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SEQUENTIAL TESTING OF SEVERAL SIMPLE HYPOTHESES FOR A DIFFUSION PROCESS AND THE CORRESPONDING FREE BOUNDARY PROBLEM

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An n-dimensional stochastic process $\xi(t)$ is observed. It is known that $\xi(t)$ has the statistics of an n-dimensional Brownian motion with any one of possibly $n+1$ drifts $\lambda_0, \cdots, \lambda_n$ (λ_i are given n-vectors). We observe the process at a running cost, per unit time, given by c_i when the drift is λ_i, and after some (stopping) time τ make a decision which hypothesis to accept; the hypothesis H_j means accepting the drift λ_j; the drift changes in time in accordance with a Markov process with $n+1$ states and a given transition probability matrix. The problem of finding the optimal stopping time and optimal final decision leads to a variational inequality for a degenerate elliptic operator. In this paper we study this variational inequality and the corresponding free boundary. We also consider, by purely probabilistic methods, the case where $\xi(t)$ is k-dimensional, $k \neq n$. The outline of the main results is given at the end of § 2.

1. **The sequential testing problem.** Let $q_{i,j} (0 \leqq i, j \leqq n)$ be real numbers such that $q_{i,j} \geqq 0$ if $i \neq j$, $q_{i,i} \leqq 0$, and $\sum_{j=0}^{n} q_{i,j} = 0$ for $0 \leqq i \leqq n$. In a probability space $(\Omega, \mathscr{F}, P)$ we are given a Markov process $\theta(t) = \theta(t, w)$ taking values $0, 1, \cdots, n$ and having the infinitesimal matrix $(q_{i,j})$. We are also given an n-dimensional Brownian motion $w(t)$ (with $w(0) = 0$) independent of the process $\theta(t)$. Let $\lambda_0, \lambda_1, \cdots, \lambda_n$ be n-dimensional vectors which span R^n, that is

$$\lambda_1 - \lambda_0, \lambda_2 - \lambda_0, \cdots, \lambda_n - \lambda_0 \text{ are linearly independent .} \tag{1.1}$$

Consider the process $\xi(t)$ in R^n given by

$$d\xi(t) = dw(t) + \sum_{j=0}^{n} I_{(\theta(t)=j)} \lambda_j dt \tag{1.2}$$

that is, on the set $\theta(t, \omega) = j$ $d\xi(t, \omega) = dw(t, \omega) + \lambda_j dt$. We set $\mathscr{F}_t = \sigma(\xi(s), 0 \leqq s \leqq t)$.

When $\theta(t, \omega) = j$ we say that *the hypothesis H_j is satisfied* (*at time t*). We shall be concerned with the problem of deciding which hypothesis to accept at a minimal cost. We follow Bayes' formulation in setting up the problem:

The observed process is $\xi(t)$. We are given an a priori probability π for $\theta(0)$, that is, we are given

$$(1.3) \qquad \pi = (\pi_0, \pi_1, \cdots, \pi_n)\ , \quad \pi_i \geqq 0\ , \quad \sum_{i=0}^{n} \pi_i = 1$$

and make the initial assumption that $\theta(0) = j$ with probability π_j. This determines a probability P^π on the space of paths $(\theta(t), w(t))$ with $w(0) = 0$, and

$$(1.4) \qquad P^\pi(\theta(0) = j) = \pi_j\ , \quad 0 \leqq j \leqq n\ .$$

We shall denote the expectation with respect to P^π by E^π. The running cost (per unit time) of the observation of $\xi(t)$ is a given positive number c_j if $\theta(t) = j$. We observe the process $\xi(t)$ for an amount of time τ, where τ is a stopping time with respect to $\mathscr{F}_t$; the incurred cost is then

$$E^\pi\left[\int_0^\tau f(\theta(t))dt\right]\ , \quad \text{where} \quad f(j) = c_j \quad (0 \leqq j \leqq n)\ .$$

At the time $t = \tau$ we make a terminal decision $d(\omega)$ as to which hypothesis to accept; $d(\omega) = j$ means accepting the hypothesis H_j. The variable $d(\omega)$ is taken to be $\mathscr{F}_\tau$ measurable.

Set

$$(1.5) \qquad W(\theta, d) = a_i \quad \text{if} \quad d = i, \theta \neq i \quad (a_i > 0)\ ,$$

i.e., a_i is the cost for erroneously accepting the hypothesis H_i. The cost of the terminal decision is

$$E^\pi[W(\theta(\tau, \omega), d(\omega))]$$

and the total cost for the decision $\delta = (\tau, d)$ is

$$J_\pi(\delta) \equiv E^\pi\left[\int_0^\tau f(\theta(t))dt + W(\theta(\tau, \omega), d(\omega))\right]\ .$$

More generally, introducing a discount factor α, $\alpha \geqq 0$, the total cost becomes

$$(1.6) \qquad J_\pi(\delta) - E^\pi\left[\int_0^\tau e^{-\alpha t} f(\theta(t))dt + e^{-\alpha\tau} W(\theta(\tau, \omega), d(\omega))\right]\ .$$

The problem is to study the least cost function

$$(1.7) \qquad V(\pi) = \inf_\delta J_\pi(\delta)$$

and to find an optimal decision $\tilde{\delta} = (\tilde{\tau}, \tilde{d})$, that is,

$$(1.8) \qquad J_\pi(\tilde{\delta}) = V(\pi)\ .$$

This problem is called a *sequential testing problem of* $n+1$ *simple hypotheses* $H_0, H_1, \cdots, H_n$. The case where

$$\begin{aligned}&\theta \text{ does not depend on } t, \text{ that is, } q_{i,j}=0 \text{ for}\\ &0 \leqq i, j \leqq n;\ c_i = c > 0 \quad \text{for} \quad \leqq i \leqq n\end{aligned} \tag{1.9}$$

will be called the *special case*; more refined results will be proved for this case.

The sequential testing problem in the special case with $n=2$ has been studied in detail (see Shiryayev [15] and the references given there). In the case of discrete times the problem (in the special case) was studied by Wald [16], Chow and Robins [7], Shiryayev [14], Kiefer and Sacks [11] and others.

Analogously to the case $n=2$ we introduce the a posteriori probability process

$$\pi(t) = (\pi_0(t), \pi_1(t), \cdots, \pi_n(t))$$

where

$$\pi_j(t) = P^\pi[\theta(t) = j \mid \mathscr{F}_t]\ .$$

Introducting the simplex in R^{n+1}

$$\Pi_{n+1} = \left\{\pi = (\pi_0, \pi_1, \cdots, \pi_n);\ \pi_i \geqq 0\ ,\quad \sum_{i=1}^{n} \pi_i = 1\right\}$$

it is clear that $\pi(t) \in \Pi_{n+1}$ for all $t > 0$. The process $\pi(t)$ was studied by Shiryayev (see [13]) and by Anderson and Friedman [2]. It is shown in these references that $\pi(t)$ is a Markov process with generator

$$\begin{aligned}\mathrm{Mu}(\pi) = \frac{1}{2}\sum_{i,j=0}^{n} \pi_i\pi_j\Big(\lambda_i - \sum_{k=0}^{n}\lambda_k\pi_k\Big)\ .\\ \Big(\lambda_j - \sum_{l=0}^{n}\lambda_l\pi_l\Big)\frac{\partial^2 u(\pi)}{\partial\pi_i\partial\pi_j} + \sum_{i,j=0}^{n} q_{i,j}\pi_i\frac{\partial u(\pi)}{\partial \pi_j}\end{aligned} \tag{1.10}$$

and (in [2]) explicit formulas are given for $\pi_j(t)$ in terms of $\xi(t)$. In particular, when (1.9) holds,

$$\begin{aligned}\pi_j(t) &= \pi_j\Big\{\sum_{k=0}^{n}\pi_k z_{j,k}(t)\Big\}^{-1}\ ,\\ z_{j,k}(t) &= \exp\Big\{(\lambda_k - \lambda_j)\cdot\xi(t) - \frac{1}{2}(\|\lambda_k|^2 - \|\lambda_j|^2)t\Big\}\ .\end{aligned} \tag{1.11}$$

As in [2] [13; p. 167] we can express $J_\pi(\delta)$ in terms of the process $\pi(t)$:

$$J_\pi(\delta) = E^\pi\left\{\int_0^\tau e^{-\alpha t}h(\pi(t))dt + e^{-\alpha\tau}\sum_{i=0}^{n}(1-\pi_i(\tau))a_i I_{[d(\omega)=i]}\right\}\ , \tag{1.12}$$

where $h(\pi) = \sum_{i=0}^{n} c_i \pi_i$.

Set

$$J_\pi(\tau) = \inf_d J_\pi(\delta) \quad \text{where} \quad (\tau, d) = \delta .$$

For a given τ the optimal $d = d(\omega)$ is such that it minimizes $\sum(1 - \pi_i(\tau(\omega)))a_i$. Consequently,

$$J_\pi(\tau) = E^\pi\left[\int_0^\tau e^{-\alpha t} h(\pi(t))dt + e^{-\alpha\tau} g(\pi(\tau))\right] \tag{1.13}$$

where

$$g(\pi) = \min_{0 \leqq i \leqq n} \{a_i(1 - \pi_i)\} . \tag{1.14}$$

The problem associated with (1.7), (1.8) thus reduces to the problem associated with

$$V(\pi) = \inf_\tau J_\pi(\tau) \tag{1.15}$$

(where $V(\pi)$ is the same as in (1.17)) and

$$J_\pi(\tilde{\tau}) = V(\pi) , \tag{1.16}$$

where τ, $\tilde{\tau}$ are stopping times with respect to $\mathscr{F}_t$.

In the sequel we shall study the hypothesis testing problem in its formulation (1.15), (1.16). For simplicity we shall also always assume that $\alpha > 0$; the results in case $\alpha = 0$ are still valid, but require some changes in the proofs; we consider this case briefly in §10.

2. The variational inequality. Let $\mathring{\Pi}_{n+1} = \operatorname{int} \Pi_{n+1}$.

As in [2], the function $V(\pi)$ in $\mathring{\Pi}_{n+1}$ can be characterized as the bounded solution u of a certain system of differential in equalities:

$$\begin{gathered} Mu - \alpha u + h \geqq 0 \quad \text{a.e.} \quad \text{in } \mathring{\Pi}_{n+1} , \\ u(\pi) \leqq h(\pi) \quad \text{in} \quad \Pi_{n+1} , \\ (Mu - \alpha u + h)(u - g) = 0 \quad \text{a.e.} \quad \text{in } \Pi_{n+1} . \end{gathered} \tag{2.1}$$

Such a system is called a *variational inequality* (for a general study of variational inequalities see, for instance, [3] [9]).

We recall [2] that, because of (1.1), M is a nondegenerate elliptic operator (in n independent variables) in $\mathring{\Pi}_{n+1}$. It degenerates however on the boundary $\partial\Pi_{n+1}$.

LEMMA 2.1. (a) *If $\pi = \pi(0)$ belongs to $\mathring{\Pi}_{n+1}$ then $\pi(t) \in \mathring{\Pi}_{n+1}$ for* *all* $t > 0$, *and* (b) *if* (1.9) *holds and if* $\pi_i = \pi_i(0) = 0$ *for some* i, *then* $\pi_i(t) = 0$ *for all* $t > 0$.

The assertion (a) follows from the formula for $\pi_j(t)$ given in [2]. The assertion (b) follows from (1.11).

From (a) it follows that no boundary Dirichlet conditions are needed to be given on $\partial\Pi_{n+1}$ in order to solve the variational inequality (2.1). The solution of (2.1) can be constructed as follows (cf. [2]):

For any $\delta > 0$, $\varepsilon > 0$, let

$$\Pi^{\delta}_{n+1} = \{\pi \in \Pi_{n+1},\ \pi_i > \delta \quad \text{for} \quad 0 \leqq i \leqq n\} \tag{2.2}$$

and let $\alpha_\varepsilon(t)$ be a C^∞ function in t satisfying:

$$\beta'_\varepsilon(t) \geqq 0,\quad \beta''_\varepsilon(t) \geqq 0\ ;\quad \beta_\varepsilon(t) \longrightarrow 0 \quad \text{if} \quad t < 0\ ,\quad \varepsilon \downarrow 0\ ,$$
$$\beta_\varepsilon(t) \longrightarrow \infty \quad \text{if} \quad t > 0\ ,\quad \varepsilon \downarrow 0\ .$$

Consider the elliptic problem

$$\begin{aligned} -Mu + \alpha u + \beta_\varepsilon(u - g) &= h \quad \text{in} \quad \Pi^{\delta}_{n+1}\ , \\ u &= \varphi \quad \text{on} \quad \partial\Pi^{\delta}_{n+1} \end{aligned} \tag{2.3}$$

where φ is any smooth function such that

$$0 \leqq \varphi \leqq g\ . \tag{2.4}$$

This problem has a unique solution $u = u_{\delta,\varepsilon}$. If $g(\pi)$ were a function in $W^{2,p}$, for any $2 \leqq p < \infty$, then one can show, by standard techniques for variational inequalities, that

$$u_{\delta,\varepsilon} \longrightarrow u_\delta \quad \text{uniformly as} \quad \varepsilon \longrightarrow 0\ , \tag{2.5}$$

where u_δ is the unique solution of the variational inequality

$$\begin{aligned} & -Mu + \alpha u \leqq h \quad \text{a.e. in} \quad \Pi^{\delta}_{n+1}\ , \\ & u \leqq g \quad \text{in} \quad \Pi^{\delta}_{n+1}\ , \\ & (-Mu + \alpha u - h)(u - g) = 0 \quad \text{a.e. in} \quad \Pi^{\delta}_{n+1}\ , \\ & u = \varphi \quad \text{on} \quad \partial\Pi^{\delta}_{n+1}\ , \\ & u \in W^{2,p}_{\text{loc}}(\Pi^{\delta}_{n+1})\ ,\quad u \in C(\overline{\Pi}^{\delta}_{n+1})\ . \end{aligned} \tag{2.6}$$

In the present case g is not even continuously differentiable. Since however it is the minimum of linear functions in the π_i, it is convex. Thus, in terms of, say, $\pi_1, \cdots, \pi_n$,

$$\left(\frac{\partial^2 g}{\partial\pi_i \partial\pi_j}\right) \text{ is negative semidefinite matrix}\ ,$$

where $\partial^2 g/\partial\pi_i\partial\pi_j$ is taken in the sense of distributions. By [4] it follows that

$$|u_{\delta,\varepsilon}|_{W^{2,\infty}(G)} \leqq C \quad \text{if} \quad \bar{G} \subset \mathring{\Pi}^{\delta}_{n+1}$$

where C is a constant independent of δ, ε, and then (2.5) is still valid. It follows that

$$|u_\delta|_{W^{2,\infty}(G)} \leqq C \,. \tag{2.7}$$

We now take $\delta \to 0$ and deduce (as in [2]) that

$$u_\delta \longrightarrow u \quad \text{uniformly in compact subsets of } \mathring{\Pi}_{n+1} \tag{2.8}$$

where u is a solution of the variational inequality (2.1); further (by a probabilistic argument),

$$u \text{ has a continuous extension into } \Pi_{n+1} \,, \tag{2.9}$$

and, by (2.7),

$$u \in W^{2,\infty}_{\mathrm{loc}}(\mathring{\Pi}_{n+1}) \,. \tag{2.10}$$

The uniqueness of the solution u subject to the smoothness conditions (2.9), (2.10) follows (as in [2]) by using Ito's formula.

