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ON THE SPECTRUM OF CARTAN-HADAMARD MANIFOLDS

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Let M be a simply-connected complete d-dimensional Riemannian manifold of nonpositive sectional curvature K. If $K \leqq -k^2 < 0$, then the infimum of the L^2 spectrum of the negative Laplacian is greater than or equal to $(d-1)^2k^2/4$ with equality in case $K \to -k^2$ sufficiently fast at infinity. This general result is obtained by analyzing a system of ordinary differential equations. If either $d=2$ or the manifold possesses appropriate symmetry, the result is obtained under weaker conditions by analyzing a Riccati equation. Finally the case $k=0$ is treated separately.

1. Description of results. The infimum of the L^2 spectrum is defined by

$$\lambda_1 = \inf_{\phi \neq 0} \frac{\int_M |d\phi|^2}{\int_M \phi^2} \tag{1.0}$$

when the infinum is taken over H_0^1, the closure of $C_0^\infty(M)$ in the norm $\int_M (\phi^2 + |d\phi|^2)$. Let $K_x(P)$ be the sectional curvature of the two-plane $P \subseteqq M_x$, the tangent space at x. Let $\gamma(t) = \gamma(t; 0, \xi)$ be the unit-speed geodesic emanating from $0 \in M$ and having initial velocity $\xi \in M_0$. Let

$$\varepsilon(t) = \sup_{|\xi|=1} \sup_{P \subseteqq M_{\gamma(t)}} |K_{\gamma(t)}(P) + k^2|$$

where k is a positive constant. Our main result is the following upper bound.

THEOREM. *Suppose that*

$$\int_1^\infty \varepsilon(t)dt < \infty . \tag{1.1}$$

Then

$$0 < \lambda_1 \leqq (d-1)^2k^2/4 .$$

This immediately implies

COROLLARY 1. *Suppose that outside of some compact set M has constant sectional curvature $K = -k^2 < 0$. Then* $0 < \lambda_1 \leqq (d-1)^2k^2/4$.

Finally, we have the result stated in the first paragraph.

COROLLARY 2. *Suppose that* (1.1) *holds and that* $K \leqq -k^2 < 0$ *everywhere on* M. *Then* $\lambda_1 = (d-1)^2k^2/4$.

2. **Proofs.** We will study Jacobi fields $J(t)$ along a geodesic $\{\gamma(t),\ t \geqq 0\}$ where $J(0) = 0$, $J(t) \not\equiv 0$, $(J(t), \gamma') = 0$. For this purpose, let $\{E_i(t),\ 2 \leqq i \leqq d\}$ be a parallel field of orthonormal vectors along γ with $(E_i, \gamma') = 0$. Write

$$J(t) = \sum_{i=2}^{d} f_i(t)E_i(t)\ . \tag{2.0}$$

From the Jacobi equation we have the following system of equations [2]

$$f_i''(t) + \sum_{j=2}^{d} (R(E_i, \gamma')\gamma', E_j)f_j(t) = 0 \quad (2 \leqq i \leqq d)\ . \tag{2.1}$$

By the representation of R in terms of sectional curvature, we have

$$(R(E_i, \gamma')\gamma', E_j) = -k^2\delta_{ij} + \varepsilon_{ij}$$

where $|\varepsilon_{ij}| \leqq \varepsilon(t)$.

We use the following result from ordinary differential equations.

PROPOSITION. *Consider the system*

$$f_i''(t) - k^2 f_i(t) = \sum_{j=2}^{d} \varepsilon_{ij}(t)f_j(t) \quad (2 \leqq i \leqq d) \tag{2.2}$$

where $\int_1^\infty |\varepsilon_{ij}(t)|\,dt < \infty$. *Then* (2.2) *has solutions* $f_i^{(1)}, f_i^{(2)}$ *with*

$$\begin{aligned} f_i^{(1)} &\sim e^{kt}, & f_i^{(1)\prime} &\sim ke^{kt} & (t &\longrightarrow \infty) \\ f_i^{(2)} &\sim e^{-kt}, & f_i^{(2)\prime} &\sim -ke^{-kt} & (t &\longrightarrow \infty)\ . \end{aligned}$$

