_0^t p(u)\Delta(u)\, du \right.$$
$$\left. l(t)\int_0^t q(u)\Delta(u)\, du\right] dt \geqq 0 \, .$$

So $\Delta(x) \leqq 0$ on $[0, x_0)$, a contradiction. Thus, $\Delta(0) = 0$ must hold.

Since $\Delta(0) = 0$, (16) yields,

$$|\Delta(x)| = \left|\int_0^x \Delta'(t)\, dt\right| \leqq \int_0^x |\Delta'(t)|\, dt$$
$$\leqq \int_0^x |h(t)|\ |\Delta(t)|\, dt + \int_0^x k(t)\int_0^t p(\sigma)|\Delta(\sigma)|\, d\sigma\, dt$$
$$+ \int_0^x l(t)\int_0^t q(\sigma)|\Delta(\sigma)|\, d\sigma\, dt \, .$$

Interchange the order of integration in the last two integrals, and use the boundedness of h, k, p, and q to obtain,

(18) $$|\Delta(x)| \leqq B\int_0^x |\Delta(t)|\, dt \, ,$$

for a fixed constant $B < \infty$.

A standard induction argument using (18) shows that,

$$|\Delta(x)| \leqq \|\Delta\| B^n x^n/n! \, ,$$

for $n = 1, 2, \cdots$ and x in $[0, 1]$. Thus, $\Delta(x) \equiv 0$, and uniqueness is established.

4. **Related results.** Consider ($\mathscr{P}$) when $f(x, C, C')$ satisfies,

(19) $$0 \leqq f(x, C, C') \leqq A + B|C| \, ,$$

on $[0, 1] \times [0, \infty) \times (-\infty, \infty)$ for some constants $A, B \geqq 0$. Thus, we allow linear rather than sublinear growth in C. We assume that the difussion coefficient satisfies (1) and that,

(20) $J(x, C)$ is continuous on $[0, 1] \times [0, \infty)$ and nonincreasing in C.

Let,

(21) $$M = \max_{z\in[0,1]}\left\{-v_0 + \int_0^z J(\tau, 0)d\tau\right\} \, ,$$

and

(22) $$\mu = \begin{cases}\inf D(x, C) \text{ if } M \geqq 0 \, , \\ \sup D(x, C) \text{ if } M < 0 \, .\end{cases}$$

Then $m(t) - m(x) \leqq M(t - x)/\mu$ from the formulas preceding (5).

Use of this inequality in (4) and some routine calculations lead to $||C|| \leqq G_1 + G_2||C||$, where G_1 is a positive constant and,

$$G_2 = \max_{x \in [0\ 1]} B\int_x^1 \frac{e^{M(t-x)/\mu}}{\varepsilon} t\, dt$$

or

$$G_2 = \frac{\mu B}{\varepsilon M}\left(1 - \frac{\mu}{M}\right)e^{M/\mu} + \frac{\mu^2 B}{\varepsilon M^2} . \tag{23}$$

If $G_2 < 1$, then,

$$||C|| \leqq G_1/(1 - G_2) , \tag{24}$$

is an apriori bound for fixed points of T defined as in § 2. Arguing as in the proof of Theorem 2.2, it follows that,

$$||C'|| \leqq G' < \infty . \tag{25}$$

Likewise, if $T_\lambda: K \to K$ (with T_λ defined as in § 2), then (24) and (25) are also a priori bounds for fixed points of T_λ for $0 \leqq \lambda \leqq 1$. Thus, we conclude, as in § 2, that T must have a fixed point.

THEOREM 4.1. *Assume* (19) *and* (20) *hold and that,*

$$\frac{\mu B}{\varepsilon M}\left(1 - \frac{\mu}{M}\right)e^{M/\mu} + \frac{\mu^2 B}{\varepsilon M^2} < 1 . \tag{26}$$