We recall that u can also be obtained as follows:

$$u = \lim_{\varepsilon \to 0} \lim_{\delta \to 0} u_{\delta,\varepsilon} \,. \tag{2.11}$$

Let

$$S = \{\pi \in \Pi_{n+1};\ u(\pi) = g(\pi)\} \,, \quad C = \{\pi \in \Pi_{n+1};\ u(\pi) < g(\pi)\} \,.$$

As in [2], $V(\pi)$ defined by (1.15) coincides in $\mathring{\Pi}_{n+1}$ with the solution u of (2.1), and an optimal stopping time $\tilde{\tau}$ (as in (1.16)) is given by

$$\tilde{\tau} = \text{hitting time of } S \text{ by the process } \pi(t) \,. \tag{2.12}$$

Thus the optimal strategy is to continue while $\pi(t)$ is in C and to stop when $\pi(t)$ hits S. For this reason the set S is called the *stopping set* and the set C is called the *continuation set.*

In the terminology of variational inequalities, S is called the *coincidence set,* C is called the *noncoincidence set,* and g is called the *obstacle.* The set

$$\Gamma = \mathring{\Pi}_{n+1} \cap \partial C \quad (\partial C = \text{boundary of } C)$$

is called the *free boundary.*

The purpose of this paper is to study the sets C, S or, equivalently, the free boundary Γ.

We shall denote by e_i the vertex

$$(\delta_{i0}, \delta_{i1}, \cdots, \delta_{in})$$

of $\Pi_{n+1}(0 \leqq i \leqq n)$.

In § 3 we prove that each vertex e_i has a Π_{n+1}-neighborhood $\tilde{S}_i$ such that $\tilde{S}_i \subset S$. In § 4 we prove some auxiliary results needed for the following section.

In § 5 we study the set

$$S_i = S \cap \{\pi \in \Pi_{n+1};\ u(\pi) = a_i(1 - \pi_i)\} \tag{2.13}$$

under the assumption that

$$q_{i,k} = 0 \quad \text{for} \quad 0 \leqq k \leqq n .$$

Introducing the coordinates

$$y_j = \frac{\pi_j}{\pi_0} \quad (1 \leqq j \leqq n) \tag{2.14}$$

we prove that $\Gamma_i \equiv \mathring{\Pi}_{n+1} \cap \partial S_i$ can be represented in the form

$$y_i = \psi_i(y_1, \cdots, y_{i-1}, y_{i+1}, \cdots, y_n) \tag{2.15}$$

where ψ_i is analytic.

In § 6 we specialize to the case (1.9) and prove that each S_i is a convex set and $u(\pi)$ is a concave function.

In §§ 7, 8 we study the asymptotic behavior of the solution when (1.9) holds and $c \to 0$. It is shown that ∂S_i lies within a $\delta_1 c$-neighborhood of e_i and outside a $\delta_2 c$-neighborhood of e_i. Further,

$$E^{\pi}\tilde{\tau} \sim \left(\sum_{i=0}^{n} \gamma_i \pi_i\right) \log \frac{1}{c}\ ,\ \frac{1}{2}\gamma_i = \{\min_{k \neq i} |\lambda_k - \lambda_i|\}^{-1} \tag{2.16}$$

where $\tilde{\tau}$ is the optimal stopping time, and

$$\frac{1}{c} u(cy) \longrightarrow \tilde{u}(y) \quad (u(y) = u(\pi)) \tag{2.17}$$

where $\tilde{u}(y)$ is the solution of a certain variational inequality; the free boundary for $\tilde{u}$ is also studied.

In § 9 we consider the behavior of the solution as $c \to \infty$. The case $\alpha = 0$ is considered in § 10. Finally, in § 11, we extend some of the results of the previous sections to the case where $w(t)$ is k-dimensional, for any k; here the methods are purely probabilistic.

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3. S contains a neighborhood of the vertices. We always denote by $u(\pi)$ the solution of (2.1) (which satisfies (2.9), (2.10); recall that $u(\pi) = V(\pi)$ if $\pi \in \mathring{\Pi}_{n+1}$.

The operator Mu can be written in terms of the tangential operators of Π_{n+1} (considered as a submanifold in R^{n+1}). Observe that on Π_{n+1}

$$\sum_{i=0}^{n} \pi_i = 1$$

and, consequently, the operator

$$\sum_{i=0}^{n} \alpha_i \frac{\partial}{\partial \pi_i}$$

is tangential if and only if $\sum_{i=0}^{n} \alpha_i = 0$. We introduce the tangential operators

$$D_{0m} = \frac{\partial}{\partial \pi_0} - \frac{\partial}{\partial \pi_m} \quad (1 \leqq m \leqq n)$$

and the normal operator

$$\tilde{D} = \sum_{i=0}^{n} \frac{\partial}{\partial \pi_i} .$$

Substituting

$$\frac{\partial}{\partial \pi_i} = -\sum_{m=1}^{n} \left(\delta_{im} - \frac{1}{n+1}\right) D_{0m} + \frac{1}{n+1} \tilde{D} \tag{3.1}$$

into Mu we discover that the coefficients of $\tilde{D}^2$, $D_{0m}\tilde{D}$ vanish (as indeed they should) and that Mu takes the form

$$\begin{aligned} Mu = \frac{1}{2} \sum_{i,j=1} \pi_i \pi_j \Big(\lambda_i - \sum_{k=0}^{n} \lambda_k \pi_k\Big) \cdot \Big(\lambda_j - \sum_{l=0}^{n} \lambda_l \pi_l\Big) D_{0i} D_{0j} u \\ - \sum_{j=1}^{n} \sum_{i=0}^{n} q_{i,j} \pi_i D_{0j} u . \end{aligned} \tag{3.2}$$

Another useful coordinate system is given by (2.14), i.e.,

$$y_i = \frac{\pi_i}{\pi_0} \quad (1 \leqq i \leqq n) . \tag{3.3}$$

(The role of π_0 is incidental; one can similarly work with the coordinates $y_i = \pi_i/\pi_j$, $0 \leqq i \leqq n$, $i \neq j$, for any fixed j.) It maps Π_{n+1} onto

$$R_n^+ = \{y = (y_1, \cdots, y_n);\ y_j \geqq 0 \quad \text{for} \quad 1 \leqq j \leqq n\} .$$

It is easy to compute that

$$\frac{\partial u}{\partial y_i} = \pi_0\Big(D_{0i} - \sum_{k=1}^{n} \pi_k D_{0k}\Big) \tag{3.4}$$

and that in the y-coordinates Mu becomes (cf. [2])

$$\begin{aligned} Lu = \frac{1}{2}\sum_{i,j=1}^{n} \mu_{ij} y_i y_j \frac{\partial^2 u}{\partial y_i \partial y_j} + \frac{1}{Y}\sum_{i,j=1}^{n} \mu_{ij} y_i y_j \frac{\partial u}{\partial y_j} \\ + \sum_{j=1}^{n}\sum_{i=0}^{n} (q_{i,j} - q_{i,0} y_j) y_i \frac{\partial u}{\partial y_j} \end{aligned} \tag{3.5}$$

where $y_0 \equiv 1$ and

$$Y = 1 + y_1 + \cdots + y_n , \tag{3.6}$$

$$\mu_{ij} = (\lambda_i - \lambda_0)\cdot(\lambda_j - \lambda_0) . \tag{3.7}$$

We shall need the following comparison lemma:

LEMMA 3.1. *Suppose that* $\tilde{u}$ *is a function satisfying the variational inequality* (2.1) *in a region* $\tilde{\Pi} \subset \Pi_{n+1}$ *with* g *replaced by* $\tilde{g}$. *If*

$$\tilde{g} \geqq g \quad on \quad \tilde{\Pi} ,$$
$$\tilde{u} \geqq u \quad on \quad \partial\tilde{\Pi} \cap \mathring{\Pi}_{n+1} ,$$

and $\tilde{u}$ *is uniformly continuous in* $\tilde{\Pi}$, *then* $\tilde{u} \geqq u$ *in* $\tilde{\Pi}$. *Similarly, if* $\tilde{g} \leqq g$ *on* $\tilde{\Pi}$, $\tilde{u} \leqq u$ *on* $\partial\tilde{\Pi} \cap \mathring{\Pi}_{n+1}$, *then* $\tilde{u} \leqq u$ *in* $\tilde{\Pi}$.

Notice that we do not assume that $\tilde{u} \geqq u$ (or $\tilde{u} \geqq u$) on $\partial\tilde{\Pi} \cap \partial\Pi_{n+1}$.

Proof. The function $\tilde{u}$ can be obtained as the limit of solutions $\tilde{u}_\delta$ of variational inequalities in $\Pi^{\delta}_{n+1} \cap \tilde{\Pi}$ (cf. (2.6)); the proof is the same as for u. By a standard comparison theorem for variational inequalities, $\tilde{u}_\delta \geqq u$. Taking $\delta \to 0$, the assertion follows.

THEOREM 3.2. *Assume that* $c_j > a_i q_{j,i}$ *for* $0 \leqq j \leqq n$ *and some* i. *Then there exists a* Π_{n+1}*-neighborhood* $\tilde{S}_i$ *of* e_i *such that* $\tilde{S}_i \subset S$.

Proof. It suffices to prove the assertion for $i = 0$. The proof is by comparison of $v \equiv u - g$ with a function z which vanishes in an R_n^+-neighborhood of $y = 0$. Notice that near $y = 0$

$$g = a_0(1 - \pi_0) = \frac{a_0}{Y}(y_1 + \cdots + y_n) . \tag{3.8}$$

Since $M(1 - \pi_0) = -\sum_{i=0}^{n} q_{i0}\pi_i$, v satisfies:

$$
\begin{aligned}
&-Lv + \alpha v \leqq \mu_1 ,\\
&v \leqq 0 ,\\
&(-Lv + \alpha v - \mu_1)v = 0
\end{aligned}
\tag{3.9}
$$

a.e., if $y \in R_n^+$, $|y| \leqq R^*$, where R^* is sufficiently small and

$$
\mu_1 = \sum_{i=0}^{n} (c_i - a_0 q_{i0})\pi_i - \alpha g > c^* , \quad c^* > 0 . \tag{3.10}
$$

We have to show that

$$
v(y) = 0 \quad \text{if} \quad y \in R_n^+ , \quad |y| < R \tag{3.11}
$$

for a sufficiently small R.

Let (cf. [10])

$$
z(y) = \begin{cases} \dfrac{N}{1-\theta}\left(\left(\dfrac{\log \frac{1}{r}}{\log \frac{1}{R}}\right)^{\theta} - \theta \dfrac{\log \frac{1}{r}}{\log \frac{1}{R}}\right) - N & \text{if} \quad R < r < R_0 , \\ 0 \quad \text{if} \quad r < R \end{cases}
$$

where $0 < \theta < 1$, $N > 0$, $r = |y|$. We compute that $(\partial z/\partial r) < 0$ if $R < r < R_0$, so that $z < 0$. Also

$$
z = \frac{\partial z}{\partial r} = 0 \quad \text{if} \quad r = R .
$$

If we show that

$$
\gamma \equiv - Lz + \alpha z \quad \text{satisfies} \quad \gamma \leqq \mu_1 \quad (R < r < R_0) \tag{3.12}
$$

and if also

$$
z(R_0) \leqq -K \quad \text{where} \quad K = \sup v \tag{3.13}
$$

then, by Lemma 3.1, $z \leqq v$ if $0 < r < R_0$. This implies that $0 \leqq v$ if $0 < r < R$, and (3.11) follows.

To establish (3.12), (3.13) we compute

$$
\begin{aligned}
&\left| L\left(\log \frac{1}{r}\right) \right| \leqq C \\
&\left| L\left(\log \frac{1}{r}\right)^{\theta} \right| \leqq \frac{C}{\left(\log \frac{1}{r}\right)^{1-\theta}} .
\end{aligned}
$$

It follows that

$$\gamma = -Lz + \alpha z \leqq \frac{CN}{\log\frac{1}{R}} + \alpha z \leqq \frac{CN}{\log\frac{1}{R}} .$$

Thus it suffices to satisfy (using (3.10))

$$\frac{CN}{\log\frac{1}{R_0}} = c_* \tag{3.14}$$

and

$$\frac{N}{1-\theta}(M^\theta - \theta M) - N \leqq -K \quad \left(M = \frac{\log\frac{1}{R_0}}{\log\frac{1}{R}} < 1 \right) . \tag{3.15}$$

Choosing M sufficiently small so that

$$\frac{M^\theta - \theta M}{1-\theta} \leqq \frac{1}{2}$$

and taking $N > 2K$, (3.15) follows. Defining R_0 by (3.14), the proof is complete.

4. Auxiliary results.

DEFINITION. A point $\pi \in \mathring{\Pi}_{n+1}$ is said to belong to the *ridge* R of the obstacle g if g is not $W^{2,\infty}$ in any neighborhood of π.

Thus, $\pi = (\pi_0, \cdots, \pi_n) \in R$ if and only if

$$a_i(1-\pi_i) = a_j(1-\pi_j) \quad \text{for some} \quad i \neq j .$$

The above definition is analogous to the definition used in elastic-plastic torsion problems [6] where g is the distance function from the boundary of the domain.

THEOREM 4.1. *The ridge is contained in* C.

Proof. Suppose $\tilde{\pi} = (\tilde{\pi}_0, \cdots, \tilde{\pi}_n) \in R$ and, say,

$$a_1(1-\tilde{\pi}_1) = a_2(1-\tilde{\pi}_2) .$$

If $\tilde{\pi} \in S$ then

$$\nabla(u - a_1(1-\pi_1)) = 0 ,$$
$$\nabla(u - a_2(1-\pi_2)) = 0$$

at $\tilde{\pi}$, since $u - a_i(1-\pi_i) \leqq 0$ in Π_{n+1} and $u(\tilde{\pi}) - a_i(1-\tilde{\pi}_i) = 0$, $i =$

1, 2. Thus

$$\nabla(a_1\pi_1 - a_2\pi_2) = 0 \quad \text{at} \quad \tilde{\pi} .$$

But

$$\Big(\frac{\partial}{\partial y_1} - \frac{\partial}{\partial y_2}\Big)(a_1\pi_1 - a_2\pi_2) = \Big(\frac{\partial}{\partial y_1} - \frac{\partial}{\partial y_2}\Big)\frac{a_1 y_1 - a_2 y_2}{Y}$$
$$= \frac{a_1 + a_2}{Y} > 0 ;$$

a contradiction.

