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CONVERGENCE THEOREMS FOR SOME SCALAR VALUED INTEGRALS WHEN THE MEASURE IS NEMYTSKII

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One of the main potential applications of Hammerstein operators is a functional analytic study of nonlinear differential equations. In fact, some connections have already been established with equations of the form $\dot{x}(t)=\phi[x(t)]$ or $\dot{x}(t)=\phi[x(t), t]$. Other applications have been made to generalized random processes and the theory of fading memory in continuum mechanics. The main purpose of the present paper is to establish and study the representation of Hammerstein operators on continuous functions. A "nonlinear" integral is introduced for this purpose. Convergence theorems for a.e. and convergence in measure are established and contrasted. The last result of the paper relates uniform integrability, a key concept in the study of martingales, to essential ranges, an important concept used to establish the differentiability of some set functions.

1. Introduction. One of the main potential applications of Hammerstein operators, as stated in [12], is a functional analytic study of nonlinear differential equations. In fact the properties of the function ϕ required to insure the existence of solutions to the differential equations

$$\dot{x}(t) = \phi[x(t)] \text{ or } \dot{x}(t) = \phi[x(t), t]$$

are closely related to the properties of the kernel ϕ used to represent an abstract Hammerstein operator T as in [12]. The representation there is given by the formula

$$T(f) = \int \phi[f(t), t]d\mu(t).$$

Other applications have already been initiated, we mention two of these; applications to the theory of generalized random processes in [9] and to the theory of fading memory in continuum mechanics in [5].

The above considerations motivated other work on the representation of Hammerstein operators over different function spaces. The reader is referred to [1], [2] and [3] for the cases where the function space is L^1, $M_E(\mathscr{B})$ or $C(K, X)$. The representation obtained in [1], [2] and [3] is of the form

$$T(f) = \int f dm$$

where the integral is (necessarily) nonlinear. Here m denotes a finitely additive set function defined on an appropriate ring with values in $M(X, C)$ which is a certain space of maps (not necessarily linear) from X (a Banach space) into C (the scalar field). The integral is of course suitably defined. These representations should be contrasted with the representation obtained in [12] where a kernel function is present and where the integral is the standard one. The nonlinearity imposed on the operator T in all of the above works is:

$$T(f + f_1 + f_2) = T(f + f_1) + T(f + f_2) - T(f)$$

whenever f_1 and f_2 have disjoint supports. This condition has been called the Hammerstein property by J. Batt in [3] and the additive property by N. A. Friedman and A. E. Tong in [8].

The main purpose of the present paper is to study the properties of the nonlinear integral as defined in [1], [2] and [3]. The main interest here focuses on Hammerstein operators that are scalar valued and defined over $C(K, X)$, the space of continuous functions over the compact space K under the supremum norm and with values in the Banach space X. Thus in our study of the properties of $\int f dm$, f will be defined over K with values in X and m will be a finitely additive set function with values in $M(X, C)$ and will satisfy certain continuity conditions. Actually most of our results can immediately be generalized to abstract Nemytskii measures (see [2]) since continuity of the functions will not be used in most of the proofs. It should be recalled that if T is a Hammerstein operator on $C(K, X)$, then $T(f) = \int f dm_T$ where m_T has values in $M(X, C)$ and moreover m_T is a measure extendable to $\mathscr{B}$, the Borel field of K (see [3]). Two types of convergence will be studied here. The first convergence will be $s_n \to f$ m a.e. where $\{s_n\}$ is a sequence of simple X-valued functions. The key property in this type of convergence will be the requirement that $\int_{(\cdot)} s_n dm$ be uniformly countably additive measures. An important technical device introduced will be a version of the Egoroff Theorem. The last theorem obtained for m a.e. convergence states that if T is a scalar valued Hammerstein operator on $C(K, X)$ that has a G-representation (see R. K. Goodrich [10]) with respect to m', then $m'(\cdot)x$ is necessarily uniformly countably additive for $\|x\| \leqq \alpha$ provided the bounded or the dominated convergence theorem holds for m'. The second type of convergence to be studied is convergence in measure. It is appropriate here to recall that W. V. Smith and D. H. Tucker [15] and also D. H. Tucker and S. G. Wayment [16] have constructed examples of X-valued functions converging in measure but having no subsequence converging m a.e. (and even an example of a sequence that converges to different functions in measure and m a.e.!) For convergence in measure we use the concept of essential range introduced by M. A. Rieffel [13] to obtain the Radon-Nikodym Theorem for Bochner integrals. Another central concept is uniform integrability. Uniform integrability is known to play a key role for martingale convergence theorems, for example see [4]. The last theorem obtained in the present paper states that if f_n are essential range functions and $\int_A f_n dm$ converges to 0 uniformly for $A \in \mathscr{B}$ then the f_n converge to 0 in measure provided the essential ranges of f_n are bounded away from 0 over appropriate sets and provided they transfer uniform integrability. Of course it is easy to give an example where $\int_A f_n dm$ converges to 0 but f_n does not converge to 0 in measure, even if m is linear. For example let $m(A)(x, y) = (\lambda(A)x, 0)$ where $(x, y) \in R^2$ and where λ is the Lebesgue measure and let $f_n(t) = (0, n)$. Then $\int_A f_n dm = 0$ yet f_n does not converge to 0 in measure. (See W. V. Smith [14], for example.)

