A CHARACTERIZATION OF THE WEIGHTS IN A
DIVIDED DIFFERENCE

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Given points $x_0, x_1, \cdots, x_n$ of the real line which satisfy $x_0 < x_1 < \cdots < x_n$ put

$$w(u) = (u - x_0) \cdots (u - x_n) \,.$$

If function values $f(x_0), \cdots, f(x_n)$ are given, the divided difference of f with respect to $x_0, \cdots, x_n$ may be defined by

$$(1) \qquad f(x_0, \cdots, x_n) = \sum_{i=0}^{n} \frac{f(x_i)}{w'(x_i)} \,.$$

If we put

$$(2) \qquad a_i = \frac{1}{w'(x_i)} \,, \qquad i = 0, \cdots, n$$

we call $a_0, \cdots, a_n$ the weights of the divided difference. Our problem is to characterize the vectors $a : (a_0, \cdots, a_n)$ which satisfy (2) for some $x : (x_0, \cdots, x_n)$ and to show that such a correspond to an essentially unique x.

Note that

$$(3) \qquad w'(x_i) = (x_i - x_0) \cdots (x_i - x_{i-1})(x_i - x_{i+1}) \cdots (x_i - x_n) \,,$$
$$i = 0, \cdots, n \,,$$

so that translating the point x leaves a unchanged. Therefore we add the further (normalizing) assumption that $x_0 = 0$. We can now state our result.

THEOREM. *If* $n > 0$ (2) *holds for unique* $x_0, \cdots, x_n$ *satisfying* $0 = x_0 < x_1 < \cdots < x_n$ *if, and only if,*
  (i) $(-1)^{n-i} a_i > 0$, $i = 0, \cdots, n$
  (ii) $\sum_{i=0}^{n} a_i = 0$
and
  (iii) $(-1)^{n-j} \sum_{i=j}^{n} a_i > 0$, $j = 1, \cdots, n$.
  Note that this theorem may be viewed as characterizing the slopes at its zeros of a polynomial $w(u) = u(u - x_1) \cdots (u - x_n)$.

Proof. (a) Suppose $x_0, \cdots, x_n$ satisfying $0 = x_0 < x_1 < \cdots < x_n$ are given. (i) follows immediately from (3). Since $f(x_0, \cdots, x_n) = 0$ if f is a polynomial of degree $< n$, taking $f \equiv 1$ in (1) yields (ii). As

for (iii) consider the fundamental polynomials of Lagrangian interpolation in $x_0, \cdots, x_n$

$$l_i(x) = \frac{w(x)}{(x - x_i)w'(x_i)} , \qquad i = 0, \cdots, n .$$

Put

$$p_j(x) = \sum_{i=0}^{j} l_i(x) , \qquad j = 0, \cdots, n - 1 .$$

Then $p_j(x_i) = 1$, $i = 0, \cdots, j$, $p_j(x_i) = 0$, $i = j + 1, \cdots, n$ and so p_j is of positive degree. If c_j is the coefficient of x^n in p_j then

$$c_j = \sum_{i=0}^{j} a_i .$$

Now p_j' is of degree at mast $n - 1$ and by Rolle's theorem has j zeros in (x_0, x_j) and $n - 1 - j$ zeros in (x_{j+1}, x_n) which accounts for all its zeros. Thus $c_j \neq 0$ and, since the behavior of $p_j(x)$ at ∞ determines the sign of c_j, this sign is that of $p_j'(x_n)$. We proceed to determine that sign. To this end note that $p_j'(x_j) \neq 0$. If $p_j'(x_j) > 0$ then since $p_j(x_j) = 1$ and $p_j(x_{j+1}) = 0$, $p_j'(x)$ has a zero in (x_j, x_{j+1}), implying that $p_j = 0$, and contradicting the fact that p_j is of positive degree. Thus $p_j'(x_j) < 0$ which implies that $p_j'(x_{j+1}) < 0$ and hence that $p_j'(x_n)$ has the sign $(-1)^{n-j}$. We have shown that

$$(-1)^{n-j} \sum_{i=0}^{j} a_i > 0 , \qquad j = 0, \cdots, n - 1 ,$$

and hence for $j = 0, \cdots, n - 1$

$$(-1)^{n-j} \sum_{i=j+1}^{n} a_i = (-1)^{n-j}\Big(\sum_{i=0}^{n} a_i - \sum_{i=0}^{j} a_i\Big) < 0$$

in view of (ii), from which (iii) follows.

(b) We prove next that for each a satisfying (i), (ii), (iii) there exist unique $x_0, \cdots, x_n$ satisfying $0 = x_0 < \cdots < x_n$ such that (2) holds.