LEMMA 4.2. *Assume that* $q_{i,k} = 0$ *for* $0 \leqq k \leqq n$ *and some* $i \geqq 1$. *Then*

$$\frac{\partial}{\partial y_i}[Y(u - a_i(1 - \pi_i))] \geqq 0 . \tag{4.1}$$

Proof. It suffices to prove (4.1) for $i = 1$. In § 2 we may replace Π^{δ}_{n+1} by any other sequence of domains which increase to Π_{n+1} and the boundary values φ on $\partial\Pi^{\delta}_{n+1}$ by any continuous function φ satisfying $0 \leqq \varphi \leqq g$. We shall choose Π^{δ}_{n+1} so that in the y-coordinates it becomes

$$G_\delta = \Big\{y; \delta < y_i < \frac{1}{\delta} \text{for} \quad 1 \leqq i \leqq n\Big\} . \tag{4.2}$$

Let

$$v = u_\delta - a_1(1 - \pi_1)$$
$$z = Yv$$

and choose φ as follows:

(4.3) $u_\delta = 0$ on $y_1 = \delta$;
Yu_δ, on each face $y_i = \delta$ or $y_i = 1/\delta$ $(2 \leqq i \leqq n)$, is a monotone increasing function of y_1 such that $Yu_\delta \leqq Yg$, and $Yu_\delta = 0$ at $y_1 = \delta$, $Yu_\delta = Yg$ at $y_1 = 1/\delta$; $Yu_\delta = Yg$ on $y_1 = 1/\delta$.

Then, on $y_1 = \delta$

$$\frac{\partial}{\partial y_1}(Yu_\delta) \geqq 0 \text{ (since } u_\delta = 0 \text{ on } y_1 = \delta,\ u_\delta \geqq 0 \text{ elsewhere) .}$$

Also

$$a_1 Y(1-\pi_1) = a_1(1+y_2+\cdots+y_n)$$

so that

$$-\frac{\partial}{\partial y_1}(a_1 Y(1-\pi_1)) = 0 \,. \tag{4.4}$$

Consequently

$$\frac{\partial z}{\partial y_1} \geqq 0 \quad \text{on} \quad y_1 = \delta \,. \tag{4.5}$$

On $y_i = \delta$ or $y_i = 1/\delta$ $(2 \leqq i \leqq n)$ we have, by (4.3),

$$\frac{\partial}{\partial y_1}(Yu_\delta) \geqq 0 \,.$$

Using (4.4) we again get

$$\frac{\partial z}{\partial y_1} \geqq 0 \quad \text{on} \quad y_i = \delta \quad \text{or} \quad y_i = \frac{1}{\delta} \quad (2 \leqq i \leqq n) \,. \tag{4.6}$$

On $y_1 = 1/\delta$, $z = 0$ by (4.3). Since $z \leqq 0$ elsewhere, we obtain

$$\frac{\partial z}{\partial y_1} \geqq 0 \quad \text{on} \quad y_1 = \frac{1}{\delta} \,. \tag{4.7}$$

Denote by C_δ the set where $u_\delta < g$. Then, in C_δ,

$$Mv - \alpha v = -\sum_{i=0}^{n}(c_i - a_i q_{i,1})\pi_i + \alpha a_1(1-\pi_1) \,.$$

Recalling that $Lv = Mv$ where L is defined by (3.5), and substituting

$$\frac{\partial v}{\partial y_i} = \frac{1}{Y}\frac{\partial z}{\partial y_i} - \frac{1}{Y^2}z \,,$$

$$\frac{\partial^2 v}{\partial y_i \partial y_j} = \frac{1}{Y}\frac{\partial^2 z}{\partial y_i \partial y_j} - \frac{1}{Y^2}\frac{\partial z}{\partial y_i} - \frac{1}{Y^2}\frac{\partial z}{\partial y_j} + \frac{2}{Y^2}z \,,$$

we find that

$$L_0 z - \alpha z = -\sum_{i=0}^{n}(c_i - a_1 q_{i,1})y_i + \alpha a_1(Y - y_1) \tag{4.8}$$

where

$$L_0 z \equiv \frac{1}{2}\sum_{i,j=1}^{n}\mu_{ij}y_i y_j \frac{\partial^2 z}{\partial y_i \partial y_j} + \sum_{j=1}^{n}\sum_{i=0}^{n}(q_{i,j} - q_{i,0}y_j)y_i \frac{\partial z}{\partial y_j} + \sum_{i=0}^{n} q_{i,0}y_i z \,. \tag{4.9}$$

Differentiating (4.8) with respect to y_1, we obtain the following equation for $w = \partial z/\partial y_1$:

$$(4.10)\qquad L_0 w + \sum_{i=1}^{n} \mu_{i1} y_i \frac{\partial w}{\partial y_i} + \sum_{i=0}^{n}\sum_{j=1}^{n} (q_{i,j} - q_{i,0} y_j) y_i \frac{\partial w}{\partial y_j} - \alpha w$$
$$= -(c_1 - a_1 q_{1,1}) = -c_1 .$$

From the maximum principle it then follows that $w > 0$ in C_δ provided $w \geqq 0$ on ∂C_δ. In view of (4.5)-(4.7), $w \geqq 0$ on $\partial C_\delta \cap G_\delta$. We next show that

$$(4.11)\qquad w(\tilde{y}) \geqq 0 \quad \text{if} \quad \tilde{y} \in \partial C_\delta \cap (\operatorname{int} G_\delta) .$$

Indeed, since $\tilde{y} \in \partial C_\delta \cap (\operatorname{int} G_\delta)$,

$$(4.12)\qquad u_\delta = a_i(1 - \pi_i) , \quad \nabla u_\delta = \nabla(a_i(1 - \pi_i)) \quad \text{at} \quad \tilde{y} ,$$

for some i for which $g = a_i(1 - \pi_i)$ at $\tilde{y}$. Writing

$$w = \frac{\partial z}{\partial y_1} = \frac{\partial}{\partial y_1}[Y u_\delta - a_i Y(1 - \pi_i)] + \frac{\partial}{\partial y_1}[a_i Y(1 - \pi_i)]$$
$$- \frac{\partial}{\partial y_1}[a_1 Y(1 - \pi_1)] ,$$

we note that the first term on the right hand side vanishes by (4.12), the third one vanishes by (4.4), and the middle one is equal to

$$a_i \frac{\partial}{\partial y_1}(Y - y_i) = a_i \quad \text{if} \quad i \neq 1 ,$$
$$= 0 \quad \text{if} \quad i = 1 ,$$

we conclude that $w(\tilde{y}) \geqq 0$.

It follows that

$$\frac{\partial}{\partial y_1}[Y(u_\delta - a_1(1 - \pi_1))] \geqq 0 \quad \text{in } C_\delta;\ \text{hence also in } G_\delta .$$

Taking $\delta \to 0$, the assertion of the lemma follows.

REMARK. Recalling (3.4), we can rewrite the assertion of Lemma 4.2 as follows:

$$(4.13)\qquad \Big(D_{0i} - \sum_{k=1}^{n} \pi_k D_{0k}\Big)[Y(u - a_i(1 - \pi_i))] \geqq 0 .$$

If we replace the role of e_0 by another vertex, say e_n, the corresponding differential inequality

$$\left(D_{ni} - \sum_{k=0}^{n-1} \pi_k D_{nk}\right)[Y(u - a_i(1 - \pi_i))] \geqq 0$$

(where $D_{nj} = \partial/\partial\pi_n - \partial/\partial\pi_j$) coincides with (4.13); thus we do not get any new inequality.

5. The free boundary is analytic. We continue to use the y coordinates (3.3).

Denote by G_i $(0 \leqq i \leqq n)$ the open components of $\Pi_{n+1} \backslash R$ with $\partial G_i \ni e_i$ and set

$$S_i = S \cap G_i \,, \quad C_i = C \cap G_i \,, \tag{5.1}$$

the definition of S_i is the same as in (2.13). Denote by $\tilde{G}_i$, $\tilde{S}_i$, $\tilde{C}_i$ the images of G_i, S_i, C_i respectively in the y-coordinates. Denote by $\tilde{R}$ the image of the ridge R in the y-coordinates. It is easy to check that if $\tilde{y} = (\tilde{y}_0, \cdots, \tilde{y}_n) \in \tilde{G}_i$ then there is a line segment

$$\gamma = \{y; y_j = \tilde{y}_j \quad \text{if} \quad j \neq i, \tilde{y}_i - \beta < y_i \leqq \tilde{y}_i\} \quad (\beta > 0)$$

which belongs to $\tilde{G}_i$ and its left end point lies on $\tilde{R}$.

Suppose now that

$$q_{i,k} = 0 \quad \text{for} \quad 0 \leqq k \leqq n \quad \text{and some} \quad i \geqq 1 \,. \tag{5.2}$$

By Lemma 4.2 we then deduce that if

$$u(\tilde{y}) - g(\tilde{y}) < 0 \,, \quad \tilde{y} \in \tilde{G}_i \tag{5.3}$$

then

$$u(y^*) - g(y^*) < 0 \quad \text{for any} \quad y^* \in \gamma \,. \tag{5.4}$$

Thus the open set C_i is connected to R. Since R belongs to C, by Theorem 4.1 it follows that C_i is connected.

The previous argument involving (5.2), (5.3) shows also that there exists a function $\psi_i(y_1, \cdots, y_{i-1}, y_{i+1}, \cdots, y_n)$ such that

$$\tilde{C}_i = \{y \in \tilde{G}_i; y_i < \psi_i(y_1, \cdots, y_{i-1}, y_{i+1}, \cdots, y_n)\} \,. \tag{5.5}$$

We can thus state:

THEOREM 5.1. *If* (5.2) *holds then* C_i *is connected and it is a subgraph in the sense of* (5.5).

We next prove that ψ_i is analytic:

THEOREM 5.2. *If* (5.2) *holds then* ψ_i *is analytic. More precisely, if*

$$\tilde{y}_i = \psi_i(\tilde{y}_1, \cdots, \tilde{y}_{i-1}, \tilde{y}_{i+1}, \cdots, \tilde{y}_n) ,$$
$$(\tilde{y}_1, \cdots, \tilde{y}_n) \in \operatorname{int} R_n^+$$

then ψ_i *is analytic at* $(\tilde{y}_1, \cdots, \tilde{y}_{i-1}, \tilde{y}_{i+1}, \cdots, \tilde{y}_n)$. *Thus the free boundary in the interior of* G_i *is analytic.*

The proof of Theorem 5.2 given below is based on a method of Alt [1].

Proof. It suffices to prove the theorem for $i = n$. Let $y' = (y_1, \cdots, y_{n-1})$ and consider the function

$$\zeta = a_n(1 - \pi_n) - u$$

in

$$D_{\rho_0} = \{\beta < y_n < \psi_n(y'), |y' - y_0'| < \rho_0\} .$$

Here y_0' is a fixed point with positive coordinates, $\beta < \psi_n(y_0')$, and $\psi_n(y_0') - \beta$, ρ_0 are sufficiently small so that D_{ρ_0} is contained in $\tilde{C}_n$. We have $\zeta > 0$ in D_{ρ_0}, $\zeta = 0$ on $y_n = \psi_n(y')$. By Lemma 4.2,

$$\frac{\partial \zeta}{\partial y_n} < 0 \quad \text{in} \quad D_{\rho_0} . \tag{5.6}$$

Consider the function

$$\eta = \sum_{k=1}^{n-1} \alpha_k \frac{\partial \zeta}{\partial y_k} - A \frac{\partial \zeta}{\partial y_n} \tag{5.7}$$

in D_{ρ_0}, where $\sum \alpha_k^2 \leqq 1$ and A is a sufficiently large positive constant to be determined later on. We have (cf. (4.8))

$$L_0\zeta - \alpha\zeta = \sum_{i=0}^{n} (c_i - a_n q_{i,n}) y_i - \alpha a_n (Y - y_n) \equiv \tilde{k} .$$

Differentiating with respect to y_k, first when $k = n$ and then when $1 \leqq k \leqq n - 1$, we get

$$L_0\Big(\frac{\partial \zeta}{\partial y_n}\Big) + \sum_{i=1}^{n} \mu_{in} y_i \frac{\partial}{\partial y_i}\Big(\frac{\partial \zeta}{\partial y_n}\Big) - \alpha \frac{\partial \zeta}{\partial y_n} = \frac{\partial \tilde{k}}{\partial y_n} = c_n ,$$
$$L_0\Big(\frac{\partial \zeta}{\partial y_k}\Big) + \sum_{i=1}^{n} \mu_{in} y_i \frac{\partial}{\partial y_i}\Big(\frac{\partial \zeta}{\partial y_k}\Big) - \alpha \frac{\partial \zeta}{\partial y_k} = \frac{\partial \tilde{k}}{\partial y_k} + M_k$$

where

$$M_k = \sum_{i=1}^{n} (\mu_{in} - \mu_{ik}) y_i \frac{\partial^2 \zeta}{\partial y_i \partial y_k} + \sum_{i=1}^{n} \lambda_{ki} \frac{\partial \zeta}{\partial y_i} + \tilde{\lambda}_k \zeta$$

where λ_{ik}, $\tilde{\lambda}_k$ are linear functions, and

$$\frac{\partial \tilde{k}}{\partial y_k} = c_k - a_n q_{k,n} - \alpha a_n \ .$$

Since $\zeta \in W^{2,\infty}_{\text{loc}}$, M_k is bounded and consequently,

$$\tilde{L}\eta \equiv L_0\eta + \sum_{i=1}^{n} \mu_{in} y_i \frac{\partial \eta}{\partial y_i} - \alpha\eta = -Ac_n + B \tag{5.8}$$

where B is bounded indepedenntly of the α_i and A.

We choose A sufficiently large so that

$$\tilde{L}\eta \leqq -1 \quad \text{in} \quad D_{\rho_0} \ . \tag{5.9}$$

Now let ρ be any number $< \rho_0$ (for instance $\rho = \rho_0/2$) and define

$$D_\rho = \{\beta < y_n < \psi_n(y'),\ |y' - y'_0| < \rho\} \ .$$

Denote by ∂D_ρ the boundary of D_ρ and set

$$\Gamma_{\rho,\sigma} = \{y \in \partial D_\rho;\ y_n < \psi_n(y') - \sigma\} \ ,$$
$$\tilde{\Gamma}_{\rho,\sigma} = \{y \in \partial D_\rho;\ \psi_n(y') - \sigma < y_n < \psi_n(y')\} \ .$$

Define $\partial D_{\rho_0}, \Gamma_{\rho_0,\sigma}$, $\Gamma_{\rho_0,\sigma}$ similarly with respect to D_{ρ_0}.

For any sufficiently small $\sigma > 0$ we have, by (5.6),

$$\eta > 0 \quad \text{in} \quad \Gamma_{\rho,\sigma} \cup \Gamma_{\rho,\sigma} \tag{5.10}$$

provided $A = A(\sigma)$ is sufficiently large. We claim that if σ is sufficiently small depending on ρ, ρ_0 then

$$\eta \geqq 0 \quad \text{on} \quad \Gamma_{\rho,\sigma} \ . \tag{5.11}$$

Indeed, suppose (5.11) is not true. Then there exists a point $y^* \in \tilde{\Gamma}_{\rho,\sigma}$ such that $\eta(y^*) < 0$.