Then ($\mathscr{P}$) *has at least one nonnegative solution.*

We further specialize to the case when $D = D(x)$ is independent of C and $J(x, C(x)) = J(x)$ is independent of C. In this situation, the estimates above show that,

$$\begin{aligned}(\mathrm{TC})(x) &= c_1 e^{-m(x)} \\ &+ \int_x^1 \frac{e^{m(t)-m(x)}}{D(x)}\left\{L + \int_0^t f(\tau, C(\tau), C'(\tau))d\tau\right\} dt ,\end{aligned}$$

satisfies,

$$||T(C_1) - T(C_2)|| \leqq G_2||C_1 - C_2|| ,$$

with G_2 given by (23). Use of Banach's fixed point theorem gives,

THEOREM 4.2. *Assume that D and J are both independent of C and that* (19) *and* (26) *hold. Then* ($\mathscr{P}$) *has a unique nonnegative solution.*

Remark. In (26) if $M<0$ and $\mu = \sup D(x, C) = +\infty$, we interpret the left side to be $B/2\varepsilon$, its limiting value as $\mu \to \infty$.

5. Comments on existence and uniqueness. The uniqueness result established in § 3 is sharp in the sense that uniqueness need not hold if f depends on C' or if $f_C \leqq 0$ fails. To see this consider,

$$\tag{27} \begin{cases} -C'' + C' = \lambda C\,, \\ -C'(0) + C(0) = 0,\ C(1) = c_1\,, \\ \qquad v'(x) = 0 \\ \qquad v(0) = 1\,, \end{cases}$$

where λ is a parameter. (Thus, $D \equiv 1$, $v_0 = 1$, $J \equiv 0$ in $(\mathscr{P})$.) If $c_1 = 0$ the resulting eigenvalue problem from C has only positive eigenvolues. Let $\lambda = \lambda_1$ be the first eigenvalue of this eigenvalue problem and let y_1 be a corresponding eigenfunction. Then (27) with $c_1 = 0$ has all multiples of y_1 as solutions. Note here that $f(x, C) = \lambda_1 C$ satisfies $f_C > 0$.

On the other hand, if $\lambda \leqq 0$ (27) has at most one solution regardless of the choice of c_1. Finally, Theorem 4.2 applies with $B = \lambda$, $M = -1$, and $\varepsilon = \mu = 1$ to guarantee that (27) has a unique solution for

$$0 \leqq \lambda < 2/(e-2)\,.$$

To obtain nonuniqueness when f depends on C' set $f(x, C, C') = \lambda_1 C - C'$, $v_0 = 0$, and specify the other parameters as above. Then $(\mathscr{P})$ is,

$$\begin{cases} -C'' = \lambda_1 C - C'\,, \\ -C'(0) + C(0) = 0\,,\ C(1) = 0\,, \\ \qquad v(x) \equiv 0\,, \end{cases}$$

and nonuniqueness holds as above.

Consideration of the eigenvalue problem above shows that in general a priori bounds cannot be obtained for solutions to $(\mathscr{P})$ when $f(x, C, C')$ grows linearly in C and C', as opposed to the sublinear growth imposed in § 1. Thus, with the hypotheses of Theorem 2.1 general topological existence theorems requiring a priori bounds cannot be used to establish existence for a problem whose source's growth in C and C' is greater than sublinear. Such methods do apply in certain cases as indicated in § 4.

On the other hand, existence of solutions can be established for sources $f(x, C, C')$ which exhibit highly nonlinear growth in C and up to quadradic growth in C' provided $f(x, C, C')$ has certain additional properties. Some results of this kind are given in the next section. Related results may be found in [7] for the case of Dirichlet boundary conditions when $f = f(x, C)$ is independent of C' and $f(x, C)$ and $J(x, C)$ are assumed suitably smooth.

6. Further existence results. Consider the following variant of $(\mathscr{P})$:

$$(\mathscr{P}')\quad \begin{cases} -(D(x)C'(x))' + (v(x)C(x))' = f(x, C(x), C'(x)),\ 0 \leqq x \leqq 1\,, \\ \alpha C(0) - \beta C'(0) = L_0,\ aC(1) + bC'(1) = L_1\,, \\ \qquad v'(x) = -J(x, C(x))\,, \\ \qquad v(0) = v_0\,. \end{cases}$$

Here $\alpha, \beta, a, b, L_0, L_1 \geqq 0$, $\alpha, a > 0$, and v_0 are given constants. We assume $D(x) > 0$, $f(x, C, C')$, and $J(x, C)$ are continuous on $[0, 1]$, $[0, 1] \times [0, \infty) \times (-\infty, \infty)$, and $[0, 1] \times [0, \infty)$, respectively. Assume also that,