We now present a summary of the results obtained. All of these results pertain to measures representing Hammerstein operators T. To stress this we will use the notation m_T in the section on results. The first result states that if $\{s_n\}$ is a sequence of X-valued simple functions converging m a.e. to f and if the measures $\int_{(\cdot)} s_n dm$ are uniformly countably additive, then there exists a unique scalar measure r such that $r(E) = \lim \int_E s_n dm$, moreover the limit is uniform for $E \in \mathscr{B}$. This allows us to define the space $L^1(m)$ as the space of functions from K into X that are limits m a.e. of a sequence of simple functions s_n with $\int_{(\cdot)} s_n dm$ uniformly countably additive. We then show that $\{s_n\}$ may be replaced by $\{f_n\}$ where $f_n \in L^1(m)$. The second result is an Egoroff type theorem. The norm used is of the type $\sup_{||x|| \leqq \alpha} |s_n(\cdot)x - f(\cdot)x|$ where s_n and f are scalar valued and x is in X. As a corollary to this result we show that if in addition $\left\{\int_{(\cdot)} s_n(\cdot)x dm\right\}$ is uniformly countably additive (in n and for $||x|| \leqq \alpha$), then there exists a scalar measure r_x such that $\sup_{||x|| \leqq \alpha} \left| r_x(E) - \int_E s_n(\cdot)x dm \right|$ converges to 0 uniformly for $E \in \mathscr{B}$. The third result shows a version of the bounded convergence theorem for the nonlinear integral. This result is followed by a version of the dominated convergence theorem. Our next result relates to G-representation as defined by R. K. Goodrich, see [10]. For our second type of convergence we use the α-semi-variation function to define convergence in measure. We already have discussed the examples obtained in [15] and [16] to point out the rather striking differences between the two modes of convergence. We define the function f to be m-integrable if there exists a sequence $\{s_n\}$ of simple functions that are uniformly integrable (see [4]) and such that $\{s_n\}$ converges to f in measure. It is then shown that $\int_A s_n dm$ converges uniformly for $A \in \mathscr{B}$. Our last result, already mentioned above, used the essential ranges of $\{f_n\}$ (see [13]) and the property of uniform integrability to yield a sufficient condition for the sequence $\{f_n\}$ to converge to 0 in measure.

II. Results. We introduce some basic notations. Let K denote a compact set, $\mathscr{B}$ the Borel sets of K, $C(K, X)$ X-valued functions that are continuous and defined on K with the topology of the sup norm. Here X denotes a Banach space. Let u be a function from X into C. For $\alpha > 0$ let u_α denote the restriction of u to the closed α-ball of X. Let $\|u_\alpha\| = \sup \|u(x)\|$ where the sup is over $\|x\| \leqq \alpha$. For $\delta > 0$ define

$$D_\delta u_\alpha = \sup \|u(x) - u(y)\| \text{ where the sup is over}$$

$\|x\| \leqq \alpha$, $\|y\| \leqq \alpha$, $\|x - y\| \leqq \delta$. Let $M(X, C)$ denote the space of all functions from X into C which are bounded on the α-balls of X, uniformly continuous on bounded sets of X and 0 at 0. Thus if $u \in M(X, C)$ then

$$u(0) = 0, \ \|u_\alpha\| < \infty, \ \lim_{\delta \to 0} D_\delta u_\alpha = 0 .$$

Let m be a finitely additive function from $\mathscr{B}$ into $M(X, C)$. We set $m_\alpha(B) = m(B)_\alpha$ and define

$$sv[m_\alpha, B] = \sup \|\Sigma m(B_i)x_i\| .$$

Here the sup is over finite partitions $\{B_i\}$ of B with $\|x_i\| \leqq \alpha$. Let

$$sv_\delta(m_\alpha, B) = \sup \|\Sigma m(B_i)x_i - \Sigma m(B_i)y_i\|$$

where the sup is over finite partitions $\{B_i\}$ of B and $\|x_i\| \leqq \alpha$, $\|y_i\| \leqq \alpha$, $\|x_i - y_i\| \leqq \delta$. Unless otherwise stated m denotes a finitely additive set function from $\mathscr{B}$ into $M(X, C)$ satisfying

$$sv[m_\alpha, K] < \infty \text{ and } \lim_{\delta \to 0} sv_\delta[m_\alpha, K] = 0 .$$