Consider the function

$$\tilde{\eta} = \eta + \gamma|y - y^*|^2 \quad (\gamma > 0) \ .$$

If γ is sufficiently small then $\tilde{L}\tilde{\eta} < 0$. Therefore, $\tilde{\eta}$ cannot take negative minimum in D_{ρ_0}. But since $\tilde{\eta}(y^*) < 0$, $\tilde{\eta} > 0$ on $\Gamma_{\rho_0,\sigma}$ (by (5.10)) and on $y_n = \psi_n(y')$, there must exist a point $\hat{y} \in \tilde{\Gamma}_{\rho_0,\sigma}$ such that $\tilde{\eta}(\hat{y}) < 0$. Thus

$$\sum_{k=1}^{n-1} \alpha_k \frac{\partial \zeta}{\partial y_k} - A \frac{\partial \zeta}{\partial y_n} + \gamma|\hat{y} - y^*|^2 < 0 \ .$$

Recalling that $-A(\partial\zeta/\partial y_n) > 0$, and that

$$\frac{\partial \zeta}{\partial y_k} = 0(\sigma) \quad \text{on} \quad \tilde{\Gamma}_{\rho_0,\sigma}$$

(since $\zeta \in W^{2,\infty}_{\text{loc}}$ and $\nabla\zeta = 0$ on $y_n = \psi_n(y')$), we deduce that

$$(\rho_0 - \rho)^2 \leqq C\sigma \tag{5.12}$$

where C is a constant independent of ρ_0, ρ, σ, A. Consequently, if σ is sufficiently small so that (5.12) is not true then the inequality (5.11) is valid. It follows that $\eta \geqq 0$ on ∂D_ρ. Applying the maximum principle we conclude that $\eta > 0$ in D_ρ, i.e.,

$$\sum_{k=1}^{n-1} \alpha_k \frac{\partial\zeta}{\partial y_k} - A\frac{\partial\zeta}{\partial y_n} > 0 \quad \text{in} \quad D_\rho . \tag{5.13}$$

Denote by K the cone

$$\left\{y;\, y_n < -\frac{|y'|}{A}\right\} .$$

The inequality (5.13) implies that if

$$\tilde{y} = (\tilde{y}', \tilde{y}_n) , \quad \tilde{y}_n = \psi_n(\tilde{y}')$$

then $\zeta > 0$ in the cone $K + \tilde{y}$. Thus, if $\hat{y}_n = \psi_n(\hat{y}')$ then

$$\hat{y} \notin K + \tilde{y} ,$$

i.e.,

$$\hat{y}_n > \tilde{y}_n - \frac{|\hat{y}' - \tilde{y}'|}{A} ,$$

or equivalently,

$$\psi_n(\hat{y}') > \psi_n(\tilde{y}') - \frac{|\hat{y}' - \tilde{y}'|}{A} .$$

Interchanging $\hat{y}$ with $\tilde{y}$ we deduce that

$$|\psi_n(\hat{y}') - \psi_n(\tilde{y}')| \leqq \frac{|\hat{y}' - \tilde{y}'|}{A} ,$$

that is, ψ_n is Lipschitz continuous.

By a general result of Caffarelli [5] it then follows that ψ_n is a C^1 function and then (by Kinderlehrer and Nirenberg [12]) also analytic.

REMARK. It is clear that Theorems 5.1, 5.2 extend to the case where (5.2) holds with $i = 0$. Instead of using the coordinate transformation (3.3), we take $y_j = \pi_j/\pi_{j_0}$ for $0 \leqq j \leqq n$, $j \neq j_0$ for any j_0, $j_0 \neq 0$.

6. **The special case (1.9).** In this section we obtain additional

results in the special case when (1.9) holds. For any $0 \leqq j \leqq n$, let

$$\pi'_j = (\pi_0, \pi_1, \cdots, \pi_{j-1}, \pi_{j+1}, \cdots, \pi_n)$$

and denote by $\tilde{u}_j(\pi'_j)$ the solution of (2.1) corresponding to the problem with n hypotheses H_i, $0 \leqq i \leqq n$, $i \neq j$.

THEOREM 6.1. *Suppose* $q_{i,j} = 0$ *for* $0 \leqq i, j \leqq n$. *If* $\pi_j > 0$, $\pi_j \downarrow 0$ *then*

$$u(\pi) \longrightarrow \tilde{u}_j(\pi'_j) . \tag{6.1}$$

REMARK. Recall that boundary values for u were not prescribed (on $\partial\Pi_{n+1}$); in fact, in Π_{n+1},

$$u(\pi) = \inf_\tau E^\pi\left[\int_0^\tau e^{-\alpha t}h(\pi(t))dt + e^{-\alpha\tau}g(\pi(\tau))\right] \equiv V(\pi)$$

and, as shown in [2], the middle term is uniformly continuous in $\mathring{\Pi}_{n+1}$. This implies that u has a continuous extension into $\partial\Pi_{n+1}$, which is denoted again by u. What we have to prove is that this extension, when restricted to $\pi_j = 0$, coincides with $\tilde{u}_j(\pi'_j)$.

Proof. If suffices to consider the case $j = n$. Let $\pi' = \pi'_n$ and $\tilde{u}(\pi') = \tilde{u}_n(\pi'_n)$. We denote by τ_δ the exit time of $\pi(t)$ from Π^δ_{n+1}. We shall compare the cost functions

$$\begin{aligned} J_\pi(\tau) = E^\pi\Big[\int_0^{\tau\wedge\tau_\delta} ce^{-\alpha t}dt &+ e^{-\alpha\tau}g(\pi(\tau))I_{\tau\leqq\tau_\delta} \\ &+ e^{-\alpha\tau_\delta}\varphi(\pi(\tau_\delta))I_{\tau\geqq\tau_\delta}\Big] \\ J_{\pi'}(\tau) = E^{\pi'}\Big[\int_0^\tau ce^{-\alpha t}dt &+ e^{-\alpha\tau}g_1(\tilde{\pi})\Big] \end{aligned}$$

where $\tilde{\pi}(t) = (\tilde{\pi}_0(t), \cdots, \tilde{\pi}_{n-1}(t), \tilde{\pi}_n(t))$ is the process $\pi(t)$ with $\tilde{\pi}(0) = (\pi', 0)$ and

$$g_1(\pi) = \min_{0\leqq i\leqq n-1} \{a_i(1 - \pi_i)\} .$$

Recall [2] that

$$\begin{aligned} u_\delta(\pi) &= \inf_\tau J_\pi(\tau) \\ \tilde{u}(\pi') &= \inf_\tau J_{\pi'}(\tau) \end{aligned} \tag{6.2}$$

where τ varies over all $\mathscr{F}_t$ stopping times.

By Lemma 2.1, $\tilde{\pi}_n(t) \equiv 0$.

In what follows we shall use a model of the Markov process associated with $\pi(t)$ in which the probability is fixed, say P, and the initial condition $\pi(0) = \pi$ varies; for each π, $\pi(t)$ is the solution of the stochastic differential system associated with the generator M, and the initial condition $\pi(0) = \pi$. Working with this model, we can replace E^{π}, $E^{\pi'}$ by E, and we shall compare $J_{\pi}(\tau)$, $J_{\pi'}(\tau)$ with the same τ. We have (see, for instance, [8]), for any $T > 0$,

$$E[\sup_{0 \leqq t \leqq T} |\pi(t) - \tilde{\pi}(t)|^2] \leqq C_T \pi_n^2 , \quad C_T \text{ constant} . \tag{6.3}$$

By Lemma 2.1, for any $\eta > 0$,

$$P[\tau_\delta < T] < \eta \quad \text{if} \quad \pi(0) = (\pi_0, \cdots, \pi_n) , \quad \pi_n > 0 \tag{6.4}$$

provided δ is sufficiently small (depending on η, π_n).

Next, by Lemma 2.1 and (6.3),

$$E|g(\pi(t)) - g_1(\tilde{\pi}(t))| \leqq C_T \pi_n \quad \text{if} \quad 0 < t \leqq T . \tag{6.5}$$

Using (6.3)-(6.5) we find that

$$|J_{\pi}(\tau) - J_{\pi'}(\tau)| \leqq CC_T \pi_n + Ce^{-\alpha T} + C\eta$$

if δ is sufficiently small, depending on η. Recalling (6.2) we get

$$|u(\pi) - \tilde{u}(\pi')| \leqq CC_T \pi_n + Ce^{-\alpha T} + C\eta .$$

Taking $\delta \to 0$ and ssing (2.8), we obtain

$$|u(\pi) - \tilde{u}(\pi')| \leqq CC_T \pi_n + Ce^{-\alpha T} + C\eta .$$

Taking $\pi_n \to 0$ we conclude that

$$\limsup |u(\pi) - \tilde{u}(\pi')| \leqq Ce^{-\alpha T} + C\eta .$$

Taking $\eta \to 0$, $T \to \infty$ the assertion (6.1) follows.

THEOREM 6.2. *If* (1.9) *holds then each set* S_i *is a convex set and the function* $u(\pi)$ *is concave.*

The proof is analogous to that for the discrete case [16].

Proof. Let π^1, π^2 belong to $\mathring{\Pi}_{n+1}$ and set

$$\tilde{\pi} = \lambda \pi^1 + (1 - \lambda)\pi^2 \quad (0 < \lambda < 1) .$$

We can write (1.6) in the form

$$J_{\pi}(\delta) = \sum_{i=0}^{n} \pi_i E^i \left[c \int_0^{\tau} e^{-\alpha t} dt + e^{-\alpha \tau} W(\theta(\omega), d(\omega)) \right] , \tag{6.6}$$

where $E^i = E^{\pi}$ for $\pi = e_i$. Writing this relation for a specific $\delta = (\tau(\omega), d(\omega))$ and $\pi = \pi^1$, $\pi = \pi^2$, and multiplying the first relation by λ and the second one by $(1 - \lambda)$ we obtain, upon adding these relations,

$$\begin{aligned}(6.7)\qquad \lambda J_{\pi^1}(\delta) + (1-\lambda)J_{\pi^2}(\delta) &= \sum_{i=0}^{n} (\lambda\pi_i^1 + (1-\lambda)\pi_i^2)E^i[\cdots] \\ &= \sum_{i=0}^{n} \tilde{\pi}_i E^i[\cdots] = E^{\tilde{\pi}}[\cdots] = J_{\tilde{\pi}}(\delta)\ ;\end{aligned}$$

here π_i^1, π_i^2, $\tilde{\pi}_i$ are the ith coodinates of π^1, π^2, $\tilde{\pi}$ respectively and the expression $[\cdots]$ is the same as in (6.6).

Suppose now that π^1 and π^2 belong to S_i. Then

$$a_i(1-\pi_i^1) \leqq J_{\pi^1}(\delta)\ ,\quad a_i(1-\pi_i^2) \leqq J_{\pi^2}(\delta)\ .$$

It follows from (6.7) that

$$a_i(1-\tilde{\pi}_i) \leqq J_{\tilde{\pi}}(\delta)\ .$$

Thus $\tilde{\pi} \in S_i$ and, consequently, S_i is a convex set.

Next, (6.7) gives

$$\inf_{\delta} J_{\tilde{\pi}}(\delta) \geqq \lambda \inf_{\delta} J_{\pi^1}(\delta) + (1-\lambda) \inf_{\delta} J_{\pi^2}(\delta)\ ,$$

i.e.,

$$u(\lambda\pi^1 + (1-\lambda)\pi^2) = u(\tilde{\pi}) \geqq \lambda u(\pi^1) + (1-\lambda)u(\pi^2)\ ,$$

so that $u(\pi)$ is concave.

REMARK 1. From Theorems 6.2, 5.2, 3.2 we deduce that each S_i is a convex domain containing a Π_{n+1}-neighborhood of e_i and $\partial S_i \cap \mathring{\Pi}_{n+1}$ is an analytic manifold.

REMARK 2. For any numbers $\alpha_{i,k} (0 \leqq i \leqq n,\ 1 \leqq k \leqq l)$ the equations

$$\sum_{i=1}^{n} \alpha_{i,k} y_i = \alpha_{0,k} \Big(1 \leqq k \leqq l,\ y_i = \frac{\pi_i}{\pi_0}\Big)$$

hold if and only if

$$\sum_{i=1}^{n} \alpha_{i,k}\pi_i - \alpha_{0,k}\pi_0 = 0 \quad 1 \leqq k \leqq l)\ .$$

Since also $\sum_{i=0}^{n} \pi_i = 1$, it follows that the mapping (3.3) maps planes onto planes and lines onto lines. It also maps segments onto segments. It follows, in particular, that

(6.8) (3.3) maps convex sets onto convex sets .

Consequently, by Theorem 6.2, *the image* $\tilde{S}_i (1 \leqq i \leqq n)$ *in the* y*-space of the coincidence set* S_i *(in the* π*-space) is a convex set.*

7. Asymptotic estimates for $c \to 0$. For any $\gamma > 0$ set

$$N_\gamma^i = \{\pi \in \Pi_{n+1};\, 1 - \gamma \leqq \pi_i < 1\}\ , \quad N_\gamma = \bigcup_{i=0}^{n} N_\gamma^i\ . \tag{7.1}$$

THEOREM 7.1. *Assume that* (1.9) *holds. Then there exist positive constants* δ_1, δ_2 *independent of* c, α, *such that for all* c *sufficiently small,*

$$N_{\delta_1 c} \subset S \subset N_{\delta_2 c}\ . \tag{7.2}$$

Proof. Set

$$r = \sum_{i=1}^{n} \pi_i\ . \tag{7.3}$$

Using (3.2) and the relation $\sum_{j=1}^{n} q_{i,j} = -q_{i,0}$ we find that (for general $q_{i,j}$)

$$M(\log r) = -\frac{1}{2r^2}\left|\sum_{i=1}^{n}\sum_{k=0}^{n}(\lambda_i - \lambda_k)\pi_i\pi_k\right|^2 - \frac{1}{r}\sum_{i=0}^{n} q_{i,0}\pi_i\ . \tag{7.4}$$

Since

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(\lambda_i - \lambda_k)\pi_i\pi_k = 0\ , \quad \text{by symmetry}\ ,$$

and since

$$\left|\sum_{i=1}^{n}(\lambda_i - \lambda_0)\pi_i\pi_0\right|^2 = \pi_0^2\left|\sum_{i=1}^{n}(\lambda_i - \lambda_0)\pi_i\right|^2\ ,$$

we obtain upon recalling (1.1), that

$$-q_0^* - K_1\pi_0^2 \leqq M(\log r) \leqq -K_2\pi_0^2 + |q_{0,0}|\pi_0/r \quad (q_0^* = \max_{1\leqq i\leqq n} q_{i,0})\ , \tag{7.5}$$

where K_1, K_2 are positive constants depending only on the λ_i.