(i)′ There is a constant $M \geqq 0$ such that, $f(x, C, 0) + CJ(x, C) \leqq 0$ for $C \geqq M$;

(ii)′ There are nonnegative functions $A(x, C)$ and $B(x, C)$, which are bounded on bounded sets, for which,

$$|f(x, C, C')| \leqq A(x, C)C'^2 + B(x, C)\,,$$

for (x, C, C') in $[0, 1] \times [0, \infty) \times (-\infty, \infty)$;

(iii)′ $f(x, 0, 0) \geqq 0$ on $[0, 1]$.

THEOREM 6.1. *Assume that* (i)′, (ii)′, (iii)′ *hold. Then* $(\mathscr{P}')$ *has at least one nonnegative solution. This solution is strictly positive if* $L_0, L_1 > 0$ *and strict inequality holds in the conditions* (i)′ *and* (iii)′.

The proof will be given in several lemmas. If (C, v) solves $(\mathscr{P}')$, then a short calculation shows that C solves,

$$(28)\quad \begin{aligned} &C'' = g(x, C, C') \\ &\alpha C(0) - \beta C'(0) = L_0,\ aC(1) + bC'(1) = L_1 \end{aligned}$$

where

$$g(x, C, C') = -\frac{f(x, C, C') + \left(D'(x) - v_0 + \int_0^x J(t, C(t))\,dt\right)C' + CJ(x, C)}{D(x)}\,.$$

Conversely, if C solves (28) and v is defined as in $(\mathscr{P}')$, then (C, v) solves $(\mathscr{P}')$.

We will derive a priori bounds for the family of problems,

$$(28)_\lambda \qquad \begin{aligned} &C'' = \lambda g(x, C, C') ,\\ &\alpha C(0) - \beta C'(0) = L_0,\ \mathrm{a}C(1) + \mathrm{b}C'(1) = L_1 , \end{aligned}$$

for λ in $[0, 1]$ and $C(x) \geqq 0$.

We extend the definitions of f and J to include $C < 0$ and preserve (i)′, (ii)′, and (iii)′ by setting $f(x, C, C') = f(x, |C|, C')$ and $J(x, C) = J(x, |C|)$ for $C < 0$. Also assume for the moment that $f(x, C, 0) + \mathrm{C}J(x, C) < 0$ in (i)′, that $f(x, 0, 0) > 0$ in (iii)′, and that $L_0, L_1 > 0$.

LEMMA 6.2. *If $C(x) \geqq 0$ solves $(28)_\lambda$ then*

$$0 < C(x) \leqq \max (L_0/\alpha, L_1/a, M) \equiv M_0 .$$

Proof. If $\lambda = 0$, the result is easily checked (see (29) below). If $C(x)$ achieves its maximum at $x=0$, then $C'(0) \leqq 0$ and so $C(0) \leqq L_0/\alpha$ from the boundary data. Likewise, if $C(x)$ achieves its maximum at $x = 1$, then $C(1) \leqq L_1/a$. If x_0 is an interior maximum, then,

$$\begin{aligned} 0 \geqq C''(x_0) &= g(x, C(x_0), 0) \\ &= -\frac{f(x_0, C(x_0), 0) + C(x_0)J(x_0, C(x_0))}{D(x_0)} \end{aligned}$$

which implies that $C(x_0) \leqq M$ by (i)′. Consequently, $C(x) \leqq M_0$. If $C(x_0) = 0$ for some x_0 in $(0, 1)$, then C has a minimum at x_0 and so,

$$0 \leqq C''(x_0) = g(x_0, 0, 0) = -\frac{f(x_0, 0, 0)}{D(x_0)} < 0$$

by the strengthened form of (i)′, a contradiction. Thus, $C(x)>0$ on $(0, 1)$. This fact and the boundary data imply that $C(0)$ and $C(1)$ are positive.