For the proof see Hartman [5, p. 381] for the case $d = 2$. To apply this to (2.1) we recall that from the Rauch comparison theorem [2] $|J(t)| \to \infty$ when $t \to \infty$. Now let

$$f_i(t) = \sum_{j=2}^{d} [c_{ij}f_j^{(1)}(t) + d_{ij}f_j^{(2)}(t)]\ . \tag{2.3}$$

We claim that $c_{ij} \neq 0$ for at least one value of (i, j). Indeed, if $c_{ij} \equiv 0$, then $f_i(t) = \mathcal{O}(e^{-kt})$, $t \to \infty$ which implies that $|J(t)| \to 0$, a contradiction. Now

$$
\begin{aligned}
\frac{(J'(t),\, J(t))}{(J(t),\, J(t))} &= \frac{\sum_{i=2}^{d} f_i(t) f_i'(t)}{\sum_{i=2}^{d} f_i(t)^2} \\
&= \frac{\sum_{i,j} c_{ij}^2 f_i^{(1)} f_i^{(1)\prime}}{\sum_{i,j} c_{ij}^2 f_i^{(1)2}} (1 + o(1)) \quad (t \longrightarrow \infty) \\
&= k(1 + o(1)) \qquad (t \longrightarrow \infty)\,.
\end{aligned} \tag{2.4}
$$

Thus we have proved the following proposition.

LEMMA 1. *Let $J(t)$ be a Jacobi field along γ with $J(0) = 0$, $(J(t), \gamma') = 0$, $J(t) \not\equiv 0$. If (1.1) is satisfied, then*

$$
\frac{(J'(t),\, J(t))}{(J(t),\, J(t))} \longrightarrow k\,, \qquad t \longrightarrow \infty\,. \tag{2.5}
$$

LEMMA 2. *Let r be the geodesic distance from $0 \in M$. Then (1.1) implies that*

$$
\Delta r(\gamma(t)) \longrightarrow (d-1)k \qquad (t \longrightarrow \infty) \tag{2.6}
$$

where the convergence is uniform over S^{d-1}.

Proof. Let $\gamma(t; 0, \xi)$ be the geodesic emanating from $0 \in M$ with initial velocity ξ. Let $\{J_i(t),\, 2 \leqq i \leqq d\}$ be Jacobi fields along γ with $J_i(0) = 0$, $J_i'(0) = E_i$ where $(\gamma'(0), E_2, \cdots, E_d)$ is an orthonormal basis of M_0. Then from the second variation of arclength [1], we have

$$
\Delta r(\gamma(t)) = \sum_{k=2}^{d} \frac{(J_k'(t),\, J_k(t))}{(J_k(t),\, J_k(t))}\,. \tag{2.7}
$$

Using Lemma 1 the result follows.

LEMMA 3. *Let $m = (d-1)k$, $0 < R_0 < R_1 < \infty$,*

$$
\phi(r) = \begin{cases} e^{-mr/2} \sin \pi \dfrac{(r - R_0)}{R_1 - R_0} & R_0 \leqq r \leqq R_1 \\ 0 & \textit{otherwise.} \end{cases} \tag{2.8}
$$

Then

$$
\Delta\phi + \left|\frac{m^2}{4} + \frac{\pi^2}{(R_1 - R_0)^2}\right| \phi = (\Delta r - m)\phi'(r)\,. \tag{2.9}
$$

Proof. Calculus and the formula $\Delta\phi = \phi'' + (\Delta r)\phi'$ [1, p. 134].

Now let B be the annular domain $R_0 \leqq r \leqq R_1$.

LEMMA 4.

$$(2.10) \qquad -\int_B (\phi')^2 + \left|\frac{m^2}{4} + \frac{\pi^2}{(R_1 - R_0)^2}\right| \int_B \phi^2 = \int_B (\Delta r - m)\phi\phi' .$$

Proof. Multiply equation (2.9) by ϕ, integrate by parts and use the boundary condition $\phi = 0$.