A property will be called true m a.e. if the property is true for all $t \notin A$ and if whenever $B \subset A$ and $B \in \mathscr{B}$ then $m(B) = 0$. Let

χ_{C_j} represents the characteristic function of C_i. For disjoint C_i we define

$$\int \Sigma \chi_{C_i} x_i dm = \Sigma m(C_i) x_i$$

$M_H[C(K, X), C]$ will denote all scalar valued maps defined on $C(K, X)$ and satisfying

$$T(f + f_1 + f_2) = T(f + f_1) + T(f + f_2) - T(f)$$

whenever f_1 and f_2 have disjoint supports

$$T(0) = 0$$

$$\| T_\alpha \| < \infty$$

$$\lim_{\delta \to 0} D_\delta T_\alpha = 0 \text{ where } D_\delta T_\alpha = \sup \| Tx_i - Ty_i \| \text{ and}$$

where the sup is over $\|x\| \leqq \alpha$, $\|y\| \leqq \alpha$, $\|x - y\| \leqq \delta$. In [3], J. Batt has shown that if $T \in M_H[C(K, X), C]$, then T can be written as $T(f) = \int f dm$ where the nonlinear integral is extended from simple functions to functions in $C(K, X)$. Moreover $\| T_\alpha \| = sv[m_\alpha, K]$ and $D_\delta T_\alpha = sv_\delta[m_\alpha, K]$. If $T \in M_H[C(K, X), C]$, m_T will denote the corresponding measure. We now prove our first result.

THEOREM 1. *Let $\{s_n\}$ be a sequence of X-valued simple functions. Assume $\{s_n\}$ converges to f pointwise and assume that the set functions $\int_{(\cdot)} s_n dm_T$ are uniformly countably additive. Then there exists a unique, countably additive, scalar measure r such that $r(E) = \lim_{n \to \infty} \int_E s_n dm_T$ uniformly for $E \in \mathscr{B}$.*

Proof. For every $E \in \mathscr{B}$ we define

$$\hat{m}(E) = \sum_{n=1}^{\infty} \frac{\int_E s_n dm_T}{2^n \left[1 + \sup_{A \in \mathscr{B}} \left| \int_A s_n dm_T \right| \right]}$$

clearly $\hat{m}$ is a finitely additive set function on $\mathscr{B}$. It is shown in [3] that $m_T(\cdot)x$ is countably additive for $x \in X$. In fact it is uniformly countably additive for $\|x\| \leqq \alpha$. It follows that $\hat{m}$ is a bounded countably additive function on $\mathscr{B}$. Since $\{s_n\}$ converges to f, by Corollary 1 to Egoroff's Theorem (see [6, p. 95]). there exists a sequence $\{A_k\}$ of disjoint sets of $\mathscr{B}$ and $N \in \mathscr{B}$ such that $\{s_n\}$ converges uniformly to f on each A_k, N is a $\hat{m}$-null set, and $K = N \cup \bigcup A_k$. Thus, $\int_N s_n dm_T = 0$ for all n. Let $B_k = \bigcup_{j=1}^k A_j$, clearly $B_k \uparrow N'$ and the convergence is uniform on each B_k. Moreover,

$$\left|\int_E (s_n - s_p)dm_T\right| \leqq \left|\int_{E\cap B_k} (s_n - s_p)dm_T\right| + \left|\int_{E\cap B'_k\cap N'} (s_n - s_p)dm_T\right|$$
$$\leqq sv_{\tau_k}[m_{T\alpha}, K] + \left|\int_{E\cap B'_k\cap N'} s_n dm_T\right| + \left|\int_{E\cap B'_k\cap N'} s_p dm_T\right|$$

where $\tau_k = \operatorname{ess\,sup} \|s_n - s_p\|$ restricted to B_k and where $\alpha = \sup |x_i|$ where x_i is a value in Range $s_n \cup$ Range s_p where s_n and s_p are restricted to $E \cap B_k$. The first term on the right goes to 0 as n and p get large. The next two terms tend to 0 by uniform countable additivity of $\int_{(\cdot)} s_n dm_T$. Thus $\int_E s_n dm_T$ converges uniformly for $E \in \mathscr{B}$ and

$$r(E) = \lim_{n\to\infty} \int_E s_n dm_T$$

has the stated properties.

Note. The above proof can be extended to the case in which pointwise convergence of $\{s_n\}$ is replaced by m_T a.e. convergence.

We now define $\int_E f dm_T = \lim \int_E s_n dm_T$. The arguments above show that $\int_E f dm_T$ is well defined. Let $L^1(m_T)$ denote all functions f: $K \to X$ where f is the limit of a sequence $\{s_n\}$ of simple functions m_T a.e. and where $\int_{(\cdot)} s_n dm_T$ are uniformly countably additive.