Let K denote the convex set in R^n defined by the inequalities

$$(4) \qquad\qquad (-1)^{n-i}t_i > 0 , \qquad i = 1, \cdots, n ,$$

and

$$(5) \qquad\qquad (-1)^{n-j} \sum_{i=j}^{n} t_i > 0 , \qquad j = 1, \cdots, n .$$

In view of (i) and (iii) K is not empty. Let S be the convex set of all $(x_1, \cdots, x_n) \in R^n$ such that $0 < x_1 < x_2 < \cdots < x_n$. Let ϕ be the continuous mapping of S into K defined by

$$\phi: (x_1, \cdots, x_n) \longrightarrow (a_1, \cdots, a_n)$$

where the a_i are as defined in (2) with $x_0 = 0$. We intend to show that ϕ is a grobal homeomorphism of S onto K. If this were so then given $a_0, \cdots, a_n$ satisfying (i), (ii) and (iii), $(a_1, \cdots, a_n)$ is in K and its unique preimage $(x_1, \cdots, x_n)$ satisfies $0 = x_0 < x_1 < \cdots < x_n$ and $a_i = (w'(x_i))^{-1}$, $i = 1, \cdots, n$. But then in view of (ii)

$$a_0 = -\sum_{i=1}^{n} a_i = -\sum_{i=1}^{n} \frac{1}{w'(x_i)} = \frac{1}{w'(x_0)},$$

and the proof of the theorem would be complete. It remains, therefore, to show that ϕ is a homeomorphism of S onto K.

We begin by showing that the Jacobian matrix of ϕ, $J(\phi)$ is nonsingular in S. Given $0 = x_0 < x_1 < \cdots < x_n$ the divided difference $f(x_0, \cdots, x_n) = 0$ if f is a polynominal of degree at most $n - 1$ while if $f = x^n$, $f(x_0, \cdots, x_n) = 1$. If we take f to be $x, x^2, \cdots, x^n$ successively we thus obtain

$$\sum_{i=1}^{n} a_i x_i^j = 0, \qquad j = 1, \cdots, n - 1,$$

$$\sum_{i=1}^{n} a_i x_i^n = 1.$$

Partial differentiation with respect to x_i gives

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & & & \\ x_1^n & x_2^n & \cdots & x_n^n \end{pmatrix} \begin{pmatrix} \dfrac{\partial a_1}{\partial x_i} \\ \vdots \\ \dfrac{\partial a_n}{\partial x_i} \end{pmatrix} + \begin{pmatrix} 1 \\ 2x_i \\ 3x_i^2 \\ \vdots \\ nx_i^{n-1} \end{pmatrix} a_i = 0$$

so that,

$$AJ(\phi) + B := \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ \vdots & & & \\ x_1^n & x_2^n & \cdots & x_n^n \end{pmatrix} J(\phi) + \begin{pmatrix} a_1 & \cdots & a_n \\ 2a_1 x_1 & & 2a_n x_n \\ \vdots & & \vdots \\ na_1 x_1^{n-1} & & na_n x_n^{n-1} \end{pmatrix} = 0.$$

Since A and B are nonzero multiplies of the Vandermonde matrix they are nonsingular and so, therefore, is $J(\phi)$.

Next we show that ϕ maps the boundary of S (which includes the point at infinity) into the boundary of K (which includes the point at infinity) in the sense that if a sequence of points of S has a limit point in ∂S then the image sequence under ϕ has its limit points in ∂K. The boundary points of S arise either from coalescence

of coordinates or coordinates tending to zero or infinity, or combinations of these. (3) reveals that if 2 or more coordinates coalesce with x_i and stay bounded away from 0 and ∞ then $w'(x_i) \to 0$ and hence $a_i \to \infty$. If $x_1 \to 0$ and $w'(x_1) \to 0$, or $x_n \to \infty$ and $w'(x_n) \to \infty$, boundary goes to boundary. The only remaining possibility is that $x_1 \to 0$ and $x_n \to \infty$ in such a way that $\{|w'(x_i)|\}$, $i = 1, \cdots, n$ stays bounded away from 0 and ∞. This can only happen if for some j satisfying $2 \leqq j \leqq n - 1$ we have $x_{j-1} \to 0$ and $x_j \to \infty$. But in this case put $p(x) = x(x - x_1) \cdots (x - x_{j-1})$ and $q(x) = (x - x_j) \cdots (x - x_n)$. Then on the one hand eventually

$$\frac{1}{2\pi i} \int_{|z|=1} \frac{dz}{p(z)q(z)} = \sum_{i=0}^{j-1} \frac{1}{w'(x_i)}$$

by the calculus of residues, while on the other the integral tends to zero since $|q| \to \infty$. Thus

$$\lim \sum_{i=0}^{j-1} a_i = \lim \sum_{i=j}^{n} a_i = 0 \ ,$$

so that the image sequence again tends to the boundary of K.