To prove the second part of (7.2), consider the function, in $\tilde{\Pi} = \Pi_{n+1} \cap \{r < 1/(n+1)\}$,

$$v(r) = \begin{cases} a_0 r & \text{if } \ 0 \leqq r \leqq R\ , \\ c\delta \log r + A - r & \text{if } \ R < r \leqq \dfrac{1}{n+1} \end{cases}$$

where A, R, δ are positive constants. We choose A, R as functions of δ so that v becomes C^1 at $r = R$, i.e.,

$$\frac{c\delta}{R} - 1 = a_0\ , \quad c\delta \log R + A - R = a_0 R\ ;$$

δ is a positive constant (independent of c) to be determined. Thus

(7.6) $$R = \frac{c\delta}{a_0 + 1} \quad \left(R < \frac{1}{n+1} \text{if } c \text{ is sufficiently small}\right),$$

(7.7) $$A = c\delta + c\delta \log \frac{a_0 + 1}{c\delta} .$$

Notice that $\tilde{\Pi} \subset G_1$. The condition

(7.8) $$v\left(\frac{1}{n+1}\right) < 0$$

is satisfied if

$$c\delta \log \frac{1}{n+1} + A - \frac{1}{n+1} < 0 ,$$

i.e., (in view of (7.7), if c is sufficiently small. We also easily find that

(7.9) $$v \leqq c\delta \log(a_0 + 1) .$$

Using (7.5), (7.7), (7.9) and the conditions (1.9), we find that, if $R < r < 1/(n + 1)$,

$$Mv - \alpha v \geqq -K_1 \frac{c\delta}{(n+1)^2} - \alpha c\delta \log(a_0 + 1) > -c$$

provided δ is sufficiently small (independently of c). We also have

$$v < a_0 r = g \quad \text{if} \quad R < r < \frac{1}{n+1} .$$

Thus, we can apply Lemma 3.1 with $\tilde{u} = v$ and conclude that $v \leqq u$ in $\tilde{\Pi}$. Since $v < g$ if $r > R$, the same is true for u. Thus $S \cap G_0$ is contained in $N_{\delta c}$. Similarly one can prove that $S \cap G_i$ is contained in $N_{\delta c}$ for any $i \geqq 1$.

To prove the first part of (7.2), let

$$w_0(r) = \begin{cases} a_0 r & \text{if} \quad 0 \leqq r \leqq R_0 , \\ c\delta \log r + A_0 & \text{if} \quad R_0 < r \leqq 1 . \end{cases}$$

This function is C^1 at $r = R_0$ if

$$R_0 = \frac{c\delta}{a_0} ,$$

$$A_0 = c\delta + c\delta \log \frac{a_0}{c\delta} .$$

Using (7.5) and (1.9) we get, for $R_0 < r < n/(n + 1,)$

$$Mw_0 - \alpha w_0 \leqq -\frac{K_2 c\delta}{(n+1)^2} < -c$$

if δ is sufficiently large (independently of c).

Similarly, we define functions w_i for each $1 \leqq i \leqq n$ and take

$$w = \min_{0 \leqq i \leqq n} w_i . \tag{7.10}$$

Note that if $r = n/(n+1)$ then certainly $w < w_0$. Thus, if $w = w_0$ then $r < n/(n+1)$ and, consequently,

$$Mw_0 - \alpha w_0 < -c \quad \text{if further} \quad r > R_0 .$$

The corresponding result is true for each w_i.

It follows that outside the $(c\delta)$-neighborhoods of the vertices e_i,

$$Mw - \alpha w < -c$$

where Mw is taken in the distribution sense.

We can now apply Lemma 3.1 (whose proof extends, by approximation, to the case where $\tilde{u}$ is only Lipschitz continuous and $M\tilde{u}$ is taken in the distribution sense). It follows that $u \leqq w$, and the first part of (7.2) is established.

REMARK 1. The proof of the second part of (7.2) extends to the case where, for some i,

$$|q_{i,i}| \geqq (n+1)\max_{k \neq i} (q_{k,i} - \alpha) ;$$

it gives the relation

$$S \cap G_i \subset N_{\delta_2 c} .$$

REMARK 2. From the proof of the first part of (7.2) we see that the function

$$W_0(r) = \log \frac{r}{\varepsilon} \quad \left(r = \sum_{i=1}^{n} \pi_i,\ \varepsilon > 0\right)$$

satisfies $MW_0 \leqq -A$ if $\varepsilon < r < n/(n+1)$, where A is a positive constant independent of ε. Define W_i in a similar manner with respect to the vertex e_i, and set

$$W = \frac{1}{A} \min_{0 \leqq i \leqq n} W_i .$$

Then $MW \leqq -1$ in

$$\tilde{N}_\varepsilon = \Pi_{n+1} \backslash N_\varepsilon .$$

Also $W = 0$ on $\partial \tilde{N}_\varepsilon \cap \mathring{\Pi}_{n+1}$. Denoting by τ_ε the hitting time of N_ε by the process $\pi(t)$ it follows, by comparison, that

$$E^\pi \tau_\varepsilon \leqq W(\pi) .$$

Thus

$$E^\pi \tau_\varepsilon \leqq A_1 \log \frac{1}{\varepsilon} \text{ for all } \pi \in \Pi_{n+1} . \tag{7.11}$$

In the following section we shall obtain a more precise result as $\varepsilon \to 0$.

REMARK 3. From Theorem 7.1 it follows that

$$u(\pi) \leqq A_2 c \tag{7.12}$$

for all c sufficiently small, where A_2 is a constant independent of c. In the following section we shall obtain a more precise result as $c \to 0$.

8. Asymptotic estimates for $c \to 0$ (continued).

THEOREM 8.1. *Suppose* (1.9) *holds. Then, for any* $\pi \in \Pi_{n+1}$,

$$E^\pi \tau_\varepsilon = \Big(\sum_{i=0}^{n} \gamma_i \pi_i\Big) \log \frac{1}{\varepsilon} + 0\Big(\Big(\log \frac{1}{\varepsilon}\Big)^{1/2}\Big), \quad \frac{1}{2}\gamma_i = \{\min_{k \neq i} |\lambda_k - \lambda_i|\}^{-1} \tag{8.1}$$

as $\varepsilon \to 0$.

The analogous result for discrete processes is given, for instance, in Kiefer and Sacks [10].

Proof. Set $\tau = \tau_\varepsilon$. Then τ is the first time t such that

$$\max_{0 \leqq j \leqq n} \pi_j(t) = 1 - \varepsilon .$$

Using (1.11), the last inequality becomes

$$\max_{0 \leqq j \leqq n} \Big\{1 + \sum_{k \neq j} \frac{\pi_k}{\pi_j} e^{(\lambda_k - \lambda_j)\cdot \xi(t) - 1/2(|\lambda_k|^2 - |\lambda_j|^2)}\Big\}^{-1} = 1 - \varepsilon ,$$

or

$$\min_{0 \leqq j \leqq n} \max_{k \neq j} e^{(\lambda_k - \lambda_j)\cdot \xi(t) - 1/2(|\lambda_k|^2 - |\lambda_j|^2)} = C\varepsilon \tag{8.2}$$

where C is a random variable, $B_1 \leqq C \leqq B_2$, and B_1, B_2 are positive constants independent of ε (but depending on the initial point π. Taking the logarithm on both sides of (8.2) we conclude that

$$\min_{0 \leqq j \leqq n} \max_{k \neq j} \left\{(\lambda_k - \lambda_j)\cdot\xi(\tau) - \frac{1}{2}(|\lambda_k|^2 - |\lambda_j|^2)\tau\right\} = -\log\frac{1}{\varepsilon} + 0(1) . \tag{8.3}$$

Recalling that

$$E^\pi\tau = \sum_{l=0}^{n} \pi_l E^l\tau , \tag{8.4}$$

we proceed to evaluate $E^l\tau$ for a fixed l. With respect to the probability P^l,

$$\xi(t) = w(t) + \lambda_l t \text{ a.s.} \tag{8.5}$$

Thus, the stopping time τ is the hitting time of some region Q by the process $w(t)$.

We claim that for any hitting time τ of a region Q,

$$E_x |w(\tau)|^2 = 2n\, E_x\tau + |x|^2 . \tag{8.6}$$

Indeed, if $Q^c = R^n \backslash Q$ is a bounded open set then, since both sides of (8.6) are harmonic functions in Q^c taking the same boundary values $|x|^2$ on ∂Q^c, they must agree in Q^c. If Q^c is unbounded then (8.6) follows by approximating Q^c by bounded open sets.

From (8.6) applied with $x = 0$ it follows that

$$E^l|w(\tau)| = (2n)^{1/2}(E_x\tau)^{1/2} \leqq C_0\left(\log\frac{1}{\varepsilon}\right)^{1/2} (C_0 \text{ constant}) \tag{8.7}$$

where (7.11) was used.

Combining (8.3) with (8.5) and using (8.7), we find that

$$\begin{aligned}\min_{0 \leqq j \leqq n} \max_{k \neq j} &\left\{(\lambda_k - \lambda_j)\cdot\lambda_l - \frac{1}{2}(|\lambda_k|^2 - |\lambda_j|^2)\right\}E^l\tau \\ &= -\log\frac{1}{\varepsilon} + 0(1) + 0\left(\left(\log\frac{1}{\varepsilon}\right)^{1/2}\right) .\end{aligned} \tag{8.8}$$

Next, one easily checks that

$$\max_{k \neq j}\left\{(\lambda_k - \lambda_j)\cdot\lambda_l - \frac{1}{2}(|\lambda_k|^2 - |\lambda_l)\right\} = \begin{cases} \max\limits_{k \neq l}\left[-\frac{1}{2}|\lambda_k - \lambda_l|^2\right] \text{if} \quad j = l \\ (\lambda_l - \lambda_j)\cdot\lambda_l - \frac{1}{2}(|\lambda_l|^2 - |\lambda_j|^2) \\ \qquad = \frac{1}{2}|\lambda_l - \lambda_j|^2 \quad \text{if} \quad j \neq l . \end{cases}$$

Using this in (8.8), we obtain

$$\left[\frac{1}{2}\min_{k \neq l}|\lambda_k - \lambda_l|^2\right]E^l\tau = \log\frac{1}{\varepsilon} + 0\left(\left(\log\frac{1}{\varepsilon}\right)^{1/2}\right) ;$$

recalling (8.4), the assertion (8.1) follows.

We wish to study the behavior of the solution $u(\pi)$ in a neighborhood of a vertex e_i as $c \to 0$. It suffices to take $i = 0$. It will be convenient to use the coordinates (3.3). We also set

$$u(y) = u(\pi)\ , \quad |y| = y_1 + \cdots + y_n\ ,$$
$$(8.9) \qquad L_0 u \equiv \frac{1}{2} \sum_{i,j=1}^{n} \mu_{ij} y_i y_j \frac{\partial^2 u}{\partial y_i \partial y_j}\ .$$

The function $u(y)$ satisfies the variational inequality, in $0 \leqq |y| < \delta_0$, $\delta_0 = 1/(n+1)$,

$$L_0 u + \frac{1}{Y} \sum_{i,j=1}^{n} \mu_{ij} y_i y_j \frac{\partial u}{\partial y_j} - \alpha u + c \leqq 0\ ,$$
$$(8.10) \qquad u \leqq \frac{a_0 |y|}{Y}\ ,$$
$$\Big(L_0 u + \frac{1}{Y} \sum_{i,j=1}^{n} \mu_{ij} y_i y_j \frac{\partial u}{\partial y_j} - \alpha u + c\Big)\Big(u - \frac{a_0|y|}{Y}\Big) = 0\ .$$

Consider the variational inequality in R_n^+;

$$L_0 \tilde{u} - \alpha \tilde{u} + 1 \geqq 0\ ,$$
$$(8.11) \qquad \tilde{u} \leqq a_0 |y|\ ,$$
$$(L_0 \tilde{u} - \alpha \tilde{u} + 1)(\tilde{u} - a_0 |y||) = 0\ ,$$

subject to the growth condition

$$(8.12) \qquad \tilde{u}(y) = 0(|y|) \quad \text{as} \quad |y| \longrightarrow \infty\ .$$

THEOREM 8.2. *Let* (1.9) *hold. Then there exists a unique solution* $\tilde{u}(y)$ *of* (8.11), (8.12); *further,* $0 \leqq u(y) \leqq C$ *for some constant* C, *and*

$$(8.13) \qquad \frac{u(cy)}{c} \longrightarrow \tilde{u}(y) \quad as \quad c \longrightarrow 0\ ,$$

uniformly in y *in compact subsets of* int R_n^+.

Proof. For any $A > 0$ let $\tilde{u}_A$ be the solution of the variational inequality, in $|y| < A$,

$$L_0 \tilde{u}_A - \alpha \tilde{u}_A + 1 \geqq 0\ ,$$
$$\tilde{u}_A \leqq a_0 |y|\ ,$$
$$(8.14) \qquad (L_0 \tilde{u}_A - \alpha \tilde{u}_A + 1)(\tilde{u}_A - a_0|y|) = 0\ ,$$
$$\tilde{u}_A = 0 \quad \text{on} \quad |y| = A\ .$$

It is easily seen that

$$\tilde{u}_A \geqq 0 , \quad \tilde{u}_A(y)\uparrow \quad \text{if} \quad A\uparrow .$$

It follows that

$$\tilde{u}(y) \equiv \lim_{A\uparrow\infty} \tilde{u}_A(y) \tag{8.15}$$

exists and it is a solution of the variational inequality (8.11).

We can represent $\tilde{u}_A(y)$ in the form

$$\tilde{u}_A(y) = \inf_\tau J_{y,A}(\tau) , \tag{8.16}$$

$$J_{y,A}(\tau) = E_y\left[\int_0^{\tau\wedge\tau^A} c^{-\alpha t}dt + a_0|y(\tau)|e^{-\alpha\tau}I_{\tau<\tau^A}\right] \tag{8.17}$$

where $y(t)$ satisfies

$$dy(t) = \sigma(y(t))d\tilde{w}(t) , \quad y(0) = y \quad (y \in \operatorname{int} R_n^+) , \tag{8.8}$$

for some n-dimensional Brownian motion $\tilde{w}(t)$, and

$$\sigma = (\sigma^{ij}) , \quad \sigma^{ij} = \nu_{ij}y_i , \quad \sum_{k=1}^n \nu_{ik}\nu_{jk} = \mu_{ij}, \, \nu_{ij} = \nu_{ji} ;$$

τ^A is the exit time of $y(t)$ from the set $|y| \leqq A$.

We claim that for all $1 < A < \infty$,

$$E_y[e^{-\alpha\tau^A}] \leqq \frac{C}{A^\lambda} \text{for some constants } \lambda > 1 , \quad C > 0 . \tag{8.19}$$

Indeed, by comparison

$$E_y[e^{-\alpha\tau^A}] \leqq W(y)$$

provided

$$L_0 W - \alpha W \leqq 0 \quad \text{for} \quad |y| < A ,$$
$$W \geqq 1 \quad \text{on} \quad |y| = A .$$

Taking

$$W(y) = \frac{C_0}{A^\lambda}(y_1^\lambda + \cdots + y_n^\lambda)$$

where C_0 is a constant independent of A and $\lambda > 1$, $\lambda - 1$ sufficiently small, the assertion (8.19) follows.