PROPOSITION 1. *If $f_n \in L^1(m_T)$ and $\{f_n\}$ converges to f m_T a.e. and $\int_{(\cdot)} f_n dm_T$ are uniformly countably additive, then $f \in L^1(m_T)$ and $\int_E f_n dm_T$ converges to $\int_E f dm_T$ uniformly for $E \in \mathscr{B}$. Moreover*

$$\left|\int_E f dm_T\right| \leqq sv_{\|f\|_E}[m_T, B] \text{ where } \|f\|_E \text{ is}$$

the essential sup *of f restricted to E.*

Proof. The first part of the proof follows from the fact that the proof of Theorem 1 remains true when $\{f_n\}$ replaces $\{s_n\}$. In addition it is clear that a sequence of simple functions $\{s_n\}$ may be found which converges to f and for which $\left\{\int_{(\cdot)} s_n dm_T\right\}$ is uniformly countably additive. Now:

$$\left|\int_E f dm_T\right| \leqq \left|\int_E (f - s_n)dm_T\right| + \left|\int_E s_n dm_T\right|$$
$$\leqq \left|\int_E (f - s_n)dm_T\right| + \left|\int_{B_k\cap E} s_n dm_T\right| + \left|\int_{E\cap B'_k\cap N'} s_n dm_T\right|$$

$$\leqq sv\|s_n\|_{B_k \cap E}[m_T, K] + \left|\int_{E \cap B_k' \cap N'} s_n dm_T\right| + \left|\int_E (f - s_n) dm_T\right|$$

where B_k are as in Theorem 1. As $n \to \infty$, the first term goes to $sv\|f\|_{B_k \cap E}$ and the last term converges to 0. As $k \to \infty$ the middle term converges uniformly to 0 in n and the first term converges to $sv\|f\|_E$.

We now prove a version of the Egoroff Theorem which is suitable for our purpose.

THEOREM 2 (Egoroff). *Let* $\{s_n\}$ *be a sequence of real valued simple functions, let* f *be a real valued function and assume that*

$$\sup_{\|x\| \leqq \alpha} |s_n(\cdot)x - f(\cdot)x| \text{ converges to } 0 \; m_T \text{ a.e.}$$

Then there exists a sequence $\{A_k\}$ *of disjoint sets of* $\mathscr{B}$ *and* $N \in \mathscr{B}$ *such that*

(1) $K = N \cup \bigcup_k A_k$.

(2) $\sup_{\|x\| \leqq \alpha} |s_n(\cdot)x - f(\cdot)x|$ *converges to* 0 *uniformly on* A_k.

(3) $\int_N s_n(\cdot)x dm = 0$ *for all* n *and* $\|x\| \leqq \alpha$.

Proof. Let

$$\hat{m}_x(E) = \sum_n \left[\frac{\int_E s_n(\cdot)x dm_T}{1 + \sup_{A \in \mathscr{B}} \left|\int_A s_n(\cdot)x dm_T\right| 2^n}\right] \text{ For each fixed } x \text{ with}$$

$\|x\| \leqq \alpha$, $\hat{m}_x$ is finitely additive, bounded and since $m_\alpha(\cdot)$ is countably additive, it follows that $\hat{m}_x(\cdot)$ is uniformly countably additive for $\|x\| \leqq \alpha$. Since $\hat{m}_x$ is uniformly bounded it follows that $\{m_x \mid \|x\| \leqq \alpha\}$ is weakly sequentially compact (see [7, p. 305]). Thus there exists a positive measure λ_α where $\lim_{\lambda_\alpha(E) \to 0} \hat{m}_x(E) = 0$ uniformly for $\|x\| \leqq \alpha$. Also λ_α may be chosen so that $\lambda_\alpha(E) \leqq \sup_{\|x\| \leqq \alpha} |\hat{m}_x(E)|$ (see [7, p. 307].). Since $\sup_{\|x\| \leqq \alpha} |s_n(\cdot)x - f(\cdot)x|$ converges to 0 $\hat{m}_x$ a.e. for $\|x\| \leqq \alpha$, it converges to 0, λ_α a.e. The proof then proceeds as in Theorem 1.

COROLLARY 1. *Let* $\{s_n\}$ *and* f *be as above. Assume* $\int_{(\cdot)} s_n(\cdot)x dm_T$ *are uniformly countably additive with respect to* n *and for* $\|x\| \leqq \alpha$. *Then there exists scalar measures* r_x *such that*

$$\sup_{\|x\| \leqq \alpha} \left|r_x(E) - \int_E s_n dm_T\right| \text{ converges to } 0 \text{ uniformly}$$

for $E \in \mathscr{B}$ *and* $\|x\| \leqq \alpha$.