We are now in a position to show that ϕ maps S *onto* K. Since $J(\phi)$ is nonsingular, $\phi(S)$ is open in K. Let $w \in \phi(S)^c \cap K$, we claim that some neighborhood of w lies in $\phi(S)^c \cap K$. Otherwise there is a sequence $w_i = \phi(z_i)$ with $w_i \to w$. If the z_i went to the boundary, ∂S, then the $\phi(z_i)(=w_i)$ would converge to the boundary ∂K and we would have $w \in \partial K$. This contradicts our assumption and we conclude, indeed, that $\phi(S)^c \cap K$ is open. Thus $K = (\phi(S)^c \cap K) \cup (\phi(S) \cap K)$ is the union of two disjoint open sets. K, being convex, is certainly connected, however, and so we are forced to the conclusion that $\phi(S) \cap K = K$ or $\phi(S) = K$.

Thus, to sum up, we have shown that ϕ is a continuous map of convex S onto convex K which is locally one-to-one and takes the boundary of S into the boundary of K.

Next we show, following Favard (Cf. J. Favard, Cours d'Analyse de l'École Polytechnique, Vol. I, Paris, 1960, pp. 295-296.) that there is a fixed integer $k \geqq 1$ such that each point of K has exactly k preimages in S.

First we observe that if $t \in K$ then $t = \phi(x)$ cannot hold for infinitely many $x \in S$. For if $t = \phi(x)$ for infinitely many $x \in S$ then the set of such x cannot have a boundary point of S as limit point since $t \in K$ and ϕ takes ∂S into ∂K. Thus the set of solutions $\{x\}$ has a limit point in S contradicting the fact that ϕ is a local homeomorphism.

Suppose that $\bar{t} \in K$ and $x^{(1)}(\bar{t}), \cdots, x^{(s)}(\bar{t})$ are its preimages. The inverse functions $x = \Psi_1(t), \cdots, x = \Psi_s(t)$ are homeomorphisms of

some neighborhood of $\bar{t}$, and these neighborhoods have a nonempty intersection. Let $r(\bar{t})$ be the radius of the largest ball centered at $\bar{t}$ in which $\Psi_1, \cdots, \Psi_s$ are homeomorphisms. It is easy to see that $|r(t^{(1)}) - r(t^{(2)})| \leq \|t^{(1)} - t^{(2)}\|$, hence $r(t)$ is continuous in K and has a positive lower bound on every compact subset of K. Now every point t' contained in the largest ball of homeomorphy centered at t has at least as many preimages as does t, i.e., $s(t') \geq s(t)$. On the other hand in the closed ball $\|t - t'\| \leq r(t)/2$, $r(t')$ has a positive lower bound, b, and if we further restrict t' so that $\|t - t'\| < b$ then the largest ball of homeomorphy centered at t' contains t so that $s(t) \geq s(t')$, hence $s(t) = s(t')$ and $s(t)$ is a continuous integer-valued function of t in the convex set K, hence a constant, $k \geq 1$.

We next conclude the proof of our theorem by exhibiting an explicit t which has a *unique* pre-image. To this end consider the points

$$\eta_j = -\cos\frac{j\pi}{n}, \qquad j = 0, \cdots, n,$$

the extrema of the Chebyshev polynomial of the first kind of degree n. An elementary computation yields

$$f(\eta_0, \cdots, \eta_n) = (-1)^n \frac{2^{n-1}}{n}\left(\frac{f(\eta_0)}{2} + \sum_{j=1}^{n-1}(-1)^j f(\eta_j) + \frac{(-1)^n}{2}f(\eta_n)\right)$$

$$= : (-1)^n\frac{2^{n-1}}{n}\sum_{j=0}^{n}{}''(-1)^j f(\eta_j)$$

and taking $f(u)$ to be $u, u^2, \cdots, u^{n-1}$ sucessively yields

(6) $$\sum_{j=0}^{n}{}''(-1)^j \eta_j^i = 0, \qquad i = 1, \cdots, n-1.$$

Suppose $y_0 < y_1 < \cdots < y_n$ has the property that,

(7) $$\frac{(-1)^n}{W'(y_i)} = \frac{(-1)^n}{V'(\eta_i)} = \begin{cases} \dfrac{2^{n-1}}{2n}; & i = 0, \\[2ex] \dfrac{2^{n-1}}{n}(-1)^i; & i = 1, \cdots, n-1, \\[2ex] \dfrac{2^{n-1}}{2n}(-1)^n; & i = n \end{cases}$$