Proof of the theorem. Let $X = \|\phi'\|^2_{L^2(B)}$, $I = \|\phi\|^2_{L^2(B)}$, $c = m^2/4 + \pi^2/(R_1 - R_0)^2$. Applying Schwarz's inequality we have

$$\left|\int_B (\Delta r - m)\phi\phi'\right| \leqq \varepsilon_1(R_0)\|\phi\|_{L_2(B)}\|\phi'\|_{L_2(B)}$$

where $\varepsilon_1(R_0) \to 0$ when $R_0 \to \infty$.

Applying this to (2.10), we have

$$(2.11) \qquad |X - cI| \leqq \varepsilon_1(R_0)\sqrt{IX} .$$

But this implies that X is smaller than the largest root of the corresponding equation, i.e.,

$$\sqrt{X} \leqq \sqrt{I}\left\{\frac{\varepsilon_1(R_0)}{2} + \sqrt{c + \frac{\varepsilon_1(R_0)^2}{4}}\right\} .$$

A glance at the definition (1.0) shows that $\lambda_1 \leqq X/I$. This holds for all $R_1 > R_0$; letting $R_1 \to \infty$, we have

$$\sqrt{\lambda_1} \leqq \frac{\varepsilon_1(R_0)}{2} + \sqrt{\frac{m^2}{4} + \frac{\varepsilon_1^2(R_0)}{4}} .$$

Finally letting $R_0 \to \infty$, we have the result $\lambda_1 \leqq m^2/4$.

To prove the lower bound, we first note that for some δ

$$(2.12) \qquad \Delta r \geqq \delta < 0 .$$

Indeed, outside of some sufficiently large compact set we can use Lemma 2. On the other hand, the proof of the Rauch comparison theorem implies that for any Jacobi field along γ with $J(0) = 0$, $(J(t), \gamma') = 0$, $J(t) \not\equiv 0$, we have $(J'(t), J(t))/(J(t), J(t)) \geqq 1/r$. Hence

$$\Delta r \geqq \frac{d-1}{r} > 0 \qquad (0 < r < \infty)$$

$$\Delta r \longrightarrow (d-1)k \qquad (r \longrightarrow \infty) .$$

Having proved (2.12), we can use the method of McKean. For this purpose let $G(t, \xi) = |J_2(t) \wedge \cdots \wedge J_d(t)|$. From (2.12) we see that $G_t/G \geqq \delta$. Now M is the image of R^d under $\exp_0$. Integrals over

M can be computed over R^d according to the following:

For any $\phi \in H_0^1$, $f \in L^1$

$$\int_M f = \int_{S^{d-1}} d\omega \int_0^\infty f(\exp_0 t\omega) G(t, \omega) dt \tag{2.13}$$

$$\int_M |d\phi|^2 \geqq \int_M |d\phi(\partial/\partial r)|^2 . \tag{2.14}$$

But

$$\begin{aligned}\int_0 \phi^2 G(t, \omega) dt &\leqq \frac{1}{\delta}\int_0^\infty \phi^2 G_t dt \\ &= -\frac{2}{\delta}\int_0^\infty \phi\phi_t G dt \\ &\leqq \frac{2}{\delta}\left(\int_0^\infty \phi^2 G dt\right)^{1/2} \left(\int_0^\infty \phi_t^2 G dt\right)^{1/2} .\end{aligned}$$

Thus

$$\int_0^\infty \phi_t^2 G dt \geqq \frac{\delta^2}{4}\int_0^\infty \phi^2 G dt .$$

Integrating this inequality on S^{d-1} and referring to (2.13)-(2.14), it is clear that we have proved

$$\int_M |d\phi|^2 \geqq \frac{\delta^2}{4}\int_M \phi^2 \quad (\phi \in H_0^1) .$$

Thus $\lambda_1 \geqq \delta^2/4 > 0$, as required.