Taking $\tau = \tau^A$ in (8.17) and using (8.19) and (8.16), (8.15) we conclude that

$$\tilde{u}(y) \leqq C \quad (C \text{ positive constant}) . \tag{8.20}$$

Using (8.10) we find that the function

$$w_c(y) = \frac{u(cy)}{c}$$

satisfies the variational inequality, in $0 < y < \delta_0/c$,

$$\begin{gathered} L_0 w_c + \frac{c}{1 + c|y|} \sum_{k,j=1}^{n} \mu_{ij} y_i y_j \frac{\partial w_c}{\partial y_j} - \alpha w_c + 1 \geqq 0 , \\ w_c \leqq \frac{a_0|y|}{1 + c|y|} , \qquad (8.21) \\ \Big(L_0 w_c + \frac{c}{1 + c|y|} \sum_{k,j=1}^{n} \mu_{ij} y_i y_j \frac{\partial w_c}{\partial y_j} - \alpha w_c + 1\Big)\Big(w_c - \frac{a_0|y|}{1 + c|y|}\Big) = 0 . \end{gathered}$$

Hence we can write

$$w_c(y) = \inf_\tau J^c_{y,A}(\tau) \qquad (8.22)$$

where

$$J^c_{y,A}(\tau) = E_y\Big[\int_0^{\tau\wedge\bar\tau^A} e^{-\alpha t} dy + \frac{a_0|y_c(\tau)|}{1 + c|y_c(\tau)|} e^{-\alpha\tau} I_{\tau<\bar\tau^A} + w_c(y_c(\bar\tau^A))e^{-\alpha\bar\tau^A} I_{\tau\geq\bar\tau^A}\Big] \qquad (8.23)$$

where $y_c(t)$ is the solution of the stochastic system

$$dy_c(t) = \sigma(y_c(t))d\tilde w(t) + b_c(t)dt , \quad y_c(0) = y , \qquad (8.24)$$

the matrix $\sigma(y)$ is defined above,

$$b_c = (b_{c,i}) , \quad b_{c,i} = \frac{c}{1 + c|y|} \sum_{j=1}^{n} \mu_{ij} y_i y_j ,$$

$\bar\tau^A = \tau^A \wedge \tau^A_c$, τ^A_c is the exit time of $y_c(t)$ from the set $|y| \leqq A$. Notice that $\bar\tau^A$ is a stopping time with respect to the σ-fields $\sigma(\tilde w(s), 0 \leqq s \leqq t)$, $t \geqq 0$; here A is any fixed positive number $\leqq \delta_0/c$.

Analogously to (8.22), (8.23) we can write

$$\tilde u(y) = \inf_\tau \tilde J_{y,A}(\tau) \qquad (8.25)$$

where

$$\tilde J_{y,A}(\tau) = E_y\Big[\int_0^{\tau\wedge\bar\tau^A} e^{-\alpha t} dt + a_0|y(\tau)|e^{-\alpha\tau} I_{\tau<\bar\tau^A} + \tilde u(y(\bar\tau^A))e^{-\alpha\bar\tau^A} I_{\tau\geqq\bar\tau^A}\Big] . \qquad (8.26)$$

By standard arguments, for any large $T > 0$ and small $\eta > 0$,

$$E_y[\sup_{0\leqq t\leqq T} |y_c(t) - y(t)|^2] \leqq \eta^2 \quad \text{if} \quad c \leqq c_0(\eta, T) . \qquad (8.27)$$

Next, the proof of (8.19) shows also that

$$A^{\lambda}E_y[e^{-\alpha\tau_c^A}] \leqq C \quad \text{for some} \quad \lambda > 1 , \quad C > 0$$

provided $cA \leqq 1/C^*$ where C^* is a suitably large positive constant (independent of c, A). Hence

$$A^{\lambda}E_y[e^{-\alpha\bar{\tau}^A}] \leqq C \text{ provided } cA \leqq \frac{1}{C^*} .$$

It follows that

$$E_y[e^{-\alpha\bar{\tau}^A}] < \eta \quad \left(\text{if} \quad c \leqq \frac{1}{C^*A}\right) \tag{8.29}$$

provided A is sufficiently large.

Note that

$$|y(\tau)| \leqq A , \quad |y_c(\tau)| \leqq A \quad \text{if} \quad \tau \leqq \bar{\tau}^A .$$

Now fix A such that (8.29) holds and then fix T sufficiently large (depending on A but not on c) such that

$$\begin{aligned} &|y_c(\tau)| - \frac{|y_c(\tau)|}{1 + c|y_c(\tau)|} < \eta \quad \text{if} \quad \tau \leqq \bar{\tau}^A , \\ &Ae^{-\alpha T} < \eta . \end{aligned} \tag{8.30}$$

Using (8.27), (8.29), (8.30) and (8.20), and recalling also (by (7.12)) that

$$|w_c| \leqq A_2 ,$$

we deduce from (8.23), (8.26) that

$$|J^c_{A,y}(\tau) - \tilde{J}_{y,A}(\tau)| \leqq C\eta \tag{8.31}$$

provided $c \leqq c_*(\eta, A)$; c_* and C are independent of τ and C is independent of c, A. Recalling (8.22), (8.25), we get

$$|w_c(y) - \tilde{u}(y)| \leqq C\eta , \tag{8.32}$$

and the assertion (8.13) follows.

It remains to prove that any solution $\tilde{\tilde{u}}(y)$ of (8.11), (8.12) must coincide with $\tilde{u}$. From (8.19) we conclude that

$$E_y[|\tilde{\tilde{u}}(y(\tau^A))|e^{-\alpha\tau^A}] \longrightarrow 0 \quad \text{if} \quad A \longrightarrow \infty . \tag{8.33}$$

Using (8.33) we can now repeat the argument which gave (8.32), with $w_c(y)$ replaced by $\tilde{\tilde{w}}(y)$. We thus deduce that

$$|\tilde{\tilde{u}}(y) - \tilde{u}(y)| \leqq C\eta \quad \text{for and} \quad \eta > 0 ;$$

hence $\tilde{\tilde{u}} = \tilde{u}$.

From Remark 2 at the end of § 6 we have that the component of the coincidence set of $w_c(y)$ which contains $y = 0$ in convex. We also have:

THEOREM 8.3. *The coincidence set $\tilde{S}$ of $\tilde{u}(y)$ is a convex set.*

By Caffarelli [5] it then follows that the free boundary $\partial\tilde{S} \cap \operatorname{int} R_n^+$ is analytic.

Proof. It is easy to check that if $y \leftrightarrow \pi$ by (3.3) then $cy \leftrightarrow \pi^c = (\pi_0^c, \cdots, \pi_n^c)$ where

$$\pi_0^c = \frac{\pi_0}{\pi_0 + c(1 - \pi_0)} , \quad \pi_i^c = \frac{c\pi_i}{\pi_0 + c(1 - \pi_0)} (1 \leqq i \leqq n) . \tag{8.34}$$

Setting $\tilde{u}(\pi) = \tilde{u}(y)$, $u(\pi) = u(y)$ we then have, by Theorem 8.2,

$$\frac{u(\pi^c)}{c} \longrightarrow \tilde{u}(\pi) \quad \text{as} \quad c \longrightarrow 0 . \tag{8.35}$$

Set $\bar{\pi} = \pi - e_0$, $\bar{\pi}^c = \pi^c - e_0$. Then, as easily checked,

$$\bar{\pi}^c = \frac{c}{1 + (1 - c)\bar{\pi}_0} \bar{\pi} \tag{8.36}$$

and

$$\bar{\pi} = \frac{1}{c + (c - 1)\bar{\pi}_0^c} \bar{\pi}^c \tag{8.37}$$

where

$$\bar{\pi} = (\bar{\pi}_0, \cdots, \bar{\pi}_n) , \quad \bar{\pi}^c = (\bar{\pi}_0^c, \cdots, \bar{\pi}_n^c) .$$

Now, by the concavity of $u(\pi)$ established in Theorem 6.2, for any two points $\hat{\pi}$, $\hat{\hat{\pi}}$ and $0 < \lambda < 1$,

$$\frac{1}{c}\bar{u}(\lambda\hat{\bar{\pi}}^c + (1 - \lambda)\hat{\hat{\bar{\pi}}}) \geqq \frac{1}{c}(\lambda\bar{u}(\hat{\bar{\pi}}^c) + (1 - \lambda(\bar{u}(\hat{\hat{\bar{\pi}}}^c)) \tag{8.38}$$

where $\hat{\bar{\pi}} = \hat{\pi} - e_0$, $\hat{\hat{\bar{\pi}}} = \hat{\hat{\pi}} - e_0$ and $\bar{u}(\bar{\pi}) = u(\pi)$ for any π. We can write

$$\lambda\hat{\bar{\pi}}^c + (1 - \lambda)\hat{\hat{\bar{\pi}}}^c = \tilde{\bar{\pi}}^c , \quad \tilde{\bar{\pi}} = \tilde{\pi} - e_0 , \tag{8.39}$$

where, by (8.37),

$$\tilde{\bar{\pi}} = \frac{\lambda \hat{\bar{\pi}}^c + (1-\lambda)\hat{\hat{\bar{\pi}}}^c}{c + (c-1)(\lambda \hat{\bar{\pi}}_0^c + (1-\lambda)\hat{\hat{\bar{\pi}}}_0)} . \tag{8.40}$$

The point $\tilde{\bar{\pi}}$ depends on c; as $c \to 0$

$$\tilde{\bar{\pi}} \longrightarrow \frac{\dfrac{\lambda}{1+\pi_*}}{\dfrac{\lambda}{1+\pi_*} + \dfrac{1-\lambda}{1+\pi_{**}}} \hat{\bar{\pi}} + \frac{\dfrac{1-\lambda}{1+\pi_{**}}}{\dfrac{\lambda}{1+\pi_*} + \dfrac{1-\lambda}{1+\pi_{**}}} \hat{\hat{\bar{\pi}}} \equiv \check{\bar{\pi}} , \tag{8.41}$$
$$\check{\bar{\pi}} = \check{\pi} - e_0 ,$$

as seen using (8.36).

Using (8.35) we see that the right hand of (8.38) converges to

$$\lambda \tilde{u}(\hat{\pi}) + (1-\lambda)\tilde{u}(\hat{\hat{\pi}}) \quad \text{as} \quad c \longrightarrow 0 .$$

As for the left hand side, using (8.39)-(8.41) we find that it converges to $\tilde{\bar{u}}(\check{\bar{\pi}})$, where $\tilde{\bar{u}}(\bar{\pi}) = \tilde{u}(\pi)$ for any π. Hence

$$\lambda \tilde{u}(\hat{\pi}) + (1-\lambda)\tilde{u}(\hat{\hat{\pi}}) \leqq \tilde{u}(\check{\pi}) \tag{8.42}$$

where $\check{\pi}$ is the same linear combination of π, $\hat{\pi}$ as $\check{\bar{\pi}}$ is of $\hat{\bar{\pi}}$, $\hat{\hat{\bar{\pi}}}$ in (8.41). As λ varies from 0 to 1, the points $\check{\pi}$ fill the entire interval connecting $\hat{\pi}$ to $\hat{\hat{\pi}}$.

For the obstacle of $\tilde{u}$ we have equality in (8.42) (since it is a linear function). It follows that if $\hat{\pi}$ and $\hat{\hat{\pi}}$ are in the coincidence set of $\tilde{u}$, then so is the entire interval connecting them.

Since the coincidence set is convex in the π-space, it is also convex in the y-space.

We denote by $\tilde{C}$ the continuation set for $\tilde{u}$.

LEMMA 8.4. *Suppose* (1.9) *holds and*

$$\mu_{1j} = 0 \quad \text{for} \quad 2 \leqq j \leqq n . \tag{8.43}$$

Then

$$\frac{\partial}{\partial y_1}(\tilde{u}(y) - a_0|y|) < 0 \quad \text{in} \quad \tilde{C} . \tag{8.44}$$

Proof. Denote by $\hat{u}_A$ the solution of (8.14) subject to a different boundary condition, namely,

$$\hat{u}_A = 0(A) . \tag{8.45}$$

Representing $\hat{u}_A$ as a cost function and using (8.27), we find that

$$\hat{u}_A(y) - \tilde{u}_A(y) \longrightarrow 0 \quad \text{if} \quad A \longrightarrow \infty .$$

Hence

$$\hat{u}_A(y) \longrightarrow \tilde{u}(y) \quad \text{if} \quad A \longrightarrow \infty .$$

Next, suppose we replace the domain $|y| < A$ by the domain

$$\{y \in R_n^+, 0 < y_i < A \quad \text{for} \quad 1 \leqq i \leqq n\} \tag{8.46}$$

and denote by u_A^* the solution of the variational inequality (8.14) in (8.46) subject to boundary condition (8.45). Then again we have

$$u_A^*(y) \longrightarrow \tilde{u}(y) \quad \text{if} \quad A \longrightarrow \infty . \tag{8.47}$$

(This follows, for instance, by working throughout the proof of Theorem 8.2 with the domains (8.46) instead of the domains $|y| < A$.)

Denote by $u_{\delta,A}(y)$ $(0 < \delta < A)$ the solution of the variational inequality (8.14) in the domain

$$\{y \in R_n^+, \delta < y_i < A \quad \text{for} \quad 1 \leqq i \leqq n\} , \tag{8.48}$$

subject to boundary conditions

$$u_{\delta,A}(y) = 0(A) . \tag{8.49}$$

Then, for each fixed A, we clearly have

$$u_{\delta,A}(y) \longrightarrow u_A^*(y) \quad \text{if} \quad \delta \longrightarrow 0 . \tag{8.50}$$

Set

$$v = u_{\delta,A} - a_0|y| . \tag{8.51}$$

We choose the boundary conditions in (8.49) such that $v \leqq 0$ and

$$\begin{gathered} v = 0 \quad \text{on} \quad y_1 = \delta , \\ v_{y_1} \leqq 0 \quad \text{on} \quad y_i = \delta \quad \text{and on} \quad y_i = A \quad (2 \leqq i \leqq n) , \\ v = -C^*A \quad \text{on} \quad y_1 = A \quad (C^* \text{ positive constant}) . \end{gathered} \tag{8.52}$$

Consider the penalized problem corresponding to the variational inequality for v, namely,

$$L_0 v_\varepsilon - \alpha v_\varepsilon - \beta_\varepsilon(v_\varepsilon) + 1 - \alpha a_0|y| = 0 \tag{8.53}$$

(where β_ε is as in (2.3) and $\beta_\varepsilon(0) = 1$). Using (8.43) and the condition $v_\varepsilon = 0$ on $y_1 = \delta$, we find that

$$\mu_{11}\frac{\partial^2 v_\varepsilon}{\partial y_1^2} = \beta_\varepsilon(0) + a_0|y| - 1 > 0 \quad \text{if} \quad y_1 = \delta , \tag{8.54}$$

and similarly

(8.55) $$\mu_{11}\frac{\partial^2 v_\varepsilon}{\partial y_1^2} = -\alpha C^* A + \beta_\varepsilon(-C^* A) + a_0|y| - 1 < 0 \quad \text{if} \quad y_1 = A$$

provided C^* is sufficiently large (independently of A).