where $V(u) = (u - \eta_0) \cdots (u - \eta_n)$, $W(u) = (u - y_0) \cdots (u - y_n)$ and we may assume, with no loss of generality, that $y_0 = -y_n$. Then we have

$$\sum_{j=0}^{n}{}''(-1)^j y_j^i = 0, \qquad i = 1, \cdots, n-1,$$

and if we put

$$x_j = \frac{y_j}{y_n} = : Cy_j , \qquad j = 0, \cdots, n$$

then

(8) $$\sum_{j=0}^{n}{}'' (-1)^j x_j^i = 0 , \qquad i = 1, \cdots, n - 1 ,$$

$$x_0 = \eta_0 = -1 ; \qquad x_n = \eta_n = 1$$

and $x_0 < x_1 < \cdots < x_n$.

Equating the left-hand sides in (6) to the corresponding ones in (8) yields the systems

(9a) $x_1^i + \eta_2^i + \cdots + x_{n-1}^i = \eta_1^i + x_2^i + \cdots + \eta_{n-1}^i , \quad i = 1, \cdots, n - 1 ;$

n even or

(9b) $x_1^i + \eta_2^i + \cdots + \eta_{n-1}^i = \eta_1^i + x_2^i + \cdots + x_{n-1}^i , \quad i = 1, \cdots, n - 1 ;$

n odd. We claim that (9a) or (9b) implies that $\eta_i = x_i$, $i = 1, \cdots$, $n - 1$. To establish this we use the following lemma.

LEMMA. *If*

$$\sum_{j=1}^{k} b_j^i = \sum_{j=1}^{k} c_j^i , \qquad i = 1, \cdots, k$$

and $b_1 < \cdots < b_k$; $c_1 < \cdots < c_k$ then $b_j = c_j$, $j = 1, \cdots, k$.

Proof. Let

$$S_i(b) = \sum_{j=1}^{k} b_j^i , \qquad i = 1, \cdots, k .$$

We assume $S_i(b) = S_i(c)$, $i = 1, \cdots, k$. The elementary symmetric functions $\sigma_1(b), \cdots, \sigma_k(b)$ are uniquely determined by $S_1(b), \cdots, S_k(b)$ hence $\sigma_i(b) = \sigma_i(c)$, $i = 1, \cdots, k$. Thus $(c_1, \cdots, c_k)$ and $(b_1, \cdots, b_k)$ are the complete set of zeros of the same polynomial. Since they are ordered by hypothesis, the lemma follows.

The lemma applied to (9a), say, implies that $x_1, \eta_2, \cdots, x_{n-1}$ is a rearrangement of $\eta_1, x_2, \cdots, \eta_{n-1}$. Thus in the sequence $\eta_1, \cdots, \eta_{n-1}$, $x_1, \cdots, x_{n-1}$ each of $\eta_1, \cdots, \eta_{n-1}$ appears twice. But then the monotonicity of the vectors η and x implies that $x_j = \eta_j$, $j = 1, \cdots, n - 1$. The same argument prevails if (9b) holds. Thus we have shown that

$$y_j = \frac{\eta_j}{C} , \qquad j = 0, \cdots, n .$$

But, in view of the analogue of (3), (7) implies that $C = 1$.

We have established that the weight vector

$$(-1)^n \frac{2^{n-1}}{n}\left(\frac{1}{2}, -1, +1, \cdots, (-1)^{n-1}, \frac{(-1)^n}{2}\right)$$

corresponds uniquely to the (normalized) points

$$1 + \eta_j, \qquad j = 0, \cdots, n$$

and so we may conclude that $((1 + \eta_1), \cdots, (1 + \eta_n))$ is the unique element of S whose image under ϕ is

$$(-1)^n \frac{2^{n-1}}{n}\left(-1, 1, \cdots, (-1)^{n-1}, \frac{(-1)^n}{2}\right).$$

This completes the proof of the theorem.

Postscript. I. J. Schoenberg reports (Cardinal Spline Interpolation, Vol. 12, Regional Conference Series in Applied Mathematics, Philadelphia, 1973, p. 9) that S. Karlin told him that he had solved the problem of characterizing the weights in a divided difference. Professor Karlin has confirmed this in a phone conversation with one of us, but not wishing to publish his solution suggested that we present ours.

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