3. On condition (1.1). In certain cases one may relax the technical condition (1.1). These are the following

$$d = 2 \tag{3.1}$$

$$M \text{ is a model [4] .} \tag{3.2}$$

The latter means that for every orthogonal transformation ϕ in M_0, there exists an isometry Φ: $M \rightarrow M$ such that $\Phi(0) = 0$, $\Phi^*(0) = \phi$.

PROPOSITION. *Suppose that the* CH *manifold* M *satisfies either* (3.1) *or* (3.2) *and in addition*

$$\varepsilon(t) \longrightarrow 0 \qquad (t \longrightarrow \infty) . \tag{3.3}$$

Then

$$0 < \lambda_1 \leqq (d-1)^2 k^2/4 .$$

Proof. Following the proof of the theorem, the result will follow once we prove Lemma 1. In case (3.1), the Jacobi equation is a single scalar equation

$$J''(t) + K(t)J(t) = 0 \tag{3.4}$$

where $K(t)$ is the Gaussian curvature. Let $h(t) = J'(t)/J(t)$. Then

$$h'(t) + h(t)^2 = -K(t) . \tag{3.5}$$

Recall the following asymptotic result [7] concerning solutions of (3.5).

$$\liminf_{t\to\infty} \sqrt{-K(t)} \leqq \liminf_{t\to\infty} h(t) \leqq \limsup_{t\to\infty} h(t) \leqq \limsup_{t\to\infty} \sqrt{-K(t)} . \tag{3.5a}$$

Thus (3.3) implies that $h(t) \to k$, which proves Lemma 1 in this case.

To treat the case (3.2), we use the following result of Greene-Wu [4, p. 25]: every proper Jacobi field $J(t)$ along a geodesic γ which is orthogonal to γ' and vanishes at 0 has the form

$$J(t) = f(t)E(t)$$

when $E(t)$ is a parallel vector field along γ and $f(t)$ is a real-valued function. The Jacobi equation then takes the form

$$f''(t) + K(t)f(t) = 0 \tag{3.6}$$

where $K(t)$ is the sectional curvature of the 2-plane spanned by $(\gamma'(t), E(t))$. Observing that (3.6) is of the same form as (3.4), we can copy the above proof for $d = 2$ to conclude Lemma 1 in this case also, thus completing the proof of the proposition.

Finally, using the method of Gage [3], we can obtain results using only Ricci curvature. Indeed, Gage has proved that

$$G_{rr} + \frac{R_{11}}{d-1} G = -\frac{G}{2(d-1)^2} \Sigma(\mu_i - \mu_j)^2 \tag{3.7}$$

where $G = |J_2 \wedge \cdots \wedge J_d|^{1/(d-1)}$, R_{11} is the Ricci curvature in the direction $\gamma(t)$ and $(\mu_2, \cdots, \mu_d)$ are the eigenvalues of the second fundamental form relative to the geodesic sphere. Ignoring the right hand member of (3.7) gives an inequality. Letting $h = G'/G$, we have the Riccati inequality

$$h'(t) + h(t)^2 \leqq -\frac{R_{11}}{d-1} .$$

Let $h_1(t)$ be the solution of the corresponding equation, with the same initial behavior. Then standard comparison methods yield

$$h(t) \leqq h_1(t) .$$

But the asymptotic result (3.5a) now applies to the $h_1(t)$. Combining all of the above, we have the following

PROPOSITION. *Suppose that for the* CH *manifold* M

$$R_{11}(\gamma(t)) \longrightarrow -(d-1)k^2 \qquad (t \longrightarrow \infty)$$

then

$$\lambda_1 \leqq \frac{(d-1)^2k^2}{4} .$$

4. Asymptotic flatness. The previous results are all formulated under the hypothesis $k \neq 0$, which we now remove.

DEFINITION. The CH manifold M is *asymptotically flat* if $k=0$ and either (1.1) holds or (3.3) holds with $d = 2$ or (3.3) holds where M is a model.

PROPOSITION 4.1. *Suppose that the* CH *manifold* M *is asymptotically flat. Then* $\lambda_1 = 0$.