Proof. Let $\{A_k\}$ and N be as previously defined. Let

$$B_k = \bigcup_{j=1}^{k} A_j \qquad \sup_{\|x\| \leqq \alpha} \left| \int_E (s_n - s_p)(\cdot) x dm_T \right|$$
$$\leqq \sup_{\|x\| \leqq \alpha} \left| \int_{E \cap B_k} (s_n - s_p)(\cdot) x dm_T \right| + \sup_{\|x\| \leqq \alpha} \left| \int_{E \cap B_k' \cap N} (s_n - s_p)(\cdot) x dm_T \right|$$
$$\leqq sv\beta_{(n,p)}[m_T, K] + \sup_{\|x\| \leqq \alpha} \left| \int_{E \cap B_k' \cap N'} s_n(\cdot) x dm_T \right|$$
$$+ \sup_{\|x\| \leqq \alpha} \left| \int_{E \cap B_k' \cap N'} s_p(\cdot) x dm_T \right|$$

where $\beta(n, p) = \sup_{\|x\| \leqq \alpha}$ ess sup $\|s_n(\cdot) x \chi_{B_k} - s_p(\cdot) x \chi_{B_k}\|$. Hence $\lim_{n \to \infty p \to \infty} \beta(n, p) = 0$, the last two terms converge to 0 by uniform countable additivity.

Note. We may write $\int_E f(\cdot) x dm_T = r_x(E)$.

Let $M(K)$ denote all scalar valued functions on K that are limits m_T a.e. of simple functions.

THEOREM 3. *Let* $\{f_n\}$ *be a sequence in* $M(K)$. *Assume that* $\{f_n(\cdot)x\}$ *converges to* $f(\cdot)xm_T$ a.e. *uniformly for* $\|x\| \leqq \alpha$. *Assume also that* $\{f_n\}$ *is uniformly bounded by some constant* M. *Then*

$$\left\{\int_E f_n(\cdot) x dm_T\right\} \text{ converges to } \int_E f(\cdot) x dm_T$$

uniformly for $E \in \mathscr{B}$.

Proof. Without loss of generality assume $f_n \geqq 0$. (Note that f_n^+ and f_n^- have disjoint supports.) Let $\{g_{n,i}\}$ be a sequence of simple functions such that $g_{n,i} \uparrow f_n$. Then

$$g_{n,i} = \sum_{i=1}^{k_n} a_i^n \chi_{E_i(n)}$$

and

$$\int_E g_{n,i} dm_T = \sum_{i=1}^{k_n} a_i^n m_T[E_i(n)]x .$$

Let $\sum_p$ and $\sum_N$ denote sums over positive and negative terms.

$$\left| \sum_{i=1}^{k_n} a_i^n m_T[E_i(n)]x \right| = \sum_p a_i^n m_T[E_i(n)]x - \sum_N a_i^n m_T[E_i(n)]x \cdot$$
$$\leqq 2M \operatorname{var} m_T(E)x .$$

Since $m_T(\cdot)x$ is countably additive (uniformly for $\|x\| \leqq \alpha$), the $\left\{\int_{(\cdot)} g_{n,i}(\cdot)xdm_T\right\}$ are uniformly countably additive (with respect to n, i and for $\|x\| \leqq \alpha$). Hence by Corollary 1, $\lim_{i\to\infty}\int_F g_{n,i}(\cdot)xdm_T = \int_F f_n(\cdot)xdm_T$ uniformly for $F \in \mathscr{B}$ and $\|x\| \leqq \alpha$. Also $\left\{\int_{(\cdot)} f_n(\cdot)xdm_T\right\}$ is uniformly countably additive. So again $\left\{\int_E f_n(\cdot)xdm_T\right\}$ converges to $\int_E f(\cdot)xdm_T$ uniformly in E.

Of course Theorem 3 is a version of the bounded convergence theorem for nonlinear integrals. We now obtain a version of the dominated convergence theorem.

PROPOSITION 2. *Let $\{f_n\}$ be a sequence in $L^1(m_T)$ and assume that $\{f_n\}$ converges to f m_T* a.e. *Moreover assume*

$$\left|\int_E f_n dm_T\right| \leqq \sup_{F\subset E}\left|\int_F gdm_T\right| \quad \textit{where } g \in L^1(m_T)\ .$$

Then $f \in L^1(m_T)$ and $\left\{\int_E f_n dm_T\right\}$ converges to $\int_E fdm_T$ uniformly for $E \in \mathscr{B}$.

Proof. Since $g \in L^1(m_T)$, by Theorem 1, $\int_{(\cdot)} gdm_T$ is countably additive and hence bounded. Since from the hypothesis $\int_{(\cdot)} f_n dm_T$ is dominated by the variation of $\int_{(\cdot)} gdm_T$, $\int_{(\cdot)} f_n dm_T$ are uniformly countably additive and by Proposition 1 the result follows.

We now proceed to establish a partial converse to the above result. In the work of R. K. Goodrich [10] the following ring of subsets plays a central role. Let R be the ring of all subsets E of K for which there exist nonincreasing sequences of continuous functions $\{f_n\}$, $\{g_n\}$ with $f_n - g_n$ converging to χ_E. It is shown in [10] that R is the ring generated by all compact G_δ subsets of K. Following [10], if T is a (not necessarily linear) operator from $C(K, X)$ into scalars we say that T has a G-type representation if

$$T(f) = \int fdm' \text{ where } m' \text{ maps the ring } R \text{ into } M(X, C)$$

and where $m'(\cdot)x$ is countably additive for each fixed x. (Of course, it is shown in [10] that every continuous linear operator on $C(K, X)$ has a G-type representation.)