Differentiating (8.53) with respect to y_1 and setting $z = \partial v_\varepsilon/\partial y_1$, we get

$$L_0 z - \alpha z - \beta_\varepsilon'(v)z - \alpha a_0 = 0 .$$

It follows that z cannot take a positive maximum at an interior point. Furthermore, from (8.54), (8.55) we deduce that z cannot take a maximum on the parts $y_1 = \delta$, $y_1 = A$ of the boundary. Since, by (8.52) $z \leqq 0$ in the remaining parts of the boundary, we conclude that

$$z(y) < 0 \quad \text{in the domain (8.48)} .$$

Taking $\varepsilon \to 0$ we get

$$\frac{\partial}{\partial y_1}(u_{\delta,A} - a_0|y|) = \frac{\partial v}{\partial y_1} \leqq 0 .$$

Taking $\delta \to 0$ and using (8.50), and then letting $A \to \infty$ and using (8.47), the assertion (8.44) follows.

THEOREM 8.5. *Suppose* (1.9) *holds. Then the stopping set* $\tilde{S}$ *of* $\tilde{u}$ *contains an* $(\text{int}\, R_n^+)$*-neighborhood of the origin. If*

(8.56) $$\mu_{ik} = 0 \quad \textit{for some } i \textit{ and all} \quad k \neq i$$

then the free boundary $\tilde{\Gamma} = \partial\tilde{S} \cap (\text{int}\, R_n^+)$ *of* $\tilde{u}$ *can be represented in the form*

(8.57) $$y_i = \varphi_i(y_1, \cdots, y_{i-1}, y_{i+1}, \cdots, y_n)$$

where φ_i *is analytic.*

The proof of the first part is the same as in the case of Theorem 3.2. To prove the second part, say for $k = 1$, we use Lemma 8.4 and proceed as in § 5.

REMARK. Denote by S_c the connected component of the coincidence set of $w_c(y) = u(cy)/c$ which contains $y = 0$. Introduce the free boundaries

$$\Gamma_c = \partial S_c \cap (\text{int}\, R_n^+) , \quad \tilde{\Gamma} = \partial\tilde{S} \cap (\text{int}\, R_n^+)$$

where $\tilde{S}$ is the coincidence set for $\tilde{u}$. The sets S_c, $\tilde{S}$ are contained in $|y| < R_0$ for some $R_0 > 0$. Introduce polar coordinates $(|y|, \theta) =$

$(|y|, \theta_1, \cdots, \theta_{n-1})$ in R_n^+ and a truncated convex cone

$$K = \{y; 0 < |y| \leqq R_0, \theta \in G_0\} ;$$

G_0 is such that $\partial K/\{0\}$ is contained in $\operatorname{int} R_n^+$. Since S_c and $\tilde{S}$ are convex sets we can represent Γ_c, $\tilde{\Gamma}$ in the form

$$\Gamma_c\colon |y| = \rho_c(\theta); \tilde{\Gamma}\colon |y| = \tilde{\rho}(\theta) . \tag{8.58}$$

From Theorem 3.2 we deduce that, for any $\varepsilon > 0$,

$$|w_c(y) - \tilde{u}(y)| < \varepsilon \quad \text{if} \quad y \in K_\delta , \quad c \leqq c(\varepsilon, \delta) , \tag{8.59}$$

where K_δ is a δ-neighborhood of K intersected with $\operatorname{int} R_n^+$; $\delta > 0$. We claim that

$$|\rho_c(\theta) - \tilde{\rho}(\theta)| < C\varepsilon^{1/2} \quad \text{if} \quad \theta \in G_0 ; \tag{8.60}$$

this gives a rate of convergence of the free boundary of $u(cy)/c$ to that of $\tilde{u}(y)$.

To prove (8.60) note first that

$$K_\delta \cap \tilde{S} \text{ contains } S_{c,\varepsilon} = \{K_\delta \cap S_c \text{ minus a } C\varepsilon^{1/2}\text{-neighborhood of } K \cap S_c\} . \tag{8.61}$$

Indeed, if $y \notin K_\delta \cap \tilde{S}$ then (cf. [5])

$$\sup_B \tilde{u} > \varepsilon$$

where B is a ball with center y and radius $C\varepsilon^{1/2}$; hence, by (8.59), $\sup_B w_c > 0$, i.e., $y \notin S_{c,\varepsilon}$.

Next $\rho_c(\theta)$ is uniformly Lipschitz in θ for $(|y|, \theta)$ in $K_{\delta/2}$ and small c, since $K_\delta \cap S_c$ is convex and contains a fixed K_δ-neighborhood of $y = 0$. Also $\rho(\theta)$ is Lipschitz in θ. These facts together with (8.61) and its counterpart with $\tilde{S}$, S_c interchanged, give the assertion (8.60) with a suitable C.

9. Asymptotic estimates when $c \to \infty$. Define, for any $\varepsilon > 0$,

$$D_\varepsilon = \{\varepsilon\text{-neighborhood of the ridge}\} \cap \Pi_{n+1} . \tag{9.1}$$

THEOREM 9.1. *Suppose that* (1.9) *holds and* $c \geqq \alpha a_i (0 \leqq i \leqq n)$. *Then there exist positive constants* B, c^* *independent of* α, *such that, if* $c > c^*$,

$$C \textit{ is contained in } D_{B/c} . \tag{9.2}$$

Thus

$$u(\pi) = g(\pi) \textit{ outside the } B/c\textit{-neighborhood of the ridge} . \tag{9.3}$$

Proof. Suppose $\pi^0 = (\pi_0^0, \pi_1^0, \cdots, \pi_n^0)$ is in G_0 and dist.$(\pi^0, R) = B/c$. Let

$$w(\pi) = -\delta c|\tilde{\pi} - \tilde{\pi}^0|^2 + a_0(1 - \pi_0)$$

where $\tilde{\pi} = (\pi_1, \cdots, \pi_n)$, $\tilde{\pi}^0 = (\pi_1^0, \cdots, \pi_n^0)$. Clearly w lies below the obstacle g in G_0. Since

$$\text{dist.}(\pi^0, G_i) > \frac{B}{c} \quad (i \geqq 1) ,$$

in each G_i w decreases at a rate

$$\geqq \delta c|\tilde{\pi} - \tilde{\pi}^0| - a_0 \geqq B\delta - a_0 > A_0 \quad (A_0 = \max_{0 \leqq i \leqq n} a_i)$$

provided

$$B\delta > 2A_0 . \tag{9.4}$$

This rate of decrease is faster than the linear rate of decrease of the obstacle g in G_i. Hence w lies below g.

Next, $w < 0$ outside some $(A_1/\sqrt{c})$-neighborhood N (in Π_{n+1}) of π^0. We now compare w with u in N. By the calculation leading to (7.5) we find that

$$Mw - \alpha w > -K_1 c\delta\pi_0^2 - \alpha w > -c$$

if δ is sufficiently small independently of c; we use the fact that

$$\alpha w \leqq \alpha a_0(1 - \pi_0) \leqq \frac{3}{4}\alpha a_0 c \leqq \frac{3}{4} c \text{ in } N .$$

Since $u \geqq w$ on ∂N, Lemma 3.1 implies that $u \geqq w$ in N.

Since $w = g$ at π^0 it follows also that $u(\pi^0) = g(\pi^0)$ provided δ is sufficiently small and provided (9.4) holds. This completes the proof of (9.2) for points in $C \cap G_0$; the proof for $C \cap G_i$ $(i \geqq 1)$ is similar.

Denote by $\breve{G}$ any compact subset of $\mathring{\Pi}_{n+1}$ and set

$$\tilde{D}_\varepsilon = \{\varepsilon\text{-neighborhood of the ridge}\} \cap \breve{G} .$$

THEOREM 9.2. *There exists a positive constant A (depending on $\breve{G}$) such that, for all c sufficiently large,*

$$C \text{ contains } \tilde{D}_{A/c} . \tag{9.5}$$

The proof is similar to the proof of a corresponding result in [6; § 4] for the elastic-plastic torsion problem.

Proof. Suppose $\pi^0 \in \tilde{D}_{A/c} \cap G_0$ and $\pi^0 \notin C$. Suppose for simplicity that π^0 is close to G_1 at least as much as it is to any other G_i, $i \geqq 1$. Take points $\pi^1 \in G_0$, $\pi^2 \in G_1$ such that π^0 is the center of the segment $\overline{\pi^1\pi^2}$, $|\pi^1 - \pi^2| = A_0\sigma$, $\sigma = \text{dist.}(\pi^0, R)$; A_0 is chosen so that

$$-[g(\pi^2) + g(\pi^1) - 2g(\pi^0)] \geqq A_1\sigma ;$$

both A_0, A_1 are positive constants depending only on a_0, a_1. Since $u(\pi^1) \leqq g(\pi^1)$, $u(\pi^2) \leqq g(\pi^2)$, $u(\pi^0) = g(\pi^0)$, we obtain

$$A_1\sigma \leqq -[u(\pi^2) + u(\pi^1) - 2u(\pi^0)] \leqq A_2\sigma^2 |u|_{W^{2,\infty}(N)} ,$$

where N is some neighborhood of π^0. By standard estimates for variational inequalities [4], the right hand side is bounded by $A_3\sigma^2 c$; here A_2, A_3 are positive constants independent of c. It follows that $\sigma \geqq 1/(A_3 c)$, and the proof is complete.

10. The case $\alpha = 0$. For simplicity we shall assume in this section that (1.9) holds. Since $c > 0$, if $E^\pi\tau$ is sufficiently large then $J_\pi(\tau) > V(\pi)$. Thus we may write

$$V(\tau) = \inf_{E^\pi\tau < K_0} J_\pi(\tau) \tag{10.1}$$

where K_0 is some sufficiently large positive constant (depending on c).

The existence of a bounded solution (and, in fact, uniformly continuous in $\mathring{\Pi}_{n+1}$) for the variational inequality (2.1) with $\alpha = 0$ is proved in the same way as for $\alpha > 0$. Theorem 3.2 remains valid with the same proof when $\alpha = 0$. Defining $\tilde{\tau}$ by (2.12) and recalling (7.11) we clonclude that $E^\pi\tilde{\tau} < \infty$. But then we can apply Ito's formula in order to deduce that $u(\pi) = J_\pi(\tilde{\tau})$. We also get, by Ito's formula,

$$u(\pi) \leqq J_\pi(\tau)$$

for any stopping time τ with $E^\pi\tau < K_0$. Using (10.1) we deduce that

$$u(\pi) = V(\pi) = J_\pi(\tilde{\tau}) \quad \text{if} \quad \pi \in \mathring{\Pi}_{n+1} . \tag{10.2}$$

This proves the uniqueness of the solution u of (2.1) when $\alpha = 0$.

Using (10.1), the proof of Theorem 6.1 can be extended with minor changes to the case $\alpha = 0$. Theorem 6.2 remains valid with the same proof.

The results of §§ 4, 5, 7, 9 extend without any changes to the case $\alpha = 0$; instead of (7.12) we now have

$$u(\pi) \leqq A_2 c \log \frac{1}{c} . \tag{10.3}$$

From Theorem 8.1 we deduce (for $\alpha = 0$) that

$$V(\pi) = \Big(\sum_{i=0}^{n} \gamma_i \pi_i\Big) c \log \frac{1}{c} + 0\Big(c\Big(\log \frac{1}{c}\Big)^{1/2}\Big) \text{as} \quad c \longrightarrow 0 . \tag{10.4}$$

To generalize Theorem 8.2, consider the variational inequality (8.11) for $\alpha = 0$:

$$\begin{gathered} L_0 \tilde{u} + 1 \geqq 0 , \\ \tilde{u} \leqq a_0 |y| , \\ (L_0 \tilde{u} + 1)(\tilde{u} - a_0 |y|) = 0 \end{gathered} \tag{10.5}$$

in R_n^+. A trivial solution is given by $a_0|y|$. We exclude this solution by requiring that

$$\tilde{u}(y) = O(|y^\theta) \quad \text{for some} \quad 0 < \theta < 1 . \tag{10.6}$$

THEOREM 10.1. *Let* (1.9) *hold. Then there exists a unique solution* $\tilde{u}$ *of* (10.5), (10.6); *further,*

$$0 \leqq \tilde{u}(y) \leqq C \log (|y| + 1) \tag{10.7}$$

for some positive constant C, *and*

$$\frac{u(cy)}{c} \longrightarrow \tilde{u}(y) \tag{10.8}$$

uniformly in y *in compact subsets of* int R_n^+.

Proof. Let

$$z(y) = \begin{cases} a_0 |y| & \text{if} \quad |y| \leqq \delta , \\ A \log |y| + B & \text{if} \quad |y| > \delta . \end{cases}$$

For suitable positive constants δ, A, B, one finds that z is a supersolution, i.e., $L_0 z + 1 \leqq 0$. Hence

$$0 \leqq \tilde{u}_A(y) \leqq z(y)$$

where $\tilde{u}_A$ is the solution of (8.14) with $\alpha = 0$. It follows that

$$0 \leqq \tilde{u}(y) \leqq C \log(|y| + 1) , \quad \tilde{u}(y) = \lim \tilde{u}_A(y) , \tag{10.9}$$

where C is a generic positive constant independent of c.

Next

$$u(\pi) \leqq w_0(r) ,$$

where w_0 appears in (7.10). Recalling the precise form of $w_0(r)$ we compute that

(10.10) $$w_c(y) = \frac{u(\pi^c)}{c} \leqq \frac{w_0(\pi^c)}{c} \leqq C\log(|y| + 1)$$

provided $|y| \leqq \delta_0/c$ where δ_0 is any positive constant (independent of c).

We are now ready to proceed with the proof of (8.31), (8.32) in the case $\alpha = 0$. From (10.9), (10.10) and the form of the cost functionals corresponding to $\tilde{u}$, w_c we see that we may restrict the τ to satisfy

(10.11) $$\tau \leqq \bar{\tau}^A\ , \quad E_y\tau \leqq C\log(|y| + 1) \leqq C\log(A + 1)\ .$$

The last term in (8.23), for $\alpha = 0$, is bounded by

(10.12) $$I_A = C\log(A + 1)P_y[\tau^A < \tau]\ .$$

Now, for any $\beta > 0$,

(10.13) $$\begin{aligned} P_y[\bar{\tau}^A < \tau] &= P_y[e^{-\beta(\bar{\tau}^A - \tau)} > 1] \leqq E_y[e^{-\beta(\bar{\tau}^A - \tau)}] \\ &\leqq \{E_y[e^{-\beta p\bar{\tau}^A}]\}^{1/p}\{E_y[e^{\beta q\tau}]\}^{1/q} \end{aligned}$$

where $1/p + 1/q = 1$, $p > 1$, $q > 1$.