LEMMA 6.3. *If $C(x) \geqq 0$ is a solution of $(28)_\lambda$, then*

$$|C'(x)| \leqq \max (2M_0, e^{2A_1M_0}(2A_1M_0 + B_1)^{1/2}) \equiv M_1$$

where $A_1 = \sup A_1(x, C)$ and $B_1=\sup B_1(x, C)$ over $[0, 1] \times [-M_0, M_0]$. Here $A_1(x, C)$ and $B_1(x, C)$ are functions which bound $|g(x, C, C')|$ just as $|f(x, C, C')|$ is bounded in (ii)′. *[Since* (ii)′ *holds it is clear that such functions exist.]*

Proof. Since $0 \leqq C(x) \leqq M_0$, the mean-value theorem implies that $|C'(x)|$ cannot be constantly greater than $2M_0$. So $-2M_0 \leqq$

$C'(x_0) \leqq 2M_0$ for some x_0 in [0, 1]. Now the reasoning used in [3, Lemma 3.1] yields the bound for $|C'(x)|$ stated above.

Use of (6.2) and (6.3) in $(28)_\lambda$ establishes,

LEMMA 6.4. *There is a constant M_2 such that,*

$$|C''(x)| \leqq M_2 ,$$

for any solution $C(x) \geqq 0$ to $(28)_\lambda$.

Define $C^2_{\mathscr{B}}$ to be the convex subset of functions in C^2 which satisfy the boundary conditions $\mathscr{B}$ in (28). Define $\hat{M} = 1 + \max(M_0, M_1, M_2)$,

$$U = \{y \in C^2_{\mathscr{B}}: 0 < y(x) < \hat{M},\ |y'|_0,\ |y''|_0 < \hat{M}\} ,$$

$L: C^2 \to C^2$ and $F: C^1 \to C$ by

$$\mathrm{Ly}(x) = y''(x),\ (\mathrm{Fw})(x) = g(x, w(x), w'(x)) ,$$

respectively. Then L is invertible with continuous inverse,

$$(L^{-1}f)(x) = \int_0^1 G(x, t)f(t)\,\mathrm{dt} + l(x) ,$$

where $G(x, t)$ is the Green's function for L together with the homogeneous boundary conditions corresponding to (28) and

$$l(x) = \frac{(\alpha L_1 - \mathrm{a}\mathrm{L}_0)x + (\beta L_1 + \mathrm{a}\mathrm{L}_0 + \mathrm{b}\mathrm{L}_0)}{(a\beta + a\alpha + b\alpha)} , \tag{29}$$

is the solution to $\mathrm{Ly} = 0$ satisfying the inhomogeneous boundary conditions $\mathscr{B}$. If $j: C^2_{\mathscr{B}} \to C^1$ is the completely continuous embedding, then $H: \bar{U} \times [0, 1] \to C^2_{\mathscr{B}}$ defined by $H(u, \lambda) = \lambda L^{-1}Fj + (1 - \lambda)l$ is a compact homotopy. If $H_\lambda u = u$, then $\lambda L^{-1}\mathrm{Fu} + (1 - \lambda)l = u$ so $\lambda\mathrm{Fu} = \mathrm{Lu}$ because $Ll = 0$. By definition of $\hat{M}$, H_λ is fixed point free on ∂U. By topological transversality, H_1 will be essential if $H_0 = l$ is essential. Since $l(0)$, $l(1) > 0$ it follows that $l(x) > 0$ so l is an interior point of U in $C^2_{\mathscr{B}}$. By Theorem 1.2 of § 1, H_0 is essential. So H_1 has a fixed point, that is, (28) has a solution C in U. By (6.2), $y(x) > 0$. This proves that last part of (6.1).

Finally, we relax the conditions that strict inequality hold in (i) and (iii) and that $L_0, L_1 > 0$. By replacing f by $f + 1/n$, J by $J - 1/n$, L_0 by $L_0 + 1/n$, and L_1 by $L_1 + 1/n$, we obtain for each $n=1, 2, \cdots$ a problem of the form (28) which has a solution $C_n(x) \geqq 0$. The arguments used to prove (6.2) and (6.3) produce a constant M' independent of n so that $|C_n|_2 \leqq M'$. A standard compactness argument yields a solution C to (28). Clearly, $C \geqq 0$. This completes the proof of (6.1).

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