We now consider two conditions related to the previous result.

(A) Theorem 3 is true with m' replacing m_T.

(B) Let $\{f_n\}$ be a sequence of scalar valued functions converging m' a.e. to f and assume that $\left|\int_E f_n dm'\right| \leqq \sup_{F\subset E}\left|\int_E g dm'\right|$ for some $g \in L^1(m')$. Then $\left\{\int_E f_n(\cdot)x dm'\right\}$ converges to $\int_E f(\cdot)x dm'$ for each fixed x.

THEOREM 4. *Let $T \in M_H[C(K, X), C]$. Assume that T has a G-representation with respect to m'. Then under condition* (A) *or* (B) *$m'(\cdot)x$ is uniformly countably additive for $\|x\| \leqq \alpha$.*

Proof. Suppose (A) holds: Let $E_i \in R$ with $E_i \downarrow \varnothing$. Then $m'(E_i)x$ converges to 0 for every fixed x. By the Kluvanek extension theorem [11], $m'(\cdot)x$ has a unique countably additive extension to $\mathscr{B}$ and since by [3] T admits the representation $T(f) = \int f dm$ where $m(\cdot)x$ is uniformly countably additive for $\|x\| \leqq \alpha$ and since $m(\cdot)x = m'(\cdot)x$, it follows that m' has the same property. The same argument can be made under assumption of condition (B).

We now initiate a study of convergence in measure. In the introduction we mentioned the works of W. V. Smith and D. H. Tucker [15] and of D. H. Tucker and S. G. Wayment [16]. Their examples highlight the great difference between convergence a.e. and convergence in measure. We now define convergence in measure. Let m_T be as above. We say $\{f_n\}$ converges to f in measure if $sv_\alpha m_T\{\|f_n(\cdot) - f(\cdot)\| \geqq \delta\}$ converges to 0 as n gets large, for every $\alpha > 0$ and $\delta > 0$ fixed. It is obvious, since the semi-variation is subadditive, that if $\{f_n\}$ converges to f in measure then for every fixed $\varepsilon > 0$, $\alpha > 0$, and $\delta > 0$, $sv_\alpha m_T\{\|f_{n_1}(\cdot) - f_{n_2}(\cdot)\| \geqq \delta\} < \varepsilon$ provided $n_1 \geqq N$ and $n_2 \geqq N$. Of course N depends on ε, α, δ. We now define an X-valued function to be m_T-integrable if there exists a sequence $\{s_n\}$ of uniformly integrable simple functions such that $\{s_n\}$ converges to f in measure and such that for every $\varepsilon > 0$, $\alpha > 0$, there exists a number u (depending on ε and α) such that

$$sv_\alpha[m_T, E] < u \text{ implies } \left|\int_E s_n dm_T\right| < \varepsilon \text{ i.e.,}$$

$\left\{\int_{(\cdot)} s_n dm_T\right\}$ is uniformly continuous with respect to the semi-variation. Recall that uniform integrability means that for every $\varepsilon > 0$ there exists $K(\varepsilon)$ such that

$$\left|\int_{\{\|s_n\| > K(\varepsilon)\}\cap A} s_n dm_T\right| < \varepsilon \text{ for all } n\,.$$

In the introduction we have already stated that uniform integrability has implications for the convergence of martingales. There are also strong implications for convergence properties in L^p. For example $\{g_n\}$ converges in μ measure to g with $|g_n|^p$ uniformly integrable is equivalent to the norm convergence in L^p (see [4, p. 185] where conditions equivalent to uniform integrability are pointed out).

PROPOSITION 3. *Let f be m_T integrable. Let s_n be as above. Then* $\lim \int_A s_n dm_T$ *exists uniformly for* $A \in \mathscr{B}$.

Proof. Pick $\varepsilon > 0$, let $K(\varepsilon)$ be chosen so that

$$\left|\int_{\{\|s_n\|\geqq K(\varepsilon)\}} s_n dm_T\right| < \varepsilon/4 \text{ for all } n. \quad \text{Now pick } u$$

so that $sv_{K(\varepsilon)}m_T(E_1) < u$ implies $\left|\int_{E_1} s_n dm_T\right| < \varepsilon/4$. Now pick u' so that

$$sv_{u'}[(m_T)_{K(\varepsilon)}, K] < \varepsilon/4 .$$

By the subadditivity of the semi-variation pick N so that

$$sv_{K(\varepsilon)}m_T\{\|s_{n_1}(\cdot) - s_{n_2}(\cdot)\| > u'\} < u$$

for $n_1 \geqq N$ and $n_2 \geqq N$.