Since the stopping times which minimize the cost functions are exit times, we may take τ to be an exit time. Using the second inequality in (10.11) it then follows by ([8; p. 43]) that

(10.14) $$E_y[e^{\beta q\tau}] \leqq C \quad \text{provided} \quad \beta = \frac{1}{C\log(A + 1)}\ .$$

From the proof of (8.19) with $\lambda - 1 = p\beta/C$ we get

$$E_y[e^{-\beta p\tau^A}] \leqq \frac{C|y|^\lambda}{A^\lambda} \leqq \frac{C|y|^\lambda}{A}$$

substituting this estimate and (10.14) into (10.13), we get

(10.15) $$P_y[\bar{\tau}^A < \tau] \leqq \frac{C|y|^{\lambda/p}}{A^{1/p}}\ .$$

Consequently, from (10.12), for any $\eta > 0$,

$$I_A < \eta \quad \text{if} \quad A \text{ is sufficiently large}\ ;$$

A is independent of c. From now on A is fixed. Hence, if c is sufficiently large (depending on A),

$$E_y\left|\frac{|y_c(\tau)|}{1 + c|y_c(\tau)|} - |y_c(\tau)|\right| < \eta\ .$$

In order to complete the proof of (8.31), (8.32), it remains to show that

$$E_y\| y_c(\tau)| - |y(\tau)\| < \eta \, . \tag{10.16}$$

Now, by (10.11), for any $T > 0$,

$$P_y[\tau > T] \leqq \frac{1}{T} E_y\tau \leqq \frac{C}{T}\log(|y| + 1) \, .$$

Hence, if y varies in a compact subset,

$$AE_y[\tau > T] < \eta \quad \text{for a suitable } T > 0 \, .$$

Since, on the other hand, (8.27) holds, the estimates (10.16) follow if c is small enough. We have thus completed the proof of (8.31), (8.32).

Suppose finally that $\tilde{\tilde{u}}$ is another solution of (10.5), (10.6). Repeating the preceding proof of (8.31), (8.32) with $w_c(y)$ replaced by $\tilde{\tilde{u}}(y)$ and choosing p in (10.15) such that $1/p > \theta$, we find that $\tilde{\tilde{u}} \equiv \tilde{u}$.

11. The case where $w(t)$ is k-dimensional. In this section we extend many of the results of the previous sections to the case where $w(t)$ is k-dimensional; the condition (1.1) is dropped. Thus the generator M is generally a degenerate elliptic operator in the entire region Π_{n+1}. We assume, however, that (1.9) holds, so that

$$J_\pi(\tau) = E^\pi\left[c\int_0^\tau e^{-\alpha t}dt + e^{-\alpha\tau}g(\pi(\tau))\right] . \tag{11.1}$$

From (11.1), (1.15) and the strong Markov property we get

$$\begin{aligned} V(\pi) = \inf_{\tau \leqq \tau_N} E^\pi\Big\{c\int_0^{\tau\wedge\tau_N} e^{-\alpha t}dt &+ e^{-\alpha\tau}g(\pi(\tau))I_{\tau<\tau_N} \\ &+ e^{-\alpha\tau_N}V(\pi(\tau_N))I_{\tau=\tau_N}\Big\} \end{aligned} \tag{11.2}$$

for any stopping time τ_N.

THEOREM 11.1. *There exists a Π^0_{n+1}-neighborhood $\tilde{S}_i$ of e_i such that $\tilde{S}_i \subset S$.*

Proof. Set $W = V - g$ in a neighborhood N of e_0 where $g(\pi) = a_0(1 - \pi_0)$, and let τ_N = exit time from N. Thus, for any stopping time $\tau \leqq \tau_N$

$$\begin{aligned} E^\pi[e^{-\alpha\tau}g(\pi(\tau)) - g(\pi)] &= E^\pi\left[\int_0^\tau ce^{-\alpha t}(M - \alpha)g(\pi(t))dt\right] \\ &= E^\pi\left[\int_0^\tau ce^{-\alpha t}(-\alpha g)(\pi(t))dt\right] . \end{aligned}$$

Therefore, by (11.2),

$$W(\pi) = V(\pi) - g(\pi) = \inf_{\tau \leqq \tau_N} E^\pi \left\{ \int_0^{\tau \wedge \tau_N} e^{-\alpha t}(c - \alpha g)(\pi(t))dt + e^{-\alpha \tau_N} W(\pi(\tau_N)) I_{\tau = \tau_N} \right\} . \tag{11.3}$$

Note also that $W \leqq 0$ and that $c - \alpha g \geqq C^* > 0$ if N is sufficiently small.

The function z defined following (3.11) satisfies

$$Mz - \alpha z + \gamma \geqq 0 \quad (\text{with } \gamma < c - \alpha g) ,$$
$$z \leqq 0 ,$$
$$(Mz - \alpha z + \gamma)z = 0 ,$$

and

$$z \leqq w \quad \text{on} \quad |y| = R .$$

Using Ito's formula we obtain

$$z(\pi) = \inf_{\tau \leqq \tau_N} E^\pi \left[\int_0^{\tau \wedge \tau_N} e^{-\alpha t} \gamma dt + e^{-\alpha \tau_N} z(\pi(\tau_N)) I_{\tau = \tau_N} \right] .$$

Comparing with (11.3) we conclude that

$$z(\pi) \leqq W(\pi) .$$

Since $z(\pi) = 0$ where π varies in some neighborhood of e_0, the same follows for W; this completes the proof.

Theorems 6.1, 6.2 remain valid with the same proof.

LEMMA 11.2. *The estimate* (7.11) *is valid.*

Proof. Because of the degeneracy of M, we need to choose the functions W_i differently than in the proof of (7.11) in § 7. For simplicity we exhibit the construction in case $n = 2$. Take

$$W_0 = \log \frac{\pi_1 + \pi_2}{\varepsilon} + \log \frac{\delta_1 \pi_1 + \delta_2 \pi_2}{\varepsilon}$$

outside an ε-neighborhood of e_0, where $\delta_1 = 1$, $\delta_2 = 3$. Thus $W_0 \geqq 0$ and

$$\begin{aligned} MW_0 &= M \log(\pi_1 + \pi_2) + M \log(\delta_1 \pi_1 + \delta_2 \pi_2) \\ &\leqq -\frac{A_0}{r^2}[(\pi_1 - \pi_2)^2 + (\delta_1 \pi_1 - \delta_2 \pi_2)^2] \\ &\leqq -\frac{A(\pi_1 + \pi^2)^2}{r^2} = -A \end{aligned}$$

where $r = \pi_1 + \pi_2$ and A_0, A are positive constants.

Similarly, we define the W_i with respect to e_i and notice that

$$MW_i \leqq 0 , \quad W_i \geqq 0 .$$

Hence $W = \sum_{i=0}^{2} W_i$ satisfies $MW < -A$, $W > 0$ outside an ε-neighborhood of the vertices. This implies, by Ito's formula,

$$E^{\pi}\tau_{\varepsilon} \leqq \frac{W}{A} ,$$

and (7.11) follows.

Using (7.11) we can now derive Theorem 8.1 as before.

Theorem 8.2 asserts that

$$\frac{u(cg)}{c} \longrightarrow \tilde{u}(g) \quad \text{as} \quad c \longrightarrow 0 \tag{11.4}$$

where $\tilde{u}(g)$ is defined by (8.25), (8.26). The proof can actually be given by comparing the cost functionals and without introducing variational inequalities at all. Notice that the crucial estimate (8.12) remains valid here (with the same proof) and that also the inequality

$$\frac{u(cg)}{c} \leqq A \tag{11.5}$$

which is needed in proving (11.4) is true (in fact, taking $\tau \to \infty$ in the cost functional which defines u we obtain (11.5)).

The proof of Theorem 3.2 extends to $\tilde{u}$ (cf. the proof of Theorem 11.1), showing that the stopping set $\tilde{S}$ contains a neighborhood of each vertex. This proves the part

$$S \supset N_{\delta_1 c}$$

of Theorem 7.1; the other part follows as in the proof of Theorem 7.1, since

$$M(\log r) \geqq -K_1\pi_0^2 .$$

The convexity of $\tilde{S}$ (Theorem 8.3) remains unchanged. Finally, the results of § 10 (the case $\alpha = 0$) extend with minor changes.

We shall now obtain additional information, taking $n = 2$ and $w(t)$ to be 1-dimensional. We also take for simplicity

$$\lambda_0 = 0 , \quad \lambda_1 = -1 , \quad \lambda_2 = 1 .$$

Writing $\pi_i(t)$ in terms of the observed process (see (1.11)) we easily compute that

$$\xi(t) = \frac{1}{2}\log\frac{\pi_1}{\pi_2}\frac{p_1}{p_2} \tag{11.6}$$
$$t = \log\frac{\pi_0^2}{\pi_1\pi_2}\frac{p_1p_2}{p_0^2}$$

where $p_j = \pi_j(0)$, $\pi_j = \pi_j(t)$.

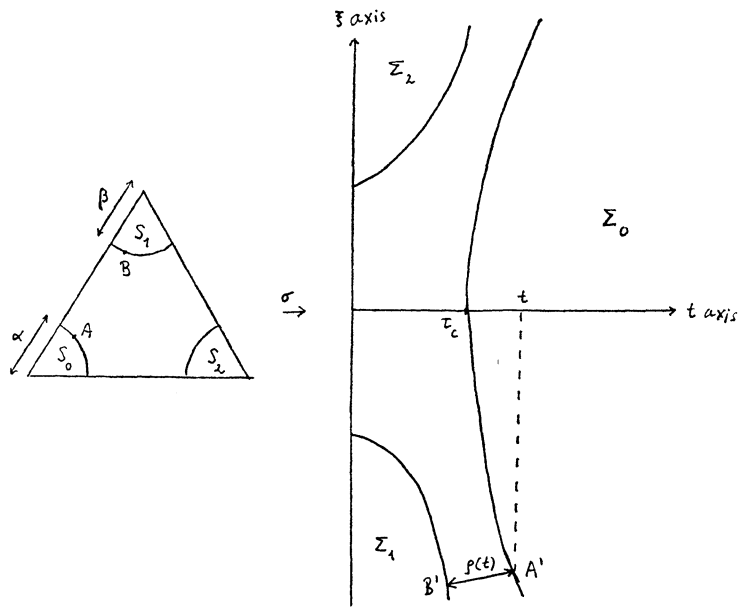
The mapping

$$\sigma: (\xi(t), t) \longrightarrow \pi(t)$$

is 1-1, mapping the half-plane $t > 0$ onto a subset of $\mathring{\prod}_3$ defined by

$$\pi_0^2 > \pi_1\pi_2\frac{p_0^2}{p_1p_2} .$$

The ridge of $\mathring{\prod}_3$ is not in the stopping set S and σ maps S_i onto a set $\sum_i$; see the accompanying figure.



Take a point t on the t-axis and mark the point $A' = (\xi', t)$ on $\partial\Sigma_0$ with $\xi' < 0$. Denote by $\rho(t)$ the distance from A' to $\partial\Sigma_1$; it is achieved at $B' \in \partial\Sigma_1$. Denote by A, B the inverse images of A', B' under σ.

THEOREM 11.3. *As* $t \to \infty$

$$\rho(t) \longrightarrow \frac{2}{\sqrt{5}}\left|\log\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right| \tag{11.7}$$

where α, β are positive constants described in the previous figure.

Proof. Write

$$A = (\pi_0, \pi_1, \pi_2) , \quad B = (\bar{\pi}_0, \bar{\pi}_1, \bar{\pi}_2) .$$

Then, as $t \to \infty$,

$$\begin{aligned} &\pi_2 \longrightarrow 0 , \quad \bar{\pi}_2 \longrightarrow 0 , \quad \pi_0 \longrightarrow \alpha , \\ &\pi_1 \longrightarrow 1 - \alpha , \quad \bar{\pi}_0 \longrightarrow \beta , \quad \bar{\pi}_1 \longrightarrow 1 - \beta . \end{aligned}$$

Setting

$$\pi_2 = \varepsilon , \quad \pi_2 = \tilde{\varepsilon} ,$$

we can then write

$$\begin{aligned} &\pi_0 = \alpha - \gamma_\varepsilon , \quad \pi_1 = 1 - \alpha + \gamma_\varepsilon - \varepsilon , \\ &\bar{\pi}_0 = \beta - \delta_{\varepsilon\tilde{\varepsilon}} , \quad \bar{\pi}_1 = 1 - \beta + \delta_{\varepsilon\tilde{\varepsilon}} - \tilde{\varepsilon} , \end{aligned}$$

where

$$\varepsilon \longrightarrow 0 , \quad \tilde{\varepsilon} \longrightarrow 0 , \quad \gamma_\varepsilon \longrightarrow 0 , \quad \delta_{\varepsilon\tilde{\varepsilon}} \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \infty .$$

From (11.6) we find that

$$\rho = \rho(t) = \left\{ \frac{1}{4}\left[\log \frac{\pi_2}{\pi_1}\frac{\bar{\pi}_1}{\bar{\pi}_2}\right]^2 + \left[\log \frac{\pi_0^2}{\pi_1\pi_2}\frac{\bar{\pi}_1\bar{\pi}_2}{\bar{\pi}_0^2}\right]^2 \right\}^{1/2} .$$

Hence

$$\rho^2 = \frac{1}{4}\left[\log\left(\lambda\frac{1-\beta}{1-\alpha}(1 + o(1))\right)\right]^2 + \left[\log\frac{1}{\lambda}\frac{\alpha^2(1-\beta)}{\beta^2(1-\alpha)}(1 + o(1))\right)\right]^2$$

where $\lambda = \tilde{\varepsilon}/\varepsilon$, $o(1) \to 0$ as $t \to \infty$.

From the definition of B' it follows that

$$\rho = \tilde{\rho}(1 + o(1))$$

where

$$\tilde{\rho} = \min_{\lambda} \left\{ \frac{1}{4}\left[\log\lambda\frac{1-\beta}{1-\alpha}\right]^2 + \left[\log\lambda\frac{\beta^2(1-\alpha)}{\alpha^2(1-\beta)}\right]^2 \right\}^{1/2} .$$

Set

$$\rho(\lambda) = \frac{1}{4}(\log k\lambda)^2 + (\log h\lambda)^2 \quad (k > 0, h > 0) .$$

Then $\min \rho(\lambda)$ is obtained at

$$\lambda = h^{-4/5}k^{-15} .$$

Using this value in our special case of $\tilde{p}$, (11.7) follows.

Consider next the point $(\xi, t) = (0, \tau_c)$ on $\partial\Sigma_0$.

THEOREM 11.4. *As* $c \to 0$

$$\tau_c \sim \log \frac{1}{\gamma^2 c^2} \tag{11.8}$$

where γ *is a positive constant.*

Proof. $(0, \tau_c)$ corresponds to (π_0, π_1, π_2) where, by Theorem 8.2 and 8.3,

$$\pi_0 \sim \frac{1}{1 + \gamma_1 c + \gamma_2 c}$$

$$\pi_i \sim \frac{\gamma_i c}{1 + \gamma_1 c + \gamma_2 c} \quad (i = 1, 2)$$

and $\gamma_1 = \gamma_2 = \gamma$ since $\pi_1 = \pi_2$. Since

$$\tau_c = \log \frac{\pi_0^2}{\pi_1 \pi_2} = \log \frac{\pi_0^2}{\pi_1^2} ,$$

the assertion follows.

REMARK. Theorems 11.3, 11.4 have the advantage of providing direct information on the observed process $(\xi(t), t)$. In case $n > 2$ we get a similar picture with $n + 1$ regions Σ_i in the half-plane $t > 0$ (of the (ξ, t) variable).

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