Proof. In this case $\Delta r \to 0$ when $r \to \infty$. Using the trial function $f = \sin \pi(r - R_0)/(R_1 - R_0)$ in the definition of λ_1, the previous proof remains unchanged, with the conclusion $\lambda_1 = 0$.

Conversely, we have the following negative result.

PROPOSITION 4.2. *There exists a* CH *manifold with* $\lambda_1 = 0$ *and curvature function* K *which satisfies* $\liminf_{r\to\infty} K < 0$.

For the proof we will construct a 2-dimensional CH manifold M with metric

$$ds^2 = dr^2 + G(r)^2 d\theta^2$$

where $G'' + KG = 0$, $G(0) = 0$, $G'(0) = 1$. The curvature function $K(r)$ is

$$K(r) = \begin{cases} 0 & r \notin (a_k, a_k + \varepsilon_k) \\ -1 & r \in (a_k, a_k + \varepsilon_k) \end{cases}$$

where a_k, ε_k are to be specified below.

Let $h = G'/G$. Then h satisfies the Riccati equation $h' + h^2 = -K$, with $h(r) = 1/r$ for $0 < r < a_1$. Note the following facts:

(i) On any interval $(a_k, a_k + \varepsilon_k)$, $h' = 1 - h^2 \leqq 1$ and thus $h(a_k + \varepsilon_k) \leqq h(a_k) + \varepsilon_k$.

(ii) On any interval $(a_k + \varepsilon_k, a_{k+1})$, the Riccati equation has the explicit solution $h(r) = (a_k + \varepsilon_k)h(a_k + \varepsilon_k)/r$.

Now let $\varepsilon_k = 1/2^{k+1} (k \geqq 1)$, $a_1 = 4$, $a_{k+1} > 4(a_k + \varepsilon_k)$. Such a choice is clearly possible, we will show that $h(r) \to 0$. First we show inductively that $h(a_k + \varepsilon_k) < 1/2^k$.

On the interval $0 < r < a_1$, $h(r) = 1/r$ and thus $h(a_1) < 1/4$. Using (i) above, we have $h(a_1 + \varepsilon_1) \leqq h(a_1) + \varepsilon_1 < 1/4 + \varepsilon_1 = 1/2$. Now if $h(a_k + \varepsilon_k) < 1/2^k$, then on the interval $a_k + \varepsilon_k < r < a_{k+1}$, $h(r) = h(a_k + \varepsilon_k)(a_k + \varepsilon_k)/r$ and thus $h(a_{k+1}) < (1/4)h(a_k + \varepsilon_k) < 1/2^{k+2}$. Using (i) again, $h(a_{k+1} + \varepsilon_{k+1}) \leqq h(a_{k+1}) + \varepsilon_{k+1} < 1/2^{k+1}$.

Finally, we check that $h(r) \to 0$ as $r \to \infty$. Indeed, on the interval $(a_k + \varepsilon_k, a_{k+1})h$ is decreasing, and thus $h(r) \leqq h(a_k + \varepsilon_k) < 1/2^k$. On the interval $(a_{k+1}, a_{k+1} + \varepsilon_{k+1})$ we have $h' \leqq 1$ and thus $h(r) \leqq h(a_{k+1}) + (r - a_{k+1}) \leqq 1/2^k + 1/2^{k+2}$.

We can now prove that $\lambda_1 = 0$. Indeed, from earlier work [8] we know that $(4\lambda_1)^{1/2} \leqq \lim_{r\to\infty} G_r/G$. Thus $\lambda_1 = 0$, as required.

REMARKS 1. By modifying the above example, it is possible to find a metric for which $\lambda_1 = 0$ and $\liminf_{r\to\infty} K(r) = -\infty$. Indeed, it suffices to replace -1 by a sequence going to $-\infty$ and choose $\varepsilon_k \to 0$ sufficiently fast.

2. It would be interesting to find a necessary condition for $\lambda_1 = 0$, expressed in terms of the curvature function. From our previous paper [8] we know that $\lambda_1 = 0$ implies $\liminf_{r\to\infty} h(r) = 0$. But we do not know what this says about $K(r)$.

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