Let $E_2 = \{\|s_{n_1}(\cdot) - s_{n_2}(\cdot)\| > u'\}$. Then

$$\left|\int_{E_2} s_{n_1} dm_T - \int_{E_2} s_{n_2} dm_T\right| < \varepsilon/2 \text{ since}$$

$sv_{K(\varepsilon)}m_T(E_2) < u$. Now

$$\left|\int_{E_2'} s_{n_1} dm_T - \int_{E_2'} s_{n_2} dm_T\right| \leqq sv_{u'}[m_T, K] + \varepsilon/4 + \varepsilon/4 < 3\varepsilon/4 .$$

Thus for all $A \in \mathscr{B}$,

$$\left|\int_A s_{n_1} dm_T - \int_A s_{n_2} dm_T\right| < 2\varepsilon .$$

Thus $\lim \int_A s_n dm_T$ exists uniformly as $A \in \mathscr{B}$.

Note. If we denote this limit by $m_f(A)$, then m_f is a measure on $\mathscr{B}$ by the Nikodym theorem (see [7, p. 160]).

Let f be a function from K into X. By the essential range of f over some set $E \subset K$ we mean $\{x \in X | m_T[\{\|x - f(\cdot)\| < \varepsilon\} \cap E] \neq 0\}$. We denote this set by $er_E(f)$. f will be called an essential range function if $er_F(f) \cap f(F) \neq \phi$ for all $F \in \mathscr{B}$ that are not m_T-null sets. The set $er_E(f)$ was introduced by M. A. Rieffel in [13]. It is shown there that if f is the pointwise limit of a sequence of X valued simple functions and if μ is a positive measure then $er_E(f) \cap f(E) \neq \phi$ provided $\mu(E) > 0$. The separability of $f(E)$ and the positivity of μ were both used in the proof of this result. $er_E(f)$ is used in [13] as a bound on local average ranges which in turn play a key role in establishing the vector valued version of the Radon-Nikodym theorem.

LEMMA. *If for every pair* (A, B) *of nonnull* m_T *sets of* $\mathscr{B}$

$$er_A(f) \cap f(A) \neq \phi \text{ and } er_B(g) \cap g(B) \neq \phi$$

where f *and* g *are essential range functions and if* $\|f(\cdot) - g(\cdot)\| < a$ *then for every* $x \in er_E(f)$ *and* $\varepsilon > 0$ *there exists* $y \in er_E(g)$ *such that* $\|y - x\| < a + \varepsilon$.

Proof. $m_T[\{x \in X \mid \|x - f(\cdot)\| < \varepsilon/2\} \cap E] \neq \phi$. Now

$$er_E(g) \cap g(E) \neq \phi \text{ so pick } y \in er_E(g) \cap g(E)$$

such that $\{\|y - g(t)\| < \varepsilon\} \cap \{t \in E\}$ has nonzero m_T measure. Thus

$$\|x - f(t)\| < \varepsilon/2, \text{ for some } t \in E\,.$$
$$\|f(t) - g(t)\| < a$$
$$\|y - g(t)\| < \varepsilon/2\,.$$

Thus $\|x - y\| < a + \varepsilon$.

Let $\{f_n\}$ be a sequence of functions from K into X. We say that $er(f_n)$ are bounded away from 0 over the sets $\{\|f_n\| \geqq a\}$ if for every $\varepsilon_2 > 0$ there exists $\varepsilon_1 > 0$ and a partition $\{A_i\}$ such that for all $L > 0$

$$\Big|\sum_i{}' m_T[A_i \cap \{\|f_n\| > a\}](y_{i,n})\Big| > \varepsilon_1 \text{ imply}$$
$$sv[m_L, A \cap \{\|f_n\| > a\}] < \varepsilon_2, \text{ where } A = \bigcup_i A_i\,.$$

Here $\sum'$ denotes the sum over $y_{i,n} \in X$ satisfying

$$\|y_{i,n}\| < L \text{ and } y_{i,n} \in er_{A_i}(f_n)\,;$$

Note. If X denotes the scalar field the above condition will be true if $0 < \delta < y_{i,n} < L$. That is, if the $y_{i,n}$ are bounded away from 0.

Finally we say that $er(h)$ transfers uniform integrability if for every $\varepsilon_2 > 0$ there exists $\varepsilon_1 > 0$ and constants $K_0 > 0$ and $u > 0$ such that if $|\sum m_T(B_i)x_i| < \varepsilon_1$ where $\|x_i\| \geqq K_0$ and $\{B_i\}$ is a partition. Then $|\sum m_T(B_i)y_i| < \varepsilon_2$ provided $\|x_i - y_i\| < u$ and $y_i \in er_{B_i}(h)$.

Note. The above condition may be rewritten

$$\left|\int_{\|s\| \geqq K_0} s dm_T \right| < \varepsilon, \text{ where } s = \sum \chi_{B_i} x_i$$

implies $\left|\int_{\|s'-s\|<u} s' dm_T\right| < \varepsilon_2$ where $s' = \sum \chi_{B_i} y_i$ and $y_i \in er_{B_i}(h)$ (in particular $\|y_i\| \geqq K_0 - u$). Roughly speaking uniform integrability is transferred to s'.

THEOREM 5. *Let $\{f_n\}$ be a sequence of m_T integrable functions we assume*

(1) *f_n is an essential range function.*

(2) *$er(f_n)$ are bounded away from 0 on sets of the form $\{\|f_n\| > a\}$ (for all $a > 0$).*

(3) *Each $er(f_n)$ transfers uniform integrability.*

If $\left\{\int_A f_n dm_T\right\}$ converges to 0 uniformly for $A \in \mathscr{B}$, then $\{f_n\}$ converges in measure to 0.

Proof. Let $\varepsilon > 0$. Then $\left|\int_A f_n dm_T\right| < \varepsilon$ for all $A \in \mathscr{B}$ and for n large enough. Let $\{s_{n,k}\}$ be a sequence of simple functions uniformly integrable with $\{s_{n,k}\}$ converging in measure to f_n as K goes to infinity. By (1), for n fixed, we can pick a constant M such that

$$\left|\int_{\{\|s_{n,k}\| > M-1\} \cap A} s_{n,k} dm_T \right| < \varepsilon \text{ for all } k \text{ and } A \in \mathscr{B} .$$

Now choose δ' so that $0 < \delta' < 1/2$ and $\delta' < u/2$ and $sv_{M,2\delta'}[m_T, K] < \varepsilon$. Let

$$A_n^k(\delta') = \{t \mid \|s_{n,k}(t) - f_n(t)\| \geqq \delta'\}$$

Since $\{s_{n,k}\}$ converges in measure,

$$sv_M m_T[A_n^k(\delta')] \text{ converges to 0 as } k \to \infty .$$

Choose k large enough so that

$$sv_M m_T[A_n^{k\prime}(\delta') \cap E_{i,n,k}] \neq 0 \text{ where}$$

$A_n^{k\prime}(\delta')$ denotes the complement of $A_n^k(\delta')$ and where $s_{n,k} = \sum_i E_{i,n,k} x_{i,n,k}$. Condition (1) implies $|\sum' m_T(E_{i,n,k})x_{i,n,k}| < \varepsilon$ where $\sum'$ is the sum over $\|x_{i,n,k}\| > M - 1$. Since $s_{n,k}$ is a simple function $x_{i,n,k} \in er_{s_{n,k}}[E_{i,n,k} \cap A_n^{k\prime}(\delta')]$. By the previous lemma choose $y_{i,n,k} \in$

$er_{f_n}[E_{i,n,k}A_n^{k'}(\delta')]$ with $\|y_{i,n,k} - x_{i,n,k}\| < 2\delta' < u$. Let $s'_{n,k} = \sum \chi_{E_{i,n,k}} x_{i,n,k}$. Then by condition (3)

$$\left|\int s_{n,k}dm_T - \int s'_{n,k}dm_T\right| \leqq \left|\int_{\{\|s_{n,k}\| \geqq M\}} s_{n,k}dm_T\right| + \left|\int_{\{\|s'_{n,k}\| \geqq M\}} s'_{n,k}dm_T\right|$$
$$+ \left|\int_{\{\|s_{n,k}\| < M\}} s_{n,k}dm_T - \int_{\{\|s'_{n,k}\| < M\}} s'_{n,k}dm_T\right|$$
$$\leqq 2\varepsilon + sv_{M,2\delta'}[m_T, K] .$$

Thus the right side is less than 3ε. Now choose n large enough so that

$$\left|\int_{\{\|f_n\| > \delta'\} \cap \{\|s'_{n,k}\| \leqq L\}} f_n dm_T\right| < \varepsilon \text{ for all } L ;$$

(This can be done since $\left\{\int_A f_n dm_T\right\}$ converges to 0 uniformly for $A \in \mathscr{B}$.) Let $A = \{\|f_n\| > \delta'\} \cap \{\|s'_{n,k}\| \leqq L\}$, then shrinking δ' if necessary

$$\left|\int_A s'_{n,k}dm_T\right| \leqq \left|\int_A f_n dm_T\right| + \left|\int_A f_n dm_T - \int_A s_{n,k}dm_T\right|$$
$$+ \left|\int_A s_{n,k}dm_T - \int_A s'_{n,k}dm_T\right|$$
$$\leqq 5\varepsilon .$$

(The second term on the right is less than ε since by Proposition 3 $\left\{\int_A s_{n,k}dm_T\right\}$ converges to $\int_A f_n dm_T$ uniformly as $A \in \mathscr{B}$.) The rest of the proof follows from condition (